Module B3

Algebra: tools for change

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Introduction

The introduction and development of algebra goes hand in hand with the development of our technological society. Today, we see the applications of algebra in every aspect of our lives from the price we pay for a loaf of bread, to the quality of our water, to the computer games children play.

All of the above involve the investigation of the properties of numbers and quantities by means of general symbols or variables. Many look on algebra as the language of mathematics and it is that language which we will use as a tool to describe, generalize, predict or make abstract relationships between changing quantities. Hence our title ‘Algebra: Tools for change’.

In this module we will build on your understanding of algebra and consolidate your techniques of algebraic manipulation developed previously (possibly in *Mathematics tertiary preparation mathematics level A*).

More formally, when you have successfully completed this module you should be able to:

- demonstrate an understanding of the nature of a variable
- manipulate algebraic formulas and expressions including fractions using processes of grouping like terms and factorizing
- develop and solve simple linear equations
- develop and solve quadratic equations by factorizing and by using the quadratic formula
- solve systems of linear equations
- solve word problems involving simple linear equations, simultaneous equations and quadratic equations
- use index laws to simplify algebraic expressions
- demonstrate an understanding of absolute value.

3.1 Describing relationships

In algebra actual numbers or quantities are replaced by letters or pronumerals. These generalized quantities are referred to as **variables** and may assume **any number of a set of values**. This enables us to examine the general properties of numbers without referring to specific examples.

So how can we use pronumerals to translate everyday language into algebraic language. Consider the following:

Example

In a particular company, the selling price of a shirt is set each year. What will the income to the company be if 250 shirts were sold in the first six months and 150 in the next six months?

Translate these statements into algebraic language.

The variables are:

- Selling price of a shirt, call it P
- Income to the company, call it I

In algebraic language this would be:

$I = 250P + 150P$

Example

The income to a company is determined by calculating the product of the quantity of goods and their selling price.

Translate this into algebraic language.

The variables:

- Income to the company, call it I
- Selling price, call it S
- Quantity of goods, call it Q

In algebraic language this would be:

$I = S \times Q$ or $I = SQ$

Example

At a constant temperature, the volume occupied by a gas is the quotient of a constant, called K, and the pressure exerted on the gas.

Translate into algebraic language.

The variables are:

- Volume of the gas, call it V
- Pressure exerted on the gas, call it P

In algebraic language this would be:

$$V = \frac{\mathrm{K}}{P}$$

Notice that in this example even though K is a letter it is not a variable but a constant as described in the original sentence. By convention we would always write variables in italics.

In each of these examples instead of an English statement, we have produced an algebraic statement. If a statement includes an equals sign, as in $I = S \times Q$, then it is termed an algebraic **equation**. If it does not contain an equals sign, then it is called an algebraic **expression** e.g. $250P + 150P$ is an expression while $I = 250P + 150P$ is an equation.

We will do more of these translations throughout this module as we develop different algebraic tools.

Activity 3.1

Translate the following into algebraic form.

(a) Cardiac output (*CO*) is the product of the stroke volume (*SV*) and the heart rate (*HR*).

(b) Aggregate demand (*AD*) is the sum of consumption demand (*C*), private investment demand (*I*) and government demand (*G*), and the difference between expenditure on exports (*X*) and imports (*M*) of goods and services.

(c) The maximum length of time (*T*) a diver can stay under water is 120 times the quotient of the volume of air (*V*) in cubic metres before compression and the depth (*d*) in metres.

(d) The average propensity to consume (*APC*) is the quotient of consumption (*C*) and disposable income (*DI*).

(e) To convert from Centigrade (*C*) to Fahrenheit (*F*), sum 32 and the product of the temperature in degrees Centigrade and 1.8.

(f) Total variable cost (*TVC*) is the product of the average variable cost (*AVC*) and the output level (*Q*).

(g) Power (*P*) produced is the product of force (*F*) and the distance travelled (*s*), divided by time (*t*).

(h) Profit (*P*) equals the difference between the revenue (*R*) and the costs (*C*).

(i) The gravitational attraction (*F*) between any two objects is the product of the mass (m) of each object and the constant (*G*), divided by the square of the distance (*d*) between them.

(j) The difference between 220 and a man's age (*A*) gives an estimate of his maximum heart rate (*HR*) while exercising.

3.2 Manipulating relationships

Just like any other language or tool, algebra comes with a set of operating instructions or conventions. We can use these to simplify expressions and later, to solve equations and inequations. Before we go any further, however, we need to recall two things.

- **Names for parts of an expression**

Consider the expression:

$-2x+3p^2-4$ (Note that this could be written as $-2x+3p^2+-4$)

- The expression has three terms, $-2x,\ 3p^2, -4$. Terms are separated by an addition sign.
- The expression has two variables, x and p. (Note p^2 is not the variable as this is really $p\times p$.)
- –2 and 3 are called coefficients, as they are associated with the variable by multiplication. –2 is the coefficient of x and 3 is the coefficient of p^2.
- –4 is called 'The Constant', as it is a constant term that is not associated with a variable.

- **Arithmetic conventions that also apply to algebra**

Arithmetic convention	Algebraic convention	Name
$2+3=3+2$	$a+b=b+a$	Commutative Law for Addition
$(2+3)+4=2+(3+4)$	$(a+b)+c=a+(b+c)$	Associative Law for Addition
$2\times3=3\times2$	$a\times b=b\times a$ or $ab=ba$	Commutative Law for Multiplication
$(2\times3)\times4=2\times(3\times4)$	$(a\times b)\times c=a\times(b\times c)$	Associative Law for Multiplication
$2+0=0+2=2$	$a+0=0+a=a$	Identity Law of Addition
$2\times1=1\times2=2$	$a\times1=1\times a=a$	Identity Law of Multiplication
$2\times(3+4)=2\times3+2\times4$ or $2(3+4)=2\times3+2\times4$	$a\times(b+c)=a\times b+a\times c$ or $a(b+c)=ab+ac$	Distributive Law
$2\times0=0\times2=0$	$a\times0=0\times a=0$	Multiplying by zero
		No need to learn these names

3.2.1 Grouping like terms

Recall in a previous example:

The selling price of a shirt is set each year. What will the return to the company be if 250 shirts were sold in the first six months and 150 in the next six months?

From this text we determined that:

$I = 250P + 150P$, where P is the selling price of a shirt, and I the income to the company.

Looking back to the original question we sold 250 and 150 shirts. This was a total of 400 shirts. The price for these would be 400*P*. So we can add 250*P* to 150*P* to get 400*P*. We can do this because 250*P* and 150*P* are **like terms**. They contain the same variable or combination of variables. We get $I = 400P$.

Example

Group like terms to simplify this expression:

$2x + 3x - 4$

In this expression 2*x* and 3*x* are like terms, –4 is a constant. So we can simplify the expression as follows.

$2x + 3x - 4$
$= 5x - 4$ add like terms

$2x + 3x - 4$ is simplified to $5x - 4$

Example

Simplify $p^2 + 2x - 2p^2$

In this expression p^2 and $-2p^2$ are the like terms because they both contain the same combination of *p*, which is p^2. So we can simplify the expression as follows.

$p^2 + 2x - 2p^2$
$= p^2 - 2p^2 + 2x$ Group like terms together.
$= (-p^2 + 2x)$ Add like terms.

Simplified form is $-p^2 + 2x$. Note we could also write this as $2x - p^2$.

Example

Write the following expression as a sum containing only two terms $3ab - 3bc + 4ab + bc$.

In this expression 3*ab* and 4*ab;* and −3*bc* and *bc* are groups of like terms.

So we can simplify the expression as follows.

$3ab - 3bc + 4ab + bc$	
$= 3ab + 4ab - 3bc + bc$	Group together like terms.
$= 7ab - 2bc$	Add or subtract like terms.

Final answer is $7ab - 2bc$ or $-2bc + 7ab$

Example

Simplify the following expression $2q^2 - q + q^2$.

In this expression $2q^2$ and q^2 are like terms.

So we can simplify the expression as follows.

$2q^2 - q + q^2$	
$= 2q^2 + q^2 - q$	Group together like terms.
$= 3q^2 - q$	Add like terms.

Simplified form is $3q^2 - q$

Note that even though q^2 and q contain the same variable, q, they are not like terms and cannot be added. Recall that $q^2 = q \times q$, and is very different from q.

Example

Remove brackets and simplify the following expression.

$x^2 + 2x(x + 1)$

$x^2 + 2x^2 + 2x$

In this expression, x^2 and $2x^2$ are like terms

$x^2 + 2x(x + 1)$

$= x^2 + 2x^2 + 2x$

$= 3x^2 + 2x$

Example

Simplify $p + 2q^2 - 2(q^2 - p)$

$p + 2q^2 - 2(q^2 - p)$

$= p + 2q^2 - 2q^2 - 2 \times {}^{-}p$

$= p + 2q^2 - 2q^2 + 2p$

$= 3p$

Special Note: If you have a negative sign outside brackets, when you remove the brackets it will have an effect on the sign that was inside the brackets.

Activity 3.2

1. Simplify the following expressions.

(a) $5y - 3y + 2$

(b) $8a + 4b - 2b + 3a$

(c) $p - 6p + 2p - 4n$

(d) $-5x^2 - x + 2x^2 + 4x$

(e) $-a^3 - 3a^2 + a^3$

(f) $6ab - 8a + 2b + 3ab$

(g) $4qr + 6st - 2qr + 5st$

(h) $-2a^2b + 3bc - 4ba^2$

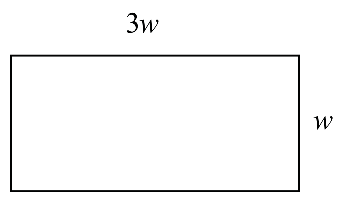
(i) $a^2 - 2a(a + 1)$

(j) $2(p - 3) - (2p + p^2)$

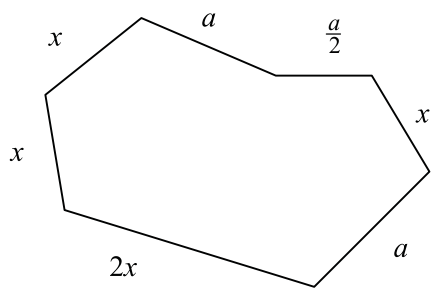
(k) $x - 3x(x + 2)$

2. Find the perimeter of the following figures and simplify the resultant expression.

(a)



(b)



3.2.2 Factors and factorization

In a previous example,

> *The income to a company is determined by calculating the product of the quantity of goods and their selling price,*

we found income by multiplying the quantity of goods by their selling price. In everyday speech we might say that quantity and price are the two factors from which we can determine income.

In algebraic speech the word factor means something more specific. A **factor** is one of the components that make up a number or an expression by multiplication. So in our example it is correct to say that quantity and price are factors of income because they multiply together to give us income e.g. $I = S \times Q$. It is often important to be able to determine what are the factors of a specific number or expression.

So the factors of:

8	are	4 and 2, or –4 and –2 or –8 and –1 and others
m	are	m and 1
$-p$	are	p and -1
$\frac{w}{2}$	are	$\frac{1}{2}$ and w
xy	are	x and y
$2(x+3)$	are	2 and $x+3$
$(a+3)(b-2)$	are	$a+3$ and $b-2$
$3(x-2)(x^2+1)$	are	$3, x-2$ and x^2+1

Factorization is the process of determining the factors of an algebraic expression.

Considering the expression $2(x+3)$ we know immediately that the factors of this are 2 and $x+3$. But if instead we had been presented with the same expression in its expanded form, $2x+6$, how would we determine its factors?

Our first step is to revisit the distributive law.

Recall in arithmetic

$2 \times (3+4) = 2 \times 3 + 2 \times 4$,

in algebra,

$a \times (b+c) = a \times b + a \times c$ or $a(b+c) = ab + ac$

So when faced with an expression like $ab + ac$, we look for a factor which occurs in both terms and use the distributive law to take that out as a common factor.

Example

Factorize $2x + 6$

$2x + 6$	
$= 2 \times x + 2 \times 3$	Look for factors common to each term.
$= 2(x+3)$	Use the distributive law to take out 2 as a common factor.

Factors are 2 and $x + 3$

Example

Find the factors of $2xy^2 - 4xy$

$2xy^2 - 4xy$	
$= 2 \times x \times y \times y - 2 \times 2 \times x \times y$	Look for factors common to each term.
$= 2xy \times y - 2xy \times 2$	
$= 2xy(y-2)$	Use the distributive law to take out $2xy$ as a common factor.

Factors are 2, x, y and y–2

Example

Convert this sum to the product of two factors $2p^2 - 4p + p^3$

$2p^2 - 4p + p^3$	
$= 2 \times p \times p - 2 \times 2 \times p + p \times p \times p$	Look for factors common to each term.
$= p(2p - 4 + p^2)$	Use the distributive law to take out p as a common factor.

