UNIVERSITY OF SOUTHERN QUEENSLAND

Limits

In mathematics, a **limit** is the value that a function 'approaches' as the input approaches some value. Limits are essential to calculus. In formulas, the *limit* is usually denoted lim. This can written as:

$$\lim_{x \to a} f(x) = L\,,$$

which means that the function, $f(x)$, can be made to be as close to the limit, L, as possible by making the input, x, sufficiently close to a. The above equation reads as 'the limit of f of x, as x approaches a, is L'.

Properties of Limits

- If c is a constant, then the limit $x \to a$ is the constant, that is symbolically:

$$\lim_{x \to a} c = c\,.$$

- If the function, $f(x)$ is continuous at $x = a$, then

$$\lim_{x \to a} f(x) = f(a)\,.$$

- If the limits of $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then
  - Addition property states:

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)\,.$$

  - Subtraction property states:

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)\,.$$

  - Scalar multiple property, if c is a constant, states:

$$\lim_{x \to a} (c \times f(x)) = c \times \lim_{x \to a} f(x)\,.$$

  - Multiplication property states:

$$\lim_{x \to a} (f(x) \times g(x)) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)\,.$$

- Division property states, if $\lim_{x \to a} g(x) \neq 0$:

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} .$$

- The Power Law states that if n is an integer, and the limit, $\lim_{x \to a} f(x)$ exists, then

$$\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x) \right)^n .$$

- The Root Law states:

$$\lim_{x \to a} \left(\sqrt[n]{f(x)} \right) = \sqrt[n]{\lim_{x \to a} f(x)} .$$

For example using the addition and scalar properties we can find the limit:

$$\begin{aligned}
& \lim_{x \to 3} \left(2x^3 + 5 \right) \\
= \; & \lim_{x \to 3} \left(2x^3 \right) + \lim_{x \to 3} 5 && \text{using the addition property,} \\
= \; & 2 \times \lim_{x \to 3} \left(x^3 \right) + \lim_{x \to 3} 5 && \text{using the scalar multiplication property,} \\
= \; & 2 \times 3^3 + 5 && \text{using the power law and the constant property,} \\
= \; & 59 .
\end{aligned}$$

Limits of trigonometric functions

The trigonometric functions have following important limit properties:

- $\lim_{x \to a} (\sin x) = \sin a$;
- $\lim_{x \to a} (\cos x) = \cos a$;
- $\lim_{x \to a} (\tan x) = \tan a$;
- $\lim_{x \to a} \left(\frac{\sin x}{x} \right) = 1$;
- $\lim_{x \to a} \left(\frac{1 - \cos x}{x} \right) = 0$.

For example,

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1+\sin x}{1+\cos x}\right) &= \frac{\lim\limits_{x\to 0} 1 + \lim\limits_{x\to 0} \sin x}{\lim\limits_{x\to 0} 1 + \lim\limits_{x\to 0} \cos x} \\
&= \frac{1+\sin 0}{1+\cos 0} \\
&= \frac{1}{2}\,.
\end{aligned}$$

Resources

- Other QuickTips flyers;
- Online resources at Study Support;
- Make a consultation with a Mathematics Learning Advisor.