Product is $p(2p - 4 + p^2)$

Activity 3.3

1. Factorize the following expressions.

   (a) $5x + 5y$

   (b) $ab + 8b$

   (c) $n^2 - 5n$

   (d) $-2y + -8$

   (e) $-3qr - 9qt$

   (f) $12pq - 28p^2$

2. Write the following sums as products.

   (a) $st^2 - t$

   (b) $7x^2y + 21xy^3 + 14xy^2$

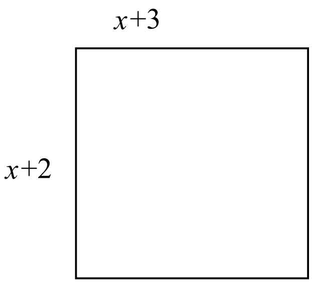
   (c) $-2a^2b^3 + 5a^2b^2 + 3a^3b^2$

   (d) $-2z^3 + 8z^2 - 4z$

The above explanation for factorization works well when one of the factors is simply a single constant or variable. But how can we use the distributive law to factorize expressions where the factors are of the form $(x + a)$.

First let's go back to the Distributive Law and expand $(x + 3)(x + 2)$. Instead of doing this just with algebra, think of it as a concrete slab, which is $x + 3$ units long and $x + 2$ units wide.

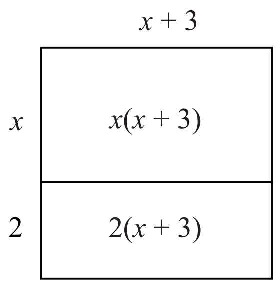
The area could be calculated for the whole shape.



Area is length multiplied by width.

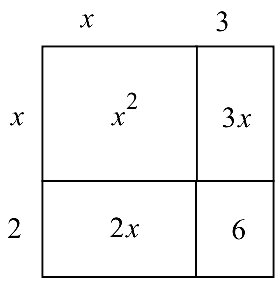
Area = $(x+3)(x+2)$

The area could be calculated from two rectangles.



Area is the sum of top rectangle and the bottom rectangle.

Area = $x(x+3)+2(x+3)$



Area is the sum of the four rectangles.

Area $= x^2 + 3x + 2x + 2 \times 3$

$= x^2 + 5x + 6$

From this we can see that the total area is the sum of the area of its parts.

So we know that:

$(x+2)(x+3)$	
$= x(x+3)+2(x+3)$	Take the first term x and multiply it with x+3, then the second term 2 with x+3.
$= x^2 + 3x + 2x + 2 \times 3$	Remove the brackets.
$= x^2 + 5x + 6$	Group and add like terms to simplify.

Notice that we start with the product of two factors and end up with the sum of three terms.

Example

Expand using the distributive law $(p+1)(p+3)$

$(p+1)(p+3)$	
$= p(p+3)+1(p+3)$	Take the first term p and multiply it with p+3, then the second term 1 with p+3.
$= p^2+3p+p+3$	Remove the brackets.
$= p^2+4p+3$	Group and add like terms to simplify.

Notice that we start with the product of two factors and end up with the sum of three terms.

Example

Expand using the distributive law $(2q+3)(4q-1)$

$(2q+3)(4q-1)$	
$= 2q(4q-1)+3(4q-1)$	Take the first term 2q and multiply it with 4q–1, then the second term 3 with 4q–1.
$= 8q^2-2q+12q-3$	Remove the brackets.
$= 8q^2+10q-3$	Group and add like terms to simplify.

Notice that we start with the product of two factors and end up with the sum of three terms.

Example

Expand using the distributive law $(a-1)(a+1)$

$(a-1)(a+1)$	
$= a(a+1)-1(a+1)$	Take the first term a and multiply it with a+1, then the second term –1 with a+1.
$= a^2+a-a-1$	Remove the brackets.
$= a^2-1$	Group and add or subtract like terms to simplify.

Note this is a useful expansion called **difference of two squares** $a^2-1^2=(a+1)(a-1)$. The generalized form of this is $a^2-b^2=(a+b)(a-b)$ …a useful tool to help speed up the expansion process. Notice you end up with only two terms.

Example

Expand using the distributive law $(2x+1)^2$

$(2x+1)^2$	Recall that squared means to multiply twice.
$=(2x+1)(2x+1)$	Take the first term $2x$ and multiply it with $2x+1$, then the second term 1 with $2x+1$.
$=2x(2x+1)+1(2x+1)$	
$=4x^2+2x+2x+1$	Remove the brackets.
$=4x^2+4x+1$	Group and add like terms to simplify.

This is another useful expansion to speed up your calculations. Its general form is $(a+b)^2=a^2+2ab+b^2$. It's often called a **perfect square**.

Activity 3.4

1. Expand the following expressions.

   (a) $5k(k+3)$

   (b) $-(2x-5)$

   (c) $(x+4)(x+3)$

   (d) $(y-2)(y+2)$

   (e) $(2z-1)(-6z+4)$

2. Write the following products as sums.

   (a) $(3y+4)^2$

   (b) $(3a-1)(2a-1)$

   (c) $(2+3x)(4x+1)$

   (d) $(-4q+2)(-3q+3)$

   (e) $(-2g-3)(-6g-5)$

Expanding and factorization are actually reverse procedures. So let's examine some of the expansions we have completed to see how they can help us to factorize.

$$
\begin{aligned}
&(p+1)(p+3)\\
&= p(p+3)+1(p+3)\\
&= p^2+3p+1p+1\times 3\\
&= p^2+(3+1)p+1\times 3
\end{aligned}
$$

We have two numbers which add to give 4 and multiply to give 3.

So if we are asked to factorize p^2+4p+3, we try to find two numbers that add to give 4 and multiply to give 3.

Numbers to try, factors of 3	Sum	Product	
–3 and –1	$-3+-1=-4$	$-3\times -1=3$	
3 and 1	$3+1=4$	$3\times 1=3$	✓

Correct factors are 3 and 1. So,

p^2+4p+3

$=p^2+(3+1)p+3\times 1$

$=p^2+3p+p+3$ Group terms in pairs.

$=p(p+3)+(p+3)$ Take out p as common factor from first two terms.

Take out 1 as a common factor of next two terms.

$=p(p+3)+1(p+3)$ Take out p + 3 as a common factor.

$=(p+3)(p+1)$

Example

Factorize $x^2 + 5x + 6$

We need two numbers which multiply to give 6 and add to give 5. Factors of 6 are 3 and 2, or –3 and –2, or 6 and 1 or –1 and –6.

Numbers to try, factors of 6	Sum	Product	
–2 and –3	$-3 + -2 = -5$	$-3 \times -2 = 6$	
3 and 2	$3 + 2 = 5$	$3 \times 2 = 6$	✓
–1 and –6	$-1 + -6 = -7$	$-1 \times -6 = 6$	
1 and 6	$1 + 6 = 7$	$1 \times 6 = 6$	

Correct factors are 3 and 2, so

$x^2 + 5x + 6$

$= x^2 + (3 + 2)x + 3 \times 2$ — Use factors to rewrite 5 and 6.

$= x^2 + 3x + 2x + 3 \times 2$ — Remove brackets.

$= x(x + 3) + 2(x + 3)$ — Take out x as a common factor from the first two terms and 2 from the next two terms. In each case the remaining factor should be the same, x+3.

$= (x + 3)(x + 2)$ — Take out x+3 as a common factor.

Check your answer by expanding:

$$(x + 3)(x + 2)$$
$$= x(x + 2) + 3(x + 2)$$
$$= x^2 + 2x + 3x + 6$$
$$= x^2 + 5x + 6$$

Therefore the final answer is $x^2 + 5x + 6 = (x + 3)(x + 2)$

Example

Factorize $x^2 + 7x - 30$

We need two numbers that multiply to give –30 and add to give 7.

Factors of –30 are 3 and –10, or –3 and 10, or 2 and –15 or –2 and 15 and others.

Numbers to try, factors of –30	Sum	Product	
3 and –10	$3 + -10 = -7$	$3 \times -10 = -30$	
–3 and 10	$-3 + 10 = 7$	$-3 \times 10 = -30$	✓
2 and –15	$2 + -15 = -13$	$2 \times -15 = -30$	
–2 and 15	$-2 + 15 = 13$	$-2 \times 15 = -30$	

Correct factors are –3 and 10, so

$x^2 + 7x - 30$

$= x^2 + (10 - 3)x - 3 \times 10$ Use factors to rewrite 7 and –30.

$= x^2 + 10x - 3x + -3 \times 10$ Remove brackets.

$= x(x + 10) - 3(x + 10)$ Take out x as a common factor from the first two terms and –3 from the next two terms. In each case the remaining factor should be the same, x+10.

$= (x + 10)(x - 3)$ Take out x+10 as a common factor.

Check your answer by expanding:

$$(x+10)(x-3)$$
$$= x(x-3) + 10(x-3)$$
$$= x^2 - 3x + 10x - 30$$
$$= x^2 + 7x - 30$$

Therefore the final answer is $x^2 + 7x - 30 = (x+10)(x-3)$

As you become more proficient at factorization you will not need to write out the table in full and will be able to do the guessing and checking in your head.

Activity 3.5

1. Factorize the following expressions.

   (a) $x^2 + 6x + 5$

   (b) $y^2 + 8y + 12$

   (c) $x^2 - 11x - 42$

   (d) $a^2 - 7a + 12$

   (e) $z^2 + 2z - 15$

2. Rewrite these expressions as the product of two factors.

   (a) $b^2 + 13b + 30$

   (b) $x^2 - 12x - 28$

   (c) $c^2 - 15c - 16$

   (d) $x^2 + 14x + 33$

   (e) $m^2 - 20m + 64$

   (f) $x^2 + 9x - 36$

   (g) $y^2 - 13y + 30$

   (h) $a^2 - a - 42$

   (i) $v^2 + 2v + 1$

   (j) $q^2 - 15q + 26$

Forward planning topic
(for those planning to enrol in more mathematics courses)

Now that you can factorize expressions in which the coefficient of the x^2 term is one, we can use similar methods to factorize expressions where the coefficient is greater than 1.

Consider what you will have to do to factorize $6x^2 + 7x + 2$.

In this instance we cannot find the factors of the constant alone, but need to find the factors of the product of the coefficient of x^2 and the constant. That is, what are the factors of 6×2 or 12?

Factors of 12 are 6 and 2, or –6 and –2, or 3 and 4 or –3 and –4 and many others.

Following the pattern of before and the knowledge we have gained by expanding using the distributive law, we need to now look for two numbers which multiply to give 12 and add to give 7.

Numbers to try, factors of 12	Sum	Product	
6 and 2	$6+2=8$	$6\times 2=12$	
–6 and –2	$-6+-2=-8$	$-6\times -2=12$	
3 and 4	$3+4=7$	$3\times 4=12$	✓
–3 and –4	$-3+-4=-7$	$-3\times -4=12$	

Correct factors are 3 and 4.

$6x^2+7x+2$	
$=6x^2+(3+4)x+2$	Use factors to rewrite 7.
$=6x^2+3x+4x+2$	Remove brackets.
$=3x(2x+1)+2(2x+1)$	Take out $3x$ as a common factor from the first two terms and 2 from the next two terms. In each case the remaining factor should be the same, $2x+1$.
$=(2x+1)(3x+2)$	Take out $2x+1$ as a common factor.

Check your answer by expanding:

$$\begin{aligned}&(2x+1)(3x+2)\\&=2x(3x+2)+1(3x+2)\\&=6x^2+4x+3x+2\\&=6x^2+7x+2\end{aligned}$$

Therefore the final answer is $6x^2+7x+2=(2x+1)(3x+2)$

Example

Factorize $10x^2+17x+3$

In this instance, again, we cannot find the factors of the constant alone but need to find the factors of the product of the coefficient of x^2 and the constant. That is, what are the factors of 3×10 or 30?

Factors of 30 are 3 and 10, or –3 and –10, or 15 and 2 or –15 and –2 and many others.

Following the pattern of before and the knowledge we have gained by expanding using the distributive law, we need to now look for two numbers which multiply to give 30 and add to give 17.

Numbers to try, factors of 30	Sum	Product	
3 and 10	$3+10=13$	$3\times 10=30$	
–3 and –10	$-3+-10=-13$	$-3\times -10=30$	
15 and 2	$15+2=17$	$15\times 2=30$	✓
–15 and –2	$-15+-2=-17$	$-15\times -2=30$	

Correct factors are 15 and 2.

$10x^2+17x+3$	
$=10x^2+(15+2)x+3$	Use factors to rewrite 17.
$=10x^2+15x+2x+3$	Remove brackets.
$=5x(2x+3)+1(2x+3)$	Take out $5x$ as a common factor from the first two terms and 1 from the next two terms. In each case the remaining factor should be the same, $2x+3$.
$=(2x+3)(5x+1)$	Take out $2x+3$ as a common factor.

Check your answer by expanding:

$(2x+3)(5x+1)$

$=2x(5x+1)+3(5x+1)$

$=10x^2+2x+15x+3$

$=10x^2+17x+3$

Therefore the final answer is $10x^2+17x+3=(2x+3)(5x+1)$

Example

Factorize $3x^2 - 10x + 3$

We need to find the factors of the product of the coefficient of x^2 and the constant. That is, what are the factors of 3×3 or 9.

Factors of 9 are 3 and 3, or –3 and –3, or 9 and 1 or –9 and –1.

Following the pattern of before and the knowledge we have gained by expanding using the distributive law we need to now look for two numbers which multiply to give 9 and add to give –10.

Numbers to try, factors of 9	Sum	Product	
3 and 3	$3 + 3 = 6$	$3 \times 3 = 9$	
–3 and –3	$-3 + -3 = -6$	$-3 \times -3 = 9$	
9 and 1	$9 + 1 = 10$	$9 \times 1 = 9$	
–9 and –1	$-9 + -1 = -10$	$-9 \times -1 = 9$	✓

Correct factors are –9 and –1.

$3x^2 - 10x + 3$

$= 3x^2 + (-9 - 1)x + 3$ Use factors to rewrite –10.

$= 3x^2 - 9x - 1x + 3$ Remove brackets.

$= 3x(x - 3) - 1(x - 3)$ Take out $3x$ as a common factor from the first two terms and –1 from the next two terms. In each case the remaining factor should be the same, x–3.

$= (x - 3)(3x - 1)$ Take out x–3 as a common factor.

Check your answer by expanding:

$(x - 3)(3x - 1)$

$= x(3x - 1) - 3(3x - 1)$

$= 3x^2 - x - 9x + 3$

$= 3x^2 - 10x + 3$

Therefore the final answer is $3x^2 - 10x + 3 = (x - 3)(3x - 1)$

You should note that in these examples we have shown one way of factorizing using our choice of which factor to work on first. You may use other methods which will be perfectly correct but different from ours. In real life, however, not all expressions will be able to be expressed as factors…but we will leave this for a later section of this module.

Activity 3.6

1. Factorize the following expressions.

   (a) $2a^2 + 3a + 1$

   (b) $3x^2 - 11x - 4$

2. Rewrite these sums as products of two linear factors.

   (a) $10a^2 - 11a - 6$

   (b) $6x^2 - 19x + 13$

   (c) $9x^2 + 30x + 25$

   (d) $3q^2 - 2q - 1$

Something to talk about…

Have you used expansion or factorization before at school or work? How did you do it? Talk with your friends or work colleagues about the quickest way to factorize or expand expressions or include something on the discussion list. You might learn a way to speed up the process.

3.2.3 Algebraic fractions

Have you ever wondered how much you would have to deposit in a bank account today to receive a certain amount in the future? To do this you could to use the formula

$$P = \frac{S}{(1+r)^n}$$

where S is the amount you want to receive
P the amount deposited
r the interest rate per year
n the number of years.

This formula involves an **algebraic fraction**, just like many others that occur in business and the sciences. For example,

in economics, price elasticity of demand is related to change in price (p) and change in quantity (q) by the equation

$$E = \frac{q_1 - q_0}{p_1 - p_0} \times \frac{p_0}{q_0},$$

in physics, Newton's Law of Gravitation is an equation relating mass (m), force (F) and distance (d) which is $F = \frac{m_1 m_2 \mathrm{G}}{d^2}$,

in chemistry, pressure (P), volume (V) and temperature (T) of a gas are related by the equation

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2},$$

in electronics, the resistances of some types of circuits are related by the equation

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}.$$

Don't try to learn any of these formulas they are just there as examples involving algebraic fractions. So, if algebraic fractions occur in such profusion in business and science studies how can we manipulate them?

You will be pleased to know that algebraic fractions are not different from the arithmetic fractions you have already worked with. Let's have a look at the some of the different operations we can perform with fractions.

Multiplication and Division

In arithmetic, when we multiply two fractions we multiply the **numerators** together (the top part of the fraction) and the **denominators** together (the bottom part of the fraction).

$$\begin{aligned}\frac{2}{3} \times \frac{1}{7} &= \frac{2 \times 1}{3 \times 7} \\ &= \frac{2}{21}\end{aligned}$$

The answer in this example is in its simplest form i.e. there are no common factors which occur in the numerator and the denominator. Let's have a look at the next example.

$$\begin{aligned}\frac{2}{3} \times \frac{1}{4} &= \frac{2 \times 1}{3 \times 4} \\ &= \frac{2}{12}\end{aligned}$$

This answer is not in its simplest form because 2 is a factor of the numerator and the denominator. When this occurs we can simplify the answer by dividing the numerator and denominator by 2 to get $\frac{1}{6}$. We call this **cancelling**.

$\frac{\not{2}^{1}}{\not{12}_{6}} = \frac{1}{6}$ The 2 is cancelled from the denominator and the numerator.

$\frac{1}{6}$ and $\frac{2}{12}$ are called **equivalent fractions**.

Usually, when multiplying fractions, this cancelling can take place much earlier in the calculation. Look at the following.

$$\frac{\not{2}^{1}}{3} \times \frac{1}{\not{4}_{2}} = \frac{1 \times 1}{3 \times 2}$$ Cancel out 2 from the 2 in the numerator and from the 4 in the denominator.

$$= \frac{1}{6}$$

Watch out though! We can only cancel out factors. That is, we need to have two terms multiplied together in the numerator and the denominator before we can consider cancelling.

Let's look at two more examples.

Consider $\frac{2x+2}{2}$, we can rewrite this as $\frac{2(x+1)}{2}$. Because 2 is multiplied by $x+1$ we can happily cancel out the 2 from the denominator and numerator to get $x+1$.

But in the expression $\frac{2x+1}{2}$ the numerator cannot be written as the product of two terms, so we cannot cancel out the 2. Two is a factor of $2x$ but is not a factor of 1. The expression cannot be simplified further.

Let's practice using some of these tools in multiplication and division of algebraic expressions.

Example

Simplify the expression $\frac{a}{2b} \times \frac{3}{4a}$

$$\frac{a}{2b} \times \frac{3}{4a} = \frac{\not{a}^{1}}{2b} \times \frac{3}{4\not{a}_{1}}$$ Cancel *a* from the numerator and denominator.

$$= \frac{1 \times 3}{2b \times 4}$$ Write as a single fraction.

$$= \frac{3}{8b}$$ Complete multiplications.

Example

Write $\frac{4q}{7p} \times \frac{-p}{8}$ as a single fraction with no common factors.

$$\frac{4q}{7p} \times \frac{-p}{8} = \frac{\not{4}^1 q}{7\not{p}_1} \times \frac{-\not{p}^1}{\not{8}_2}$$ Cancel p and 4 from the numerator and denominator.

$$= \frac{q \times -1}{7 \times 2}$$ Write as a single fraction.

$$= \frac{-q}{14}$$ Complete multiplications.

Example

Write $\frac{a}{b} \div \frac{1}{2}$ in its simplest form.

$$\frac{a}{b} \div \frac{1}{2} = \frac{a}{b} \times \frac{2}{1}$$ Division is the same as multiplying by the reciprocal.

$$= \frac{2a}{b}$$ Complete multiplications.

Example

Divide $2x$ by $\frac{1}{x}$. Write your answer in its simplest form.

$$\frac{2x}{\frac{1}{x}} = 2x \div \frac{1}{x}$$ Write with a division sign.

$$= 2x \times \frac{x}{1}$$ Division is the same as multiplying by the reciprocal.

$$= 2x^2$$ Complete multiplications.

Activity 3.7

1. Simplify the following rational expressions.

   (a) $\dfrac{6c}{9d}$

   (b) $\dfrac{16qr}{4qs}$

   (c) $\dfrac{28xy}{7xyz}$

2. Perform the indicated operation and express the following fractions with no common factors.

   (a) $\dfrac{3p}{7p} \times \dfrac{2}{3}$

   (b) $\dfrac{3x}{y} \div \dfrac{-3}{4y}$

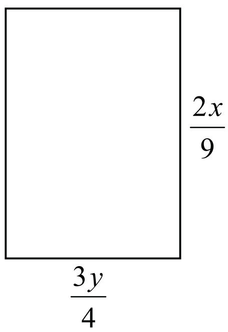
   (c) $\dfrac{-2q}{7qr} \times \dfrac{-r}{12}$

   (d) $\dfrac{-5xy}{3a} \div \dfrac{25ax}{18}$

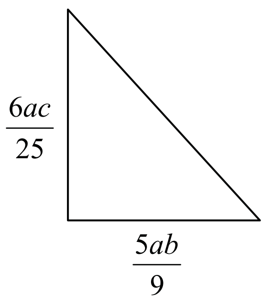
   (e) $\dfrac{3p}{5p+2} \times \dfrac{5p+2}{9}$

3. Calculate the area of the following figures.

   (a)



   (b)



Addition and subtraction

In arithmetic we can only add or subtract fractions when they have the same denominator. So it is easy to add $\frac{5}{7}$ to $\frac{1}{7}$ to get $\frac{6}{7}$ because they both have the same denominator of 7. But if we have to add $\frac{5}{7}$ to $\frac{3}{5}$ we would have to write each fraction as an equivalent fraction with the same denominator. For example

$$\begin{aligned}\frac{5}{7}+\frac{3}{5}&=\frac{5}{7}\times\frac{5}{5}+\frac{3}{5}\times\frac{7}{7}\\&=\frac{25}{35}+\frac{21}{35}\\&=\frac{46}{35}\\&=1\frac{11}{35}\end{aligned}$$

The same procedures apply to addition and subtraction of algebraic fractions.

Example

$$\begin{aligned}\frac{p}{3}+\frac{x}{4}&=\frac{p}{3}\times\frac{4}{4}+\frac{x}{4}\times\frac{3}{3}\\&=\frac{4p}{12}+\frac{3x}{12}\\&=\frac{4p+3x}{12}\end{aligned}$$

Example

Simplify, $\frac{a+b}{a}-\frac{b}{c}$ expressing your answer as a single fraction.

$$\frac{a+b}{a}-\frac{b}{c}=\frac{(a+b)}{a}\times\frac{c}{c}-\frac{b}{c}\times\frac{a}{a}$$

Make *ac* the common denominator by appropriate multiplications of each term.

$$=\frac{c(a+b)}{ac}-\frac{ba}{ac}$$

Complete multiplications for each term.

$$=\frac{ca+cb}{ac}-\frac{ba}{ac}$$

Remove brackets from first term.

$$=\frac{ca+cb-ba}{ac}$$

Subtract numerators.

Example

Subtract $\frac{x-1}{x+2}$ from $\frac{x}{x+1}$. Express your answer with no common factors.

$$\frac{x}{x+1}-\frac{x-1}{x+2}=\frac{x}{(x+1)}\times\frac{(x+2)}{(x+2)}-\frac{(x-1)}{(x+2)}\times\frac{(x+1)}{(x+1)}$$

Make $(x+1)(x+2)$ the common denominator by appropriate multiplications of each term. Note terms are bracketed to keep them together.

$$=\frac{x(x+2)}{(x+1)(x+2)}-\frac{(x-1)(x+1)}{(x+1)(x+2)}$$

$$=\frac{x^2+2x}{(x+1)(x+2)}-\frac{x^2-1}{(x+1)(x+2)}$$

Remove brackets from each numerator.

$$=\frac{x^2+2x-(x^2-1)}{(x+1)(x+2)}$$

Subtract numerators, put subtracted numerator in brackets so that entire numerator is subtracted.

$$=\frac{x^2+2x-x^2+1}{(x+1)(x+2)}$$

Remove brackets from numerator.

$$=\frac{2x+1}{(x+1)(x+2)}$$

Simplify numerator.

Example

Simplify this expression $\frac{1}{s-2}-\frac{s}{2(s-2)}$

$$\frac{1}{s-2}-\frac{s}{2(s-2)}=\frac{1}{s-2}\times\frac{2}{2}-\frac{s}{2(s-2)}$$

Make $2(s-2)$ the common denominator by appropriate multiplications of each term. Note because $(s-2)$ is common to both terms we only have to include it once in the common denominator.

$$=\frac{2}{2(s-2)}-\frac{s}{2(s-2)}$$

Simplify each numerator.

$$=\frac{2-s}{2(s-2)}$$

Subtract numerators. Notice that $2-s$ is nearly the same as $s-2$.

$$=\frac{-1\times(-2+s)}{2(s-2)}$$

Take out −1 as a common factor of the numerator.

$$=\frac{-1(s-2)}{2(s-2)} \text{ or } \frac{-(s-2)}{2(s-2)}$$

Rewrite $-2+s$ as $s-2$, and cancel $s-2$ from the numerator and denominator.

$$=\frac{-1}{2} \text{ or } -\frac{1}{2}$$

Activity 3.8

1. Write the following expressions as single fractions with no common factors.

   (a) $\dfrac{2x}{5}+\dfrac{4x}{7}$

   (b) $\dfrac{2}{5x}+\dfrac{4}{7x}$

   (c) $\dfrac{1}{ab}-\dfrac{a}{bc}$

   (d) $\dfrac{2a-b}{b}-\dfrac{4b}{c}$

2. Express each of the following as a single fraction.

   (a) $\dfrac{2}{x}+\dfrac{5}{x^2}$

   (b) $3-\dfrac{2a}{a+2}$

   (c) $\dfrac{x}{2x-3}+\dfrac{1}{x+2}$

   (d) $\dfrac{x+4}{x-4}-\dfrac{x-4}{x+4}$

3. If $x = 4$, find the value of

   (i) $\dfrac{1}{x-3}-\dfrac{x}{x^2-9}$

   (ii) $\dfrac{3}{x^2-9}$

   What do you notice about the answers for (i) and (ii)? Explain in your own words why this is so.

4. Each of the following problems has an error. Find the error and solve correctly.

   (a) $\dfrac{5}{2x+3} - \dfrac{x+1}{2x+3}$

   $= \dfrac{5-x+1}{2x+3}$

   $= \dfrac{6-x}{2x+3}$

   (b) $\dfrac{1}{3x} + \dfrac{4}{3x}$

   $= (3x)(\dfrac{1}{3x}) + (3x)(\dfrac{4}{3x})$

   $= 1+4$

   $= 5$

   (c) $\dfrac{x+1}{x-2} + \dfrac{5}{x+1}$

   $= \dfrac{x+1}{x-2} + \dfrac{5}{x+1}$

   $= \dfrac{1}{x-2} + \dfrac{5}{1}$

3.2.4 Working with powers

Powers are everywhere. In arithmetic they are used to help represent large and small numbers. Look at this example from *Australian Organizational Behaviour* by W. S. Sherman.

Ethnic background	**Number of Australians**
British	4.7 million
Irish	3.5 million
German/Austrian	1.2 million
Italian	1 million
Greek	6.5×10^5
Maltese	4×10^5
Yugoslav	2.5×10^5
Dutch	2×10^5
Aboriginal	1.6×10^5
Spanish Speaking	1.5×10^5
Arabic speaking	1.5×10^5
Polish	1×10^5

(Source: Ainsworth W.M. & Willis, Q.F. 1985, *Australian organizational behaviour: readings,* 2nd edn, Macmillan, Melbourne.)

Power notation is used in arithmetic and algebra to make repeated operations, like multiplication, quicker to write out. So that $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$ is written as 2^9. Other abbreviations are also used.

In business calculations you might see formulas like, $S = p\dfrac{(1+i)^n - 1}{i}$ or $PV = p\dfrac{1-(1+i)^{-n}}{i}$

or $G = p\dfrac{i}{(1+i)^{\frac{1}{k}} - 1}$ (aspects of annuity calculations)

In statistics you will see, $s = \sqrt{\dfrac{\sum_{i=1}^{n}(x_i - x)^2}{n-1}}$ (standard deviation for a sample)

In chemistry you might see $y = (\frac{m}{2\pi kT})^{\frac{1}{2}} c^2 e^{(\frac{-mc^2}{2kT})}$ (Maxwell-Boltzmann distribution function for molecular speeds).

Don't be put off by these equations they do look a bit daunting but they are just examples at this stage. Do not learn them. But what do these positive, negative and fractional **indices** mean?

Have a look at the table below which summarizes some power notations you might already be familiar with.

Arithmetic example	Algebraic example	Generalization
$2^0 = 1$	$a^0 = 1 \qquad a \neq 0$	
$2^1 = 2$	$a^1 = a$	
$2^2 = 2 \times 2 = 4$ 2 is multiplied 2 times, called 2 squared	$a^2 = a \times a$ *a* is multiplied 2 times, called *a* squared	
$2^3 = 2 \times 2 \times 2 = 8$ 2 is multiplied 3 times, called 2 cubed	$a^3 = a \times a \times a$ *a* is multiplied 3 times, called *a* cubed	$a^n = a \times a \times a \ldots\ldots \times a$ *a* is multiplied together n times, called *a* to the power n
$4^{\frac{1}{2}} = \sqrt{4} = \sqrt{2 \times 2} = 2$ the square root of 4	$a^{\frac{1}{2}} = \sqrt{a}$ the square root of *a*	
$8^{\frac{1}{3}} = \sqrt[3]{8} = \sqrt[3]{2 \times 2 \times 2} = 2$ the cube root of 8	$a^{\frac{1}{3}} = \sqrt[3]{a}$ the cube root of *a*	$a^{\frac{1}{n}} = \sqrt[n]{a}$ the nth root of *a*
$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$	$a^{-3} = \frac{1}{a^3}$	$a^{-n} = \frac{1}{a^n}$

Activity 3.9

1. Rewrite the following using power notation.

   (a) $7 \times 7 \times 7$

   (b) $\sqrt[4]{16}$

   (c) $\dfrac{1}{4 \times 4 \times 4 \times 4}$

   (d) $-3 \times 3 \times 3 \times 3 \times 3$

   (e) $(-3) \times (-3) \times (-3) \times (-3) \times (-3)$

   (f) $\dfrac{1}{3 \times 3 \times 3 \times 3 \times 3}$

   (g) $\sqrt{16} \times \sqrt{16} \times \sqrt{16}$

   (h) $-\dfrac{1}{2 \times 2 \times 2 \times 2}$

2. Describe in your own words what the power notation does in the following.

   (a) 8^3

   (b) -4^2

   (c) $(-4)^2$

   (d) $512^{\frac{1}{3}}$

   (e) 7^{-4}

When calculating with powers in algebra we use exactly the same principles as we developed earlier in calculating with powers in arithmetic. Let's summarize these now.

	Arithmetic	Algebra
Adding and subtracting powers	$2^2 + 2^3$ Cannot be simplified further only evaluated to get 4 + 8 =12	$a^m + a^n$ Cannot be simplified further.
Multiplying powers When multiplying powers which have the same base we add the indices.	$7^3 \times 7^5 = 7^{3+5} = 7^8$	$a^m \times a^n = a^{m+n}$ or $a^m a^n = a^{m+n}$
Dividing powers When dividing powers which have the same base we subtract the indices.	$6^3 \div 6^2 = 6^{3-2} = 6^1 = 6$	$a^m \div a^n = a^{m-n}$ or $\frac{a^m}{a^n} = a^{m-n}$
Raising a power to a power When raising a power to another power we multiply the indices.	$(3^2)^5 = 3^{2\times5} = 3^{10}$	$(a^m)^n = a^{m\times n} = a^{mn}$
Power of a product If a product of two or more factors is raised to a power then each factor is raised to that power.	$(3\times2)^4 = 3^4 \times 2^4$	$(ab)^n = a^n b^n$

Example

Evaluate $8^{\frac{2}{3}}$ without using a calculator.

You could complete this question one of two ways, both of which use your knowledge of raising a power to a power.

$$\begin{aligned} 8^{\frac{2}{3}} &= 8^{\frac{1}{3}\times2} \\ &= (8^{\frac{1}{3}})^2 \\ &= 2^2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} 8^{\frac{2}{3}} &= 8^{2\times\frac{1}{3}} \\ &= (8^2)^{\frac{1}{3}} \\ &= 64^{\frac{1}{3}} \\ &= 4 \end{aligned}$$

In both cases the answer is 4. Check it on your calculator.

Example

Simplify $3a^2 \times a^x$

$3a^2 \times a^x = 3 \times a^2 \times a^x$ Separate into a series of multiplications.

$= 3 \times a^{2+x}$ Group a's together and add the indices.

$= 3a^{2+x}$ Remove multiplication sign for simplicity.

Example

Divide $2x^3$ by x^4 and write in its simplest form with positive indices.

$\frac{2x^3}{x^4} = \frac{2 \times x^3}{x^4}$ Separate numerator into a series of multiplications.

$= 2 \times x^{3-4}$ Group together the x's and subtract indices.

$= 2 \times x^{-1}$ Rewrite the negative index in its positive form.

$= \frac{2}{x^1}$

$= \frac{2}{x}$ Recall that x raised to the power 1 is x.

Example

Simplify $\frac{(2ab)^2}{a^4}$

$\frac{(2ab)^2}{a^4} = \frac{2^2 a^2 b^2}{a^4}$ Remove brackets from numerator using power of a product.

To divide powers with the same base subtract the indices.

$= 2^2 a^{2-4} b^2$ Simplify.

$= 4a^{-2} b^2$ Rewrite a^{-2} as a positive power.

$= \frac{4b^2}{a^2}$ Note that this question does not ask for positive powers so either of the last two lines would be correct.

Example

Simplify $\dfrac{(a^{-1}b^{\frac{1}{3}})^3}{(3^{\frac{1}{2}}a^{-2})^2}$

$\dfrac{(a^{-1}b^{\frac{1}{3}})^3}{(3^{\frac{1}{2}}a^{-2})^2} = \dfrac{a^{-1\times 3}b^{\frac{1}{3}\times 3}}{3^{\frac{1}{2}\times 2}a^{-2\times 2}}$	Remove brackets from numerator and denominator by using power of a product rule.
$= \dfrac{a^{-3}b^1}{3^1a^{-4}}$	Simplify indices.
$= \dfrac{a^{-3--4}b^1}{3^1}$	To divide powers subtract indices.
$= \dfrac{a^1b}{3}$	Simplify indices.
$= \dfrac{ab}{3}$ or $\dfrac{1}{3}ab$	Both alternatives are correct.

Activity 3.10

1. Simplify the following.

   (a) $x^{\frac{3}{5}} \times x^{\frac{4}{5}}$

   (b) $(x^4)^{\frac{n}{2}}$

   (c) $(a^{\frac{3}{m}}b^{\frac{2}{n}})^{mn}$

   (d) $\dfrac{x^2}{x^{\frac{1}{2}}}$

   (e) $(a^2b)^0$

2. Simplify. Write your answers with positive powers.

   (a) $a^{-3} \times a^{-5}$

   (b) $\dfrac{1}{z^{-n}}$

   (c) $-x^{-n}$

   (d) $\dfrac{a^{-3}}{a}$

   (e) $x^{n} \times x^{-2n}$

3. Apply the rules of powers to simplify the following.

   (a) $\dfrac{8x^8y^2}{x^2y^4}$

   (b) $\dfrac{-36x^6y^{-1}}{-3x^{-3}y^2}$

   (c) $\dfrac{16(s^2t)^2}{4s^{-3}t^2}$

   (d) $(5^3x^{-2}y^{-4})^0$

   (e) $\left(a^{\frac{5}{2}} \div b^{\frac{2}{3}}\right)^6$

   (f) $\left(\dfrac{a^3b^{-2}}{b^3c^{-1}}\right)^{-3}$

   (g) $(16x^4y^2)^{\frac{1}{4}}(x^3y^6z^9)^{\frac{1}{6}}$

   (h) $\dfrac{(36a^4b^2)^{\frac{3}{2}}}{(64a^6b^3)^{\frac{2}{3}}}$

Time to reassess…

You are over half way through this module now. Check to see if an assignment is due? How is it going? Now is a good time to reassess your action plan for study. Are you on schedule or do you need to upgrade it? You may need to contact your tutor if you are having problems keeping to your plan. If you have forgotten how to manage your action plan return to module 1 where you will find details. The key to success in this course is keeping on track and keeping in touch with your tutor.

3.3 Special relationships

3.3.1 Equations

In section 3.1 of this module we generated several relationships which connected a number of variables together into an equation. Equations are really just formulas which assert the equality of two expressions or quantities.

Words	Equation	Variables
The selling price of a shirt is set each year. What will the income to the company be if 250 shirts were sold in the first six months and 150 in the next six months?	$I = 250P + 150P$	P=Selling price of a shirt I=Income to the company
The income to a company is determined by calculating the product of the quantity of goods and their selling price.	$I = S \times Q$	I=Income to the company S=Selling price Q=Quantity of the goods
At a constant temperature, the volume occupied by a gas is the quotient of a constant, called K, and the pressure exerted on the gas.	$V = \frac{\mathrm{K}}{P}$	V=Volume of the gas P=Pressure exerted on the gas K is a constant

The question now is, once we have generated an equation how can we use it? In the previous section we acquired lots of tools – grouping like terms, factorizing, algebraic fractions and powers that will, along with the standard operations of addition, subtraction, multiplication and division, help us work with equations.

Solving equations

Solving equations is like a balancing act. You need to keep both sides of the equation balanced. So whatever you do to one side you must do to the other side. This section assumes that you have solved simple equations in the past. Have a look at the revision module if you are feeling uncertain. Let's have a look at some examples.

Example

Solve this equation for x, $2x + 3 = x - 4$

$2x + 3 = x - 4$	
$2x - x + 3 = x - 4 - x$	Group x's on one side by subtracting x from both sides.
$x + 3 = -4$	Simplify.
$x + 3 - 3 = -4 - 3$	Group constants on one side by subtracting 3 from both sides.
$x = -7$	

Check:

Left hand side when $x = -7$, $2 \times -7 + 3 = -11$

Right hand side when $x = -7$, $-7 - 4 = -11$

Left hand side (LHS) equals right hand side (RHS) so solution is correct.

Solution is $x = -7$.

Example

What value of p satisfies the equation, $4(p+1) - 2p = p + 1$

$4(p+1) - 2p = p + 1$	
$4p + 4 - 2p = p + 1$	Remove brackets.
$2p + 4 = p + 1$	Group like terms on left hand side.
$2p - p + 4 = p + 1 - p$	Group the p's on left hand side by subtracting p from both sides.
$p + 4 = 1$	
$p + 4 - 4 = 1 - 4$	Group the constants on the right hand side by subtracting 4 from both sides.
$p = -3$	

Check:

Left hand side when $p = -3$, $4(-3+1) - 2 \times -3 = -2$

Right hand side when $p = -3$, $-3 + 1 = -2$

Left hand side (LHS) equals right hand side (RHS) so solution is correct.

Solution is $p = -3$.

Example

Isolate q on the left hand side of the equation, $\frac{q+3}{2}=\frac{q+1}{4}$

$\frac{q+3}{2}=\frac{q+1}{4}$	
$\frac{q+3}{2}\times 2=\frac{q+1}{4}\times 2$	To remove 2 from the denominator on LHS multiply both sides by 2.
$q+3=\frac{2(q+1)}{4}$	
$(q+3)\times 4=\frac{2(q+1)}{4}\times 4$	To remove 4 from the denominator on RHS multiply both sides by 4.
$4(q+3)=2(q+1)$	
$4q+12=2q+2$	Remove brackets.
$4q+12-2q=2q+2-2q$	To group q's on the LHS subtract $2q$ from both sides.
$2q+12=2$	
$2q+12-12=2-12$	To group constants on the RHS subtract 12 from both sides.
$2q=-10$	
$\frac{2q}{2}=\frac{-10}{2}$	To isolate q divide both sides by 2.
$q=-5$	

Check:

Left hand side when $q=-5$, $\frac{-5+3}{2}=-1$

Right hand side when $q=-5$, $\frac{-5+1}{4}=-1$

Left hand side (LHS) equals right hand side (RHS) so solution is correct.

Solution is $q=-5$.

Example

Find the value of x in the equation, $2x-\frac{1}{3}=1-10x$

$2x-\frac{1}{3}=1-10x$

$2x-\frac{1}{3}+\frac{1}{3}=1-10x+\frac{1}{3}$ To group constants on the RHS add one third to both sides.

$2x=\frac{4}{3}-10x$

$2x+10x=\frac{4}{3}-10x+10x$ To group x's on LHS add $10x$ to both sides.

$12x=\frac{4}{3}$

$12x\div 12=\frac{4}{3}\div 12$ To remove 12 from the numerator of LHS divide both sides by 12.

$x=\frac{{}^{1}\not{4}}{3}\times\frac{1}{\not{12}_{3}}$ Cancel 4 from numerator and denominator.

$x=\frac{1}{3}\times\frac{1}{3}$

$x=\frac{1}{9}$

Left hand side when $x=\frac{1}{9}$, $2\times\frac{1}{9}-\frac{1}{3}=-\frac{1}{9}$

Right hand side when $x=\frac{1}{9}$, $1-10\times\frac{1}{9}=-\frac{1}{9}$

Left hand side (LHS) equals right hand side (RHS) so solution is correct.

Solution is $x=\frac{1}{9}$

Activity 3.11

1. Solve the following equations and test that your solutions are correct.

   (a) $11m - 4 = 2 - 3m$

   (b) $38 = 14 - \frac{6x}{11}$

   (c) $3(2x - 1) = x + 4$

   (d) $5b = 4 - 2(b + 1)$

   (e) $\frac{3r - 4}{10} = 3$

2. Two students attempted a mathematics question in different ways. Investigate their solutions by completing the following steps.

   (i) Elizabeth began solving the problem as follows:

   $$\frac{3x - 4}{9} = \frac{2x - 2}{3}$$
   $$\frac{9(3x - 4)}{9} = \frac{9(2x - 2)}{3}$$

   Complete this solution.

   (ii) Mark began solving the problem as follows:

   $$\frac{3x - 4}{9} = \frac{2x - 2}{3}$$
   $$\frac{3x}{9} - \frac{4}{9} = \frac{2x}{3} - \frac{2}{3}$$

   Complete the solution.

   (iii) Which solution do you prefer? Why?

3. For each question find the value of x which satisfies the equation.

   (a) $\frac{4x + 3}{10} = -\frac{2x}{5}$

   (b) $\frac{x}{3} + \frac{x + 1}{4} = 2$

   (c) $\frac{3x - 1}{4} - \frac{5 - 2x}{6} = 1 - \frac{3x - 5}{2}$

   (d) $\frac{5}{x} + \frac{3}{2x} = 2$

   (e) $\frac{x - 2}{x + 3} = \frac{3}{5}$

4. Isolate x on the left hand side of the equation.

   (a) $\dfrac{2x-1}{3} - 5 = \dfrac{x}{6}$

   (b) $\dfrac{8}{x} = \dfrac{2}{3}$

   (c) $\dfrac{2x-1}{8} = \dfrac{3x+1}{4}$

   (d) $\dfrac{5}{x-4} = \dfrac{2}{x-2}$

   (e) $\dfrac{1}{x+2} + \dfrac{1}{x-3} = \dfrac{1}{(x+2)(x-3)}$

5. Word problems using equations involve two parts:

   - Setting up the equation for the information given.
   - Solving the equation.

   Translate the following words into equations. Remember to state what your variable represents. Do not solve the equations.

   (a) Three plus an unknown number is equal to 15 decreased by the unknown number.

   (b) An 85 cm piece of rope is to be cut into two pieces. One piece is to be 15 cm shorter than the other.

   (c) Andrew buys four more cans of red paint than white paint. All together he buys 18 cans of these two products. How many cans of red paint does he buy?

   (d) Mark buys 5 times as many pens as pencils. If he purchases 24 of these items in total, how many of each does he buy?

   (e) The length of a rectangle is 25 cm and its perimeter is 96 cm. What is the width of the rectangle?

   (f) A formula 1 driver travels at an average speed of x km/h for 30 minutes and then increases his speed by 20 km/h for the next hour. If the total distance travelled in this time is 240 km, find the value of x.

   (g) Michelle invested two amounts of money that differed by \$150. The greater part was invested at 8% and the other at 12%. If the total interest earned was \$60, how much money was invested at each rate?

   (h) An inheritance of \$22 000 dollars is to be divided among three people so that the first person has twice as much money as the second and three times as much money as the third. How much does each person receive?

6. Use the equations you have set up in 5 (a) – (e) to find the solutions to the questions.

7. Set up each of the following problems algebraically, solve and check. Remember to state what your variables represent.

   (a) If \$186 was spent on fencing which cost \$3 per metre, how many metres of fencing were purchased?

   (b) A proposed feature window has a width that is one third of its length. If the width of the frame is increased by 4 m, and the length is decreased by 2 m, then the area remains the same. Find the original dimensions of the frame.

Rearranging formulas

When we were solving equations we usually ended up with the variable equal to some quantity. Sometimes we don't necessarily want to solve an equation, but rather to just rewrite it so one variable is in terms of another variable. This process is called **rearranging the formula**.

For example, the formula for calculating temperature in the Fahrenheit scale, given the temperature in the Centigrade scale, is $F = \frac{9}{5}C + 32$. If you wanted to calculate the Centigrade temperature quickly, given the Fahrenheit temperature, you could rearrange the formula to make *C* the subject instead of the *F*. The principles for this are the same as for solving equations because formulas are just special types of equations.

Let's rearrange $F = \frac{9}{5}C + 32$ to make *C* the subject.

$$F = \frac{9}{5}C + 32$$

$$F - 32 = \frac{9}{5}C + 32 - 32$$ To isolate *C* on RHS subtract 32 from both sides.

$$F - 32 = \frac{9}{5}C$$

$$(F - 32) \times 5 = \frac{9}{5}C \times 5$$ To remove 5 from denominator of RHS multiply both sides by 5.

$$5(F - 32) = 9C$$

$$\frac{5(F - 32)}{9} = \frac{9C}{9}$$ To isolate *C* completely divide both sides by 9.

$$\frac{5(F - 32)}{9} = C$$

$$C = \frac{5(F - 32)}{9}$$ Convention says *C* should be on the LHS of the equation.

$$C = \frac{5}{9}F - \frac{160}{9}$$ This is an alternative way of writing previous line, both are correct.

Let's have a look at some further examples that will use all of the tools we have to date.

Example

Make T the subject of the formula, $I = PRT$

$$I = PRT$$

$$\frac{I}{PR} = \frac{PRT}{PR}$$ To isolate T on RHS divide both sides by PR.

$$T = \frac{I}{PR}$$ Convention says that T should be on LHS.

Example

Transpose the formula to make t the subject; $d = \frac{1}{2}gt^2$

$$d = \frac{1}{2}gt^2$$

$$d \times 2 = \frac{1}{2}gt^2 \times 2$$ To remove 2 from denominator of RHS multiply both sides by 2.

$$2d = gt^2$$

$$\frac{2d}{g} = \frac{gt^2}{g}$$ To isolate t on the RHS divide both sides by g.

$$\frac{2d}{g} = t^2$$ To remove the square from t take the square root of both sides.

$$\pm\sqrt{\frac{2d}{g}} = \pm\sqrt{t^2}$$ Recall that you should always include positive and negative square roots.

$$\pm\sqrt{\frac{2d}{g}} = t$$

$$t = \pm\sqrt{\frac{2d}{g}}$$ Put t on the LHS of the equation.

Note that in real life the t in this formula represents time and could not have negative values.

Example

Isolate the variable l on the left hand side of the formula, $T = 2\pi\sqrt{\frac{l}{g}}$

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$\frac{T}{2\pi} = \frac{2\pi\sqrt{\frac{l}{g}}}{2\pi}$$ To remove 2π from RHS divide both sides by 2π.

$$\frac{T}{2\pi} = \sqrt{\frac{l}{g}}$$

$$(\frac{T}{2\pi})^2 = (\sqrt{\frac{l}{g}})^2$$ To remove the square root from RHS square both sides.

$$(\frac{T}{2\pi})^2 = \frac{l}{g}$$

$$(\frac{T}{2\pi})^2 \times g = \frac{l}{g} \times g$$ To isolate the l on RHS multiply both sides by g.

$$(\frac{T}{2\pi})^2 \times g = l$$

$$l = g(\frac{T}{2\pi})^2$$ Put l on the LHS of the equation.

Example

Rearrange the formula to make *i* the subject, $V = P(1+\frac{i}{100})^n$

$$V = P(1+\frac{i}{100})^n$$

$$\frac{V}{P} = \frac{P(1+\frac{i}{100})^n}{P}$$ To remove *P* from RHS divide both sides by *P*.

$$\frac{V}{P} = (1+\frac{i}{100})^n$$

$$\sqrt[n]{\frac{V}{P}} = \sqrt[n]{(1+\frac{i}{100})^n}$$ To undo the power of n take the nth root of both sides. Note that at this stage this could be either positive or negative or both depending on the value of n.

$$\sqrt[n]{\frac{V}{P}} = 1+\frac{i}{100}$$

$$\sqrt[n]{\frac{V}{P}} - 1 = 1+\frac{i}{100} - 1$$ To isolate *i* subtract 1 from both sides.

$$\sqrt[n]{\frac{V}{P}} - 1 = \frac{i}{100}$$

$$(\sqrt[n]{\frac{V}{P}} - 1)\times 100 = \frac{i}{100}\times 100$$ To completely isolate the *i* multiply both sides by 100.

$$i = 100\,(\sqrt[n]{\frac{V}{P}} - 1)$$ Put *i* on the LHS of the equation.

Example

Make R the subject of the following formula instead of $\frac{1}{R}$; $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$\frac{1}{R} = \frac{1}{R_1} \times \frac{R_2}{R_2} + \frac{1}{R_2} \times \frac{R_1}{R_1}$$

Make R_1R_2 the denominator by appropriate multiplications of both terms.

$$\frac{1}{R} = \frac{R_2}{R_1R_2} + \frac{R_1}{R_1R_2}$$

$$\frac{1}{R} = \frac{R_1 + R_2}{R_1R_2}$$

Add the numerators.

$$R = \frac{R_1R_2}{R_1 + R_2}$$

Switch numerator and denominator of both sides to make R the subject of the formula (take reciprocals of both sides).

Recall that if $\frac{1}{x} = 2$ then $x = \frac{1}{2}$.

Activity 3.12

1. Rearrange the formula for the variable indicated.

(a) $V = LWH$, H

(b) $v^2 = u^2 + 2as$, u

(c) $v = \frac{4}{3}\pi r^3$, r

(d) $v = u + at$, t

(e) $s = ut + \frac{1}{2}at^2$ a

(f) $s = \frac{1}{2}\left(\frac{Ee}{m}\right)\left(\frac{l}{v}\right)^2$ e

(g) $P = (1+r)^n$ r

(h) $\frac{1}{\lambda} = R\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$ λ

(i) $T = 2\pi\sqrt{\frac{m}{k}}$ k

(j) $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$ v

2. The formula for the diminishing value method for calculating the depreciation of an item is $A = P(1-i)^n$ where,

   P is the original purchase price

   A is the current value

   i is the depreciation rate

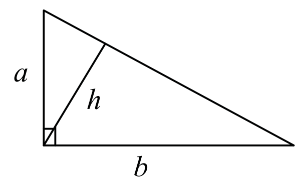
   n is the number of periods

   (a) Rearrange the formula to make i the subject.

   (b) If the original purchase price of a car is \$2 800, the number of periods is 3 and the current purchase price is \$1 500, what is the depreciation rate?

3. The variables a, b and h are related as shown for a right angled triangle.

$$a = \frac{h\sqrt{a^2 + b^2}}{b}$$



   (i) Rearrange to make h the subject of the formula.

   (ii) If $a = 12$ cm, $b = 5$ cm, what is the height of the triangle?

4. Artificial gravity is produced in a space station through the rotation of the station. The number of rotations per minute, N, to simulate Earth's gravity is given by the formula

   $N = \frac{147}{11}\sqrt{\frac{5}{r}}$ where r is the radius of the space station.

   (i) Isolate r on the left hand side of the formula.

   (ii) What is the radius of a space station if it is rotating at 3 rotations per minute to simulate earth's gravity?

5. A particle undergoing uniform circular motion is subjected to a centripetal force given by: $F = \frac{mv^2}{r}$, where m is the mass in kg, v is the velocity in m/s, r is the length of string in m.

   (i) Make r the subject of the formula.

   (ii) What is the length of string, r, attached to a rock with a mass, m, of 0.05kg, which generates a centripetal force, F, of 10 kN, when travelling at a velocity, v, of, 8 m/s?

3.3.2 Inequations

When faced with a real world problem to translate to algebraic language, not all will result in simple expressions or equations. Consider the following.

The total cost of landscaping a garden includes the cost of plants (call it P) and the cost of the soil and mulch (call it S). The owner declares that the total cost of the job should not be greater than $2 500. Express this relationship in algebraic form.

There is no way that this would be an equation. But rather this would be an **inequation** or an **inequality**.

We would write it as $P + S \leq 2500$, and say it like this:

The sum of the cost of plants and the cost of soil and mulch is less than or equal to two thousand five hundred dollars.

Recall from your previous experiences the meaning of the following signs:

$<$	Less than
$\leq$	Less than or equal to
$>$	Greater than
$\geq$	Greater than or equal to

Simple versions of these inequations can be solved in ways similar to those used to solve equations. There are a couple of differences so look at the following examples for the pitfalls.

Example

Find the value of x when $2x - 1 < 4$

$$2x - 1 < 4$$

$$2x - 1 + 1 < 4 + 1$$ To remove –1 from the LHS add 1 to both sides.

$$2x < 5$$

$$\frac{2x}{2} < \frac{5}{2}$$ To isolate the x divide both sides by 2.

$$x < \frac{5}{2}$$

Check:

One possible solution would be $x = -3$ (you could choose any number less than $2\frac{1}{2}$),

LHS = $2(-3) - 1 = -7$, this answer is less than 4 so solution is reasonable.

Example

Rearrange the following formula to make i the subject, $j \le \frac{i}{2} + 4$

$j \le \frac{i}{2} + 4$	
$j - 4 \le \frac{i}{2} + 4 - 4$	To remove 4 from the RHS subtract 4 from both sides.
$j - 4 \le \frac{i}{2}$	
$2 \times (j - 4) \le \frac{i}{2} \times 2$	To isolate the i multiply both sides by 2.
$2(j - 4) \le i$	When you switch i to the LHS of the inequation you must change the direction of the inequality sign.
$i \ge 2(j - 4)$	

> When you switch sides in an inequality you must reverse the sign. For example:
>
> $2 < 3$ must become $3 > 2$, otherwise it is not true.

Example

Solve the following inequation for x, $1 - 2x < x + 2$

$1 - 2x < x + 2$	
$1 - 2x - x < x + 2 - x$	To group the x's on the LHS subtract x from both sides.
$1 - 3x < 2$	
$1 - 3x - 1 < 2 - 1$	To group the constants on the RHS subtract 1 from both sides.
$-3x < 1$	
$\frac{-3x}{-3} > \frac{1}{-3}$	To isolate x on the LHS divide both sides by –3.
$x > -\frac{1}{3}$	When you divide an inequation by a negative number you must reverse the inequality sign.

> When you divide or multiply by a negative number you must reverse the inequality sign. For example:
>
> $2 < 3$ when multiplied on both sides by –1 must become $-2 > -3$, otherwise it is not true.

Check:

One possible solution would be $x = 0$

LHS = $1 - 2(0) = 1$

RHS = $0 + 2 = 2$

LHS < RHS, so solution is reasonable.

Example

In the following double inequality find the values of *x* which will satisfy it; $-2 < 3x + 1 < 4$

$$-2 < 3x + 1 < 4$$

We can rearrange all three parts of this inequation at the same time.

$$-2 - 1 < 3x + 1 - 1 < 4 - 1$$

To remove the 1 from the centre part subtract 1 from all three parts.

$$-3 < 3x < 3$$

To isolate the *x* in the centre divide all three parts by 3.

$$\frac{-3}{3} < \frac{3x}{3} < \frac{3}{3}$$

$$-1 < x < 1$$

This means that *x* lies between –1 and 1.

Check:

One possible solution is $x = 0$.

Centre expression is $3(0) + 1 = 1$. This is between –2 and 4 so solution is reasonable.

Activity 3.13

1. Solve the following inequations for *x*.

   (a) $\frac{x}{3} + 2 < 5$

   (b) $22 \le 5x - 3 \le 32$

   (c) $\frac{3x}{5} - \frac{2x}{3} > -7$

   (d) $3x - 2(2x - 7) \le 2(3 + x) - 4$

   (e) $-3 \le \frac{2x - 1}{3} < 3$

2. (a) Solve $6 - 3(2y - 3) = 4(2y - 2)$ for *y*

   (b) Solve $6 - 3(2y - 3) \le 4(2y - 2)$ for *y*

   (c) How are the solutions in (a) and (b) different?

3. (a) A mixed indoor cricket team must have 5 more women than men and the total number of people must be at least 9 but not more than 15 (including reserves). What is the possible number of women on the team?

   (b) The sum of two consecutive integers is less than 13. What positive values can the smaller integer have?

   (c) A food stall at a football ground sells on average 10 more sausage rolls than pies. If each food stall must allow for at least 60 customers but not more than 100, what is the possible number of sausage rolls that should be heated?

Absolute value

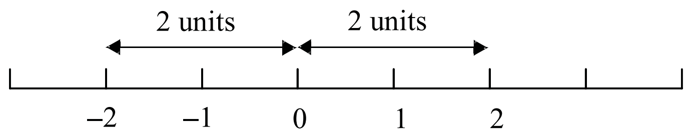
In many instances when you solve an inequality, especially if it is a double inequality, you might get a range of answers. The example $-2 < 3x + 1 < 4$ produced the answer $-1 < x < 1$.

We can summarize this type of inequality by introducing a new notation called **absolute value**. So that

$-1 < x < 1$ can be written as $|x| < 1$.

We can say this as 'the absolute value of *x* is less than 1'.

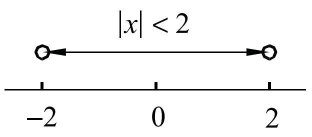
Recall now where you might have seen absolute value in the past. We use absolute value when we are interested in the magnitude or size of a number without regard to its sign. For example, if we return to the number line we know that –2 and 2 are the same distance from zero.



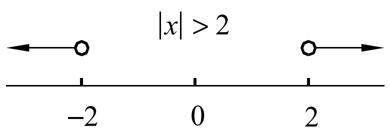
So using absolute value notation we would say that $|2| = 2$ and $|-2| = 2$

If we introduce a variable into this type of notation and say that $|x| = 2$ then that means that *x* has two possible values, $x = 2$ or $x = -2$, which are both two units from zero. You might be familiar with this being written as $x = \pm 2$.

If we extend this even further to an inequality and say $|x| < 2$, then we know that *x* must lie somewhere between –2 and 2 on the number line, i.e. $-2 < x < 2$



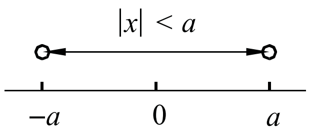
Similarly if we write $|x| > 2$, then we know that *x* must lie further than 2 on one side of the number line and also further than –2 on the other side of the number line i.e. $x > 2$ or $x < -2$



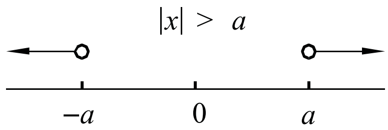
So in a general way

If $|x| < a$ then $-a < x < a$

If $|x| > a$ then $x > a$ or $x < -a$



Let's look at some examples which use this notation.



Example

Find the values of x which satisfy $|x+1| \le 2$

$|x+1| \le 2$	Replace absolute value notation with inequality.
$-2 \le x+1 \le 2$	
$-2-1 \le x+1-1 \le 2-1$	To remove 1 from the centre part subtract 1 from all three parts.
$-3 \le x \le 1$	

Check:

Two possible solutions are –3 and 1.

When $x = -3, \text{LHS} = |-3+1| = |-2| = 2 = \text{RHS}$

When $x = 1, \text{LHS} = |1+1| = |2| = 2 = \text{RHS}$

Example

Solve the following inequation for p; $|3p-4|>5$

$|3p-4|>5$. This will produce two inequations

$$\begin{aligned} 3p-4 &> 5 \\ 3p-4+4 &> 5+4 \\ 3p &> 9 \\ \frac{3p}{3} &> \frac{9}{3} \\ p &> 3 \end{aligned}$$

or

$$\begin{aligned} 3p-4 &< -5 \\ 3p-4+4 &< -5+4 \\ 3p &< -1 \\ \frac{3p}{3} &< \frac{-1}{3} \\ p &< -\frac{1}{3} \end{aligned}$$

Note: This cannot be written as $-5>3p-4>5$ because –5 is not greater than 5. It must be written as two inequations.

Final answer is $p>3$ or $p<-\frac{1}{3}$

Check:

Two possible solutions are $p=4$ and $p=-1$

When $p=4, \text{LHS}=|3\times 4-4|=|8|=8>\text{RHS}$

When $p=-1, \text{LHS}=|3\times -1-4|=|-7|=7>\text{RHS}$

Activity 3.14

1. Solve the following inequalities.

   (a) $|x|\le 3$

   (b) $|2x|>8$

   (c) $|x-4|\le 1$

   (d) $|8-4x|>0$

   (e) $|3x-4|<8$

   (f) $|2-(x+1)|\le 3$

   (g) $|x-1|>2$

3.3.3 Quadratic equations

In section 3.1 we looked at describing relationships. The relationships described in this section all involved variables which were raised only to the power of 1. Of course, not all relationships are that simple.

In economics, the average cost equation for a particular production plant is $C = 0.3Q^2 - 3.3Q + 15.3$ where C is the average cost in dollars of producing each unit and Q is the quantity produced.

In engineering, the positioning of a suspension bridge cable can be approximated by the quadratic equation $H = \frac{3d^2}{722}$ where H is height above the road in metres and d is the distance from the centre of the cable in metres.

In applied biology, the relationship $C = 30t^2 - 240t + 500$ is used to predict the number of bacteria (C) in a swimming pool, measured as count per cm^3, given the time (t) in days from treatment.

The right hand side of all of the above relationships contains an expression in one variable. This variable is always raised to a power which is either 2, 1 or zero. Recall that any variable raised to the power of zero is equal to one ($x^0 = 1$). If an expression of this type is put equal to zero or a constant then it is called a quadratic equation.

The general form of a quadratic equation is $\mathrm{a}x^2 + \mathrm{b}x + \mathrm{c} = 0$.

In this type of equation a, b and c are all constant. The constants can take any value except for ‘a’ which must not be equal to zero.

Have a look at the table below and notice which equations are quadratic equations.

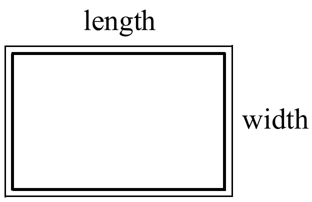
Equation	**Highest power of variable**	**Other powers of variable**	**Quadratic equation?**
$x^2 + 2x - 3 = 0$	2	1, 0	Yes
$-\frac{1}{2}x^2 - x + \frac{1}{3} = 0$	2	1, 0	Yes
$4x^2 - 2 = 0$	2	0	Yes
$3x^2 = 0$	2	0	Yes
$4x + 1 = 0$	1	0	No
$x^2 + \sqrt{x} + x = 0$	2	$\frac{1}{2}$, 1	No
$2x^2 + \frac{1}{x} + 3 = 0$	2	−1, 0	No

But how do such equations come about. Let's look now at how we can derive a quadratic equation.

Consider the following:

> *A signwriting company wants to manufacture a series of signs which their product suppliers (timber and paint) say must have a perimeter of 10 metres and an area of 6 square metres. What length and width should they make these signs?*

First draw a diagram of this situation.



Area is length multiplied by width.
Perimeter is the sum of all the sides.

If we call length, l, width, w, area, A and perimeter, P, then we would have the following equations.

$$P = l + l + w + w$$
$$P = 2(l + w)$$

But the perimeter was 10 metres, so

$$10 = 2(l + w)$$
$$5 = l + w$$
$$w = 5 - l$$

So width is written in terms of length, $w = 5 - l$.

We also know that

$$Area = length \times width$$
$$A = l \times w$$

But the area was 6 square metres and width could be written as $5 - l$, so

$$A = l \times w$$
$$6 = l \times (5 - l)$$
$$6 = 5l - l^2$$
$$l^2 - 5l + 6 = 0$$

The question now arises, how can we solve quadratic equations?

Think about the previous work we have covered in this module. Do you recall the section on factorization?

Go to section 3.2.2 for a refresher on factorization.

Returning to the signwriting example, we ended up with the equation $l^2 - 5l + 6 = 0$. Let's see how we can solve it.

The left hand side of this equation should be familiar. We can factorize this to get:

$$l^2 - 5l + 6 = (l - 3)(l - 2)$$

So the equation becomes:

$$l^2 - 5l + 6 = 0$$
$$(l - 3)(l - 2) = 0$$

But why would you want to factorize the left hand side? Think about this:

> *If you have two numbers that multiply together to give zero, then at least one of those numbers must be zero.*

Algebraically this could be written as:

If $a \times b = 0$ then $a = 0$ or $b = 0$, or both.

So in the case of $(l - 3)(l - 2) = 0$,

then $l - 3 = 0$ or $l - 2 = 0$.

Solving both these equations for l, we get $l = 3$ or $l = 2$.

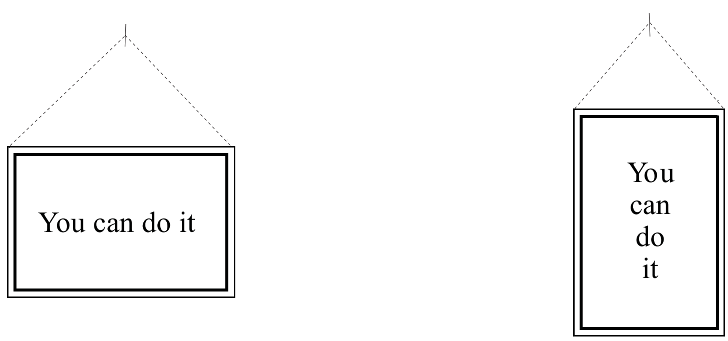
So the solutions to our original equation $l^2 - 5l + 6 = 0$, are $l = 3$ or $l = 2$.

To check this answer we substitute these values back into the original equation.

Check: $l = 3$; $\text{LHS} = 3^2 - 5 \times 3 + 6 = 0 = \text{RHS}$

$l = 2$; $\text{LHS} = 2^2 - 5 \times 2 + 6 = 0 = \text{RHS}$

So in reference to the original question, both 3 metres and 2 metres are possible lengths of the sign. Notice that if the length is 2 metres then the width must be 3 metres. While if the length is 3 metres the width is 2 metres. We will leave it to the signwriters as to which way they orientate the sign.



Factorization

Let's have a look at some examples that use factorization to solve quadratic equations.

Example

Solve this equation for *x*, $x^2 + 8x + 15 = 0$

$x^2 + 8x + 15 = 0$	
$x^2 + (5 + 3)x + 5 \times 3 = 0$	Use factors to rewrite 8.
$x^2 + 5x + 3x + 5 \times 3 = 0$	Remove brackets.
$x(x + 5) + 3(x + 5) = 0$	Take out *x* as a common factor from the first two terms and 3 from the next two terms. In each case the remaining factor should be the same, $x+5$.
$(x + 3)(x + 5) = 0$	
$x + 3 = 0$, or $x + 5 = 0$	Take out $x+5$ as a common factor.
$x = -3$, or $x = -5$	

Check: $x = -3$, $\text{LHS} = (-3)^2 + 8 \times -3 + 15 = 0 = \text{RHS}$

$x = -5$, $\text{LHS} = (-5)^2 + 8 \times -5 + 15 = 0 = \text{RHS}$

Solutions are $x = -3$ or $x = -5$.

Example

What values of *a* satisfy the quadratic equation, $6a^2 + 7a - 5 = 0$

$6a^2 + 7a - 5 = 0$	
$6a^2 + (10 - 3)a - 5 = 0$	Use factors to rewrite 7.
$6a^2 + 10a - 3a - 5 = 0$	Remove brackets.
$2a(3a + 5) - 1(3a + 5) = 0$	Take out $2a$ as a common factor from the first two terms and -1 from the next two terms. In each case the remaining factor should be the same, $3a+5$.
$(2a - 1)(3a + 5) = 0$	
$2a - 1 = 0$, $3a + 5 = 0$	Take out $3a+5$ as a common factor.
$a = \frac{1}{2}$, or $a = -\frac{5}{3}$	

Check: $a = \frac{1}{2}$, $\text{LHS} = 6 \times (\frac{1}{2})^2 + 7 \times \frac{1}{2} - 5 = 0 = \text{RHS}$

$a = -\frac{5}{3}$, $\text{LHS} = 6 \times (-\frac{5}{3})^2 + 7 \times -\frac{5}{3} - 5 = 0 = \text{RHS}$

So $a = \frac{1}{2}$ or $a = -\frac{5}{3}$ satisfy the equation.

Activity 3.15

1. Solve each equation by factoring.

   (a) $x^2 + 5x + 4 = 0$

   (b) $x^2 - 7x - 30 = 0$

   (c) $v^2 = 4(v + 24)$

   (d) $x^2 = \dfrac{3 - 5x}{2}$

   (e) $(6x + 2)(x - 4) = 2 - 11x$

2. Solve the following quadratics.

   (a) $x^2 - 3x = 0$

   (b) $6x^2 = 24x$

   (c) $9x^2 = 25$

3. Find the solutions to the following.

   (a) The height, h, metres of a stone t seconds after being thrown vertically upwards is given by $h = 64t - 12t^2$. At what times is the stone at a height of 80 m.

   (b) A rectangular swimming pool 12 m by 9 m is surrounded by a concrete path of uniform width. If the area of the path is 100 m^2, find its width.

Quadratic formula

In the real world not all practical problems will result in equations that are easy to factorize. To help us solve all types of quadratic equations, maths practitioners have developed a quadratic formula. This will allow us to solve all types of quadratic equations, if they have solutions.

It goes like this:

If $ax^2 + bx + c = 0$ **then** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$**, where a, b and c are constants and** $a \neq 0$

A bit of history… **For interest only**

The quadratic formula was developed after centuries of work. Inspired by the work of the ancient Greeks a form was seen in ancient Iraqi writings. Today we use the form developed by Descartes and his contemporaries in the 16th and 17th centuries. The currently accepted derivation of the formula goes like this. **Do not learn this**.

$$\mathrm{a}x^2 + \mathrm{b}x + \mathrm{c} = 0$$

$$a(x^2 + \frac{b}{a}x + \frac{c}{a}) = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + 2 \times \frac{b}{2a}x + \frac{c}{a} = 0$$

$$x^2 + 2 \times \frac{b}{2a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a} = 0$$

$$(x + \frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a} = 0$$

$$(x + \frac{b}{2a})^2 = (\frac{b}{2a})^2 - \frac{c}{a}$$

$$(x + \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a}\frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example

Use the quadratic formula to solve this equation for x, $x^2 + x - 3 = 0$

Compare this equation with the general form, $ax^2 + bx + c = 0$, then a = 1, b = 1 and c = –3.

Using the formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times -3}}{2 \times 1}$$
$$x = \frac{-1 \pm \sqrt{13}}{2}$$
$$x = \frac{-1 + \sqrt{13}}{2}, \text{ or } x = \frac{-1 - \sqrt{13}}{2}$$
$$x \approx 1.3028, \text{ or } x \approx -2.3028$$

Check: $x \approx 1.3028$, $\text{LHS} \approx (1.3028)^2 + 1 \times 1.3028 - 3 \approx 0.00009 \approx \text{RHS}$

$x \approx -2.3028$, $\text{LHS} \approx (-2.3028)^2 + 1 \times -2.3028 - 3 \approx 0.00009 \approx \text{RHS}$

This check will always be approximate because of the rounding of the x values.

Final solutions are $x \approx 1.3028$ or $x \approx -2.3028$.

Example

What values of q satisfy the following equation, $3q^2 - 2q - 4 = 0$

Compare this equation with the general form, $ax^2 + bx + c = 0$. In this equation the variable is q not x, and a = 3, b = –2 and c = –4.

Using the formula,

$$q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$q = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 3 \times -4}}{2 \times 3}$$
$$q = \frac{2 \pm \sqrt{52}}{6}$$
$$q = \frac{2 + \sqrt{52}}{6}, \quad q = \frac{2 - \sqrt{52}}{6}$$
$$q \approx 1.5352, \quad \text{or } q \approx -0.8685$$

Check: $x \approx 1.5352$, $\text{LHS} \approx 3 \times (1.5352)^2 - 2 \times 1.5352 - 4 \approx 0.0001 \approx \text{RHS}$

$x \approx -0.8685$, $\text{LHS} \approx 3 \times (-0.8685)^2 - 2 \times -0.8685 - 4 \approx 0.00012 \approx \text{RHS}$

This check will always be approximate because of rounding the q values.

Final solutions are 1.5352 or –0.8685.

Example

Find the root (s) of the quadratic equation, $x^2 - 8x + 16 = 0$

The solutions to an equation are sometimes called the **roots** of an equation. This question is asking you to solve the quadratic equation for x.

Compare this equation with the general form, $\mathrm{a}x^2 + \mathrm{b}x + \mathrm{c} = 0$, then a = 1, b = –8 and c = 16.

Using the formula,

$$x = \frac{-\mathrm{b} \pm \sqrt{\mathrm{b}^2 - 4\mathrm{ac}}}{2\mathrm{a}}$$
$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \times 1 \times 16}}{2 \times 1}$$
$$x = \frac{8 \pm \sqrt{64 - 64}}{2}$$
$$x = \frac{8 \pm \sqrt{0}}{2}$$
$$x = \frac{8}{2}$$
$$x = 4$$

Check: $x = 4, \quad \mathrm{LHS} = (4)^2 - 8 \times 4 + 16 = 0 = \mathrm{RHS}$

Final solution is 4.

(Note you might recognise that you could have solved this equation by factorization.)

Example

Solve the equation $t^2 + 2t + 3 = 0$ for t

Compare this equation with the general form, $\mathrm{a}x^2 + \mathrm{b}x + \mathrm{c} = 0$. In this equation the variable is t not x, and a = 1, b = 2 and c = 3.

Using the formula,

$$t = \frac{-\mathrm{b} \pm \sqrt{\mathrm{b}^2 - 4\mathrm{ac}}}{2\mathrm{a}}$$
$$t = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 3}}{2 \times 1}$$
$$t = \frac{-2 \pm \sqrt{4 - 12}}{2}$$
$$t = \frac{-2 \pm \sqrt{-8}}{2}$$

In our number system we cannot calculate the square root of a negative number, so this equation does not have a solution.

Before we go on to practice solving quadratic equations, using the four examples above complete the following table.

Quadratic equation	**Value of** $b^2 - 4ac$	**Number of solutions**
$x^2 + x - 3 = 0$	13	2
$3q^2 - 2q - 4 = 0$		
$x^2 - 8x + 16 = 0$		
$t^2 + 2t + 3 = 0$		

What did you find? It looks like quadratic equations can be divided into three groups depending on the value of $b^2 - 4ac$.

If $b^2 - 4ac$ **is positive (**$b^2 - 4ac > 0$**) then the equation will have two solutions.**

If $b^2 - 4ac$ **is zero (**$b^2 - 4ac = 0$**) then the equation will have one solution.**

If $b^2 - 4ac$ **is negative (**$b^2 - 4ac < 0$**) then the equation has no solutions and cannot be solved.**

Activity 3.16

1. The quantity $b^2 - 4ac$ is called the **discriminant** as it discriminates among the characters of the roots. Use the discriminant to determine the nature of the roots in the following equations.

   (a) $x^2 - 3x + 2 = 0$

   (b) $x^2 + 10x + 25 = 0$

   (c) $x^2 - 3x + 7 = 0$

   (d) $4x^2 - 4x + 1 = 0$

   (e) $x^2 + 1 = 0$

2. Use the quadratic formula to solve the following equations.

   (a) $4x^2 - 12x + 9 = 0$

   (b) $x^2 + 6x + 9 = 0$

   (c) $2q^2 + q - 6 = 0$

   (d) $3x^2 - x - 2 = 0$

   (e) $y^2 = 4y + 1$

   (f) $3x^2 = 6x - 2$

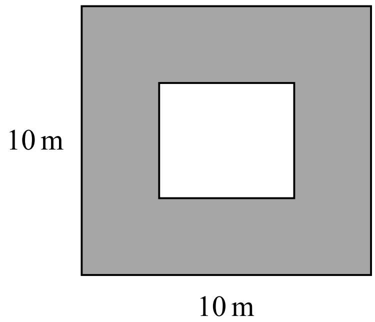
   (g) $9b^2 = 2 + 6b$

   (h) $x^2 = \dfrac{3 - 5x}{2}$

Quadratic equations in the real world

Example

A square courtyard measuring 10 m by 10 m, is designed so that the tiles around the edge of the courtyard are patterned and cover half of the courtyard as in the plans below. How wide must the area of patterned tiles be?



- What do we know?

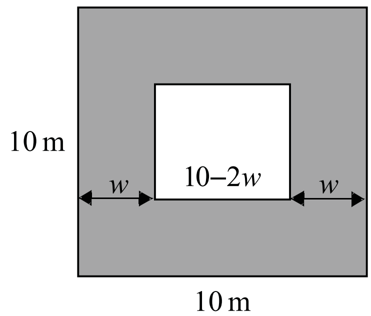
We know that the total area of the courtyard is 10 m ×10 m, which is 100 m^2. We also know that the area of the patterned tiles must be half of this or 50 m^2.

- What do we need to find out?

We need to know what will the width of the band of patterned tiles be. Let's call this *w.*

- Is there a relationship between width, *w*, and other measurements?

If *w* is the width of the band of patterned tiles then the dimension of the area left with plain tiles must be $10 - 2w$.



So,

Total area = 100

Area of plain tiles = $(10 - 2w)(10 - 2w)$

The area of the patterned tiles must be the total area minus the area of the plain tiles.

Area of patterned tiles = $100 - (10 - 2w)(10 - 2w)$

But we know that this area must be equal to 50 m^2

So

$$100-(10-2w)(10-2w)=50$$
$$100-(100-40w+4w^2)=50$$
$$100-100+40w-4w^2=50$$
$$40w-4w^2=50$$
$$4w^2-40w+50=0$$
$$2(2w^2-20w+25=0$$
$$2w^2-20w+25=0$$

To solve this quadratic equation we could try to factorize, however, the factors are not quickly apparent.

Using the quadratic formula, $x=\frac{-\mathrm{b}\pm\sqrt{\mathrm{b}^2-4\mathrm{ac}}}{2\mathrm{a}}$, where our variable is w and a = 2, b = –20 and c = 25, we get

$$w=\frac{-(-20)\pm\sqrt{(-20)^2-4\times2\times25}}{2\times2}$$
$$w=\frac{20\pm\sqrt{400-200}}{4}$$
$$w=\frac{20\pm\sqrt{200}}{4}$$
$$w=\frac{20\pm\sqrt{100\times2}}{4}$$
$$w=\frac{20\pm\sqrt{100}\times\sqrt{2}}{4}$$
$$w=\frac{20\pm10\times\sqrt{2}}{4}$$
$$w=\frac{2(10\pm5\sqrt{2})}{4}$$
$$w=\frac{10\pm5\sqrt{2}}{2}$$
$$w=\frac{10+5\sqrt{2}}{2},\quad w=\frac{10-5\sqrt{2}}{2}$$
$$w\approx8.5,\quad \text{or } w\approx1.5$$

Possible solutions are then, widths of 1.5 m or 8.5 m. The only acceptable one is 1.5 m, as the other will not fit into the 10 m dimensions of the courtyard.

Then width of the patterned tiles must be approximately 1.5 m.

Now that we have accumulated some strategies and techniques for deriving and solving quadratic equations let's put it all together in a summary.

- Look for what information is given in a question. Draw a diagram.
- Look for what the question is asking you to find.
- Look for variables embedded in the sentences. Give them a symbol.
- Look for relationships between these variable. Write these relationships out in algebraic form.
- Develop a quadratic equation in one variable, perhaps using another relationship.
- Solve the quadratic equation. First try factorization if this doesn't work quickly, then use the quadratic formula.
- Check your answer, especially if the quadratic formula is used.
- Realize that not all equations will have a solution.

We will return to this topic later in module 4 when we look at quadratic equations from a graphical perspective. But for now have a go at some more quadratic equations.

Activity 3.17

1. Use the quadratic formula to solve the following.

   (a) The length of a rectangle is 2 metres more than its width. If its area is $2m^2$, find its dimensions.

   (b) The percent markup on the cost of a dress was the same as the cost in dollars. If the dress sold for $156 dollars, what was the cost of the dress?

   (c) A radiation control point is set up near a solid waste disposal facility. The pad on which the facility is set is 6m × 10m. A walkway is set up on the pad around its edges, which reduces the available area to 40 sq.m. How wide is the walkway?

   (d) A farmer has 200m of fencing with which he wishes to enclose a rectangular field. One side of the fence can make use of an existing fence. If he knows that he needs to have an area of 5 000 sq. m., what dimensions should he use so that he will utilize all the fencing material?

   (e) A piece of wire of length 40cm is bent into a rectangular shape to enclose an area of 100cm. What will the lengths of the sides of the rectangle?

   (f) A new labour contract provided for an increase in wages of $2 per hour and a reduction of 4 hours in the work week. A worker who had been receiving $330 per week would get a $50 per week raise under the new agreement. How long was the work week before the new contract?

(g) Decreasing the speed of a car by 5 km/h to prevent overheating, resulted in a trip of 312 km taking 20 minutes longer. What was the original speed of the car?

Something to talk about…

The quadratic formula is one of the most complex ones you will come across in this course. What is the best way to learn and apply it? Share your ideas with us in our discussion group.

3.3.4 Simultaneous equations

Previously, we have concerned ourselves with one unknown variable, but in real practice there are often two or more variables to contend with. For example:

A city bakery sold 1 500 bread rolls on Sunday, with sales receipts of $1 460. Plain rolls sold for 90 cents each while gourmet rolls sold for $1.45 each. How many of each type of roll were sold?

What information is given in this question?

The total number of rolls sold was 1 500.

The total money received from selling these rolls was $1 460.

The price of each plain and gourmet roll.

What is the question asking?

This question asks us how many of each type of roll so there are two variables involved.

The **number** of plain bread rolls, let's call it P.

The **number** of gourmet rolls, let's call it G.

Is there a relationship between the variables?

If the total number of rolls sold is 1 500 then number of plain rolls and number of gourmet rolls must be 1 500. In algebraic language this is $P + G = 1500$

If the total receipts is $1 460 and plain rolls cost 90 cents each and gourmet rolls $1.45 each then the total receipts from plain rolls and the total receipts from gourmet rolls must be $1 460. In algebraic language this is $0.9P + 1.45G = 1460$. (Remember 90 cents is $0.90)

We have generated two equations in two unknown variables.

$$P + G = 1500$$

$$0.9P + 1.45G = 1460$$

Such equations are called a set of **simultaneous equations**. You can have a pair with two unknown variables or a set of three with three unknown variables. In fact, you can have as many equations as you need as long as you have at least as many equations as you have unknown variables. Only then do you have a chance of finding values for each of the variables if solutions exist.

In this module we will focus only on two equations with two unknown variables. In the next module we will look at how we can display these graphically, and in a module 6 we will look at an entirely different way of solving such equations, but for now have a go at developing your own sets of simultaneous equations. You will have done a bit of this previously in this module.

Activity 3.18

Set up equations for each of the following problems algebraically **using two variables**. Remember to state what your variables represent. **Don't solve at this stage**.

(a) The velocity, v, of a moving object is given by the equation $v = u + at$, where u is the starting velocity; a is acceleration and t is travelled. If $v = 5m/s$, at $t = 2s$ and $v = 7m/s$ at $t = 3s$, set up two equations that would allow you to calculate u and a.

(b) 80 tyres and 53 tubes cost a tyre company $959 and 106 tyres plus 75 tubes cost $1,285. Find the cost of a tyre and a tube?

(c) A mother is 4 times as old as her son. In six years time a mother will be 3 times as old as her son. Find their present ages.

(d) The weekly wages of 6 carpenters and 8 apprentices amount to $5 160 and the wages of 8 carpenters and 6 apprentices amount to $5 760. Find the weekly wages of a carpenter and an apprentice.

(e) A box contains 40 coins made up of 10 cent coins and 20 cent coins. How many of each are in the box if the value of the coins is $5.00?

Once you have mastered generating equations the next stage is to solve them. You might also have done this previously.

There are a number of different ways that you can solve a set of simultaneous equations. We will look at two here and another method in module 6 of this course. All of the methods described are equally valid. It's up to you to choose the method that is easiest for you to use.

Substitution

In this method we will substitute one variable from one equation into the other.

The steps involved are:

1. Using either of the equations, express one variable in terms of the other.
2. This expression is then substituted into the other equation to form an equation in one variable only.

3. Solve this equation to find the value of one of the variables.
4. Substitute the value of this variable into the equation formed in the first step to find the value of the other variable.
5. Check your answer in both original equations.

Let's follow the steps through in a pair of simple equations.

$$2x + 5y = 6 \qquad (1)$$
$$3x + 2y = -2 \qquad (2)$$

Step 1: Express one variable in terms of the other.

Using equation (1) rearrange the equation to make x the subject.

$$\begin{aligned} 2x + 5y &= 6 \\ 2x &= 6 - 5y \\ x &= \frac{6 - 5y}{2} \end{aligned}$$

Step 2 and Step 3: Substitute this into equation (2) and solve for the single variable.

$$\begin{aligned} 3x + 2y &= -2 \\ 3\left(\frac{6 - 5y}{2}\right) + 2y &= -2 \\ 3(6 - 5y) + 2 \times 2y &= 2 \times -2 \\ 18 - 15y + 4y &= -4 \\ 18 - 11y &= -4 \\ -11y &= -22 \\ y &= 2 \end{aligned}$$

Step 4: Substitute this value into the other equation. (You could also substitute it into the equation formed in step 1 if you find this easier.)

$$\begin{aligned} 2x + 5y &= 6 \\ 2x + 5(2) &= 6 \\ 2x + 10 &= 6 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

From our calculations our answers are $x = -2$ and $y = 2$.

Step 5: Check the answers in both original equations.

Check: $x = -2, y = 2$

Equation 1

$$2x + 5y = 6$$
$$\text{LHS} = 2 \times -2 + 5 \times 2 = -4 + 10 = 6 = \text{RHS}$$

Equation 2

$$3x + 2y = -2$$
$$\text{LHS} = 3 \times -2 + 2 \times 2 = -6 + 4 = -2 = \text{RHS}$$

Both values substitute correctly into both equations so the answer must be correct.

Elimination

In this method we will eliminate one variable and form one equation.

The steps involved are:

1. Multiply one or both equations by constants so that one of the variables has the same coefficient.

2. Subtract one equation from the other so that the variable with the same coefficient is eliminated.

3. Solve this equation to find the value of the variable.

4. Substitute the value of this variable into one of the equations to find the value of the other variable.

5. Check your answer in both the original equations.

Let's look at the previous equations.

$$2x + 5y = 6 \qquad (1)$$
$$3x + 2y = -2 \qquad (2)$$

Step 1: Multiply one or both equations by constants.

The coefficients of x are different in both equations but if we multiplied equation (1) by 3 and equation (2) by 2, the coefficient of x in both equations would be 6. Remember, however that we cannot simply multiply the x term by a constant we must multiply every term in the equation by it. Let's do it.

Multiply equation (1) by 3 $\quad 3 \times 2x + 3 \times 5y = 3 \times 6$

$$6x + 15y = 18 \qquad (3)$$

Multiply equation (2) by 2 $\quad 2 \times 3x + 2 \times 2y = 2 \times -2$

$$6x + 4y = -4 \qquad (4)$$

Step 2: Subtract the equations to eliminate a variable.

$$6x + 15y = 18$$
$$\underline{6x + 4y = -4}$$
$$0x + 11y = 22$$

Equation is $11y = 22$

Step 3: Solve the equation.

$$11y = 22$$
$$y = 2$$

Step 4: Substitute to find the other variable.

$$2x + 5y = 6$$
$$2x + 5(2) = 6$$
$$2x + 10 = 6$$
$$2x = -4$$
$$x = -2$$

From our calculations our answers are $x = -2$ and $y = 2$.

Step 5: Check the answers in both original equations

Check: $x = -2, y = 2$

Equation 1

$$2x + 5y = 6$$
$$\text{LHS} = 2 \times -2 + 5 \times 2 = -4 + 10 = 6 = RHS$$

Note it is not absolutely essential to substitute into this equation to check because we used it to originally find the value of the variable, but it is good practice as a check on your arithmetic.

Equation 2

$$3x + 2y = -2$$
$$\text{LHS} = 3 \times -2 + 2 \times 2 = -6 + 4 = -2 = \text{RHS}$$

Both values substitute correctly into both equations so the answer must be correct.

You should note that this is just one way of eliminating a variable, we could have just as easily decided to eliminate *y* by multiplying one equation by 5 and the other by 2……the choice is yours.

Example

Solve these equations simultaneously to find the values of *x* and *y*.

$$6x + 5y = 4 \qquad (1)$$
$$8x + 3y = -2 \qquad (2)$$

Multiply equation (1) by 3

$$3 \times 6x + 3 \times 5y = 3 \times 4$$
$$18x + 15y = 12 \qquad (3)$$

Multiply equation (2) by 5

$$5 \times 8x + 5 \times 3y = 5 \times -2$$
$$40x + 15y = -10 \qquad (4)$$

Subtract equation (4) from equation (3)

$$\begin{array}{l} 18x + 15y = 12 \\ \underline{40x + 15y = -10} \\ -22x + 0y = 22 \end{array}$$

Equation is $-22x = 22$

$x = -1$

To find the value of y substitute $x = -1$ into (1)

$$\begin{aligned} 6x + 5y &= 4 \\ 6(-1) + 5y &= 4 \\ -6 + 5y &= 4 \\ 5y &= 10 \\ y &= 2 \end{aligned}$$

The solution is $x = -1$ and $y = 2$

Check:

In equation 1 $\quad \text{LHS} = 6(-1) + 5(2) = 4 = \text{RHS}$

In equation 2 $\quad \text{LHS} = 8(-1) + 3(2) = -2 = \text{RHS}$

Solution is correct.

Example

Refer back to the example at the beginning of this section.

A city bakery sold 1 500 bread rolls on Sunday, with sales receipts of $1 460. Plain rolls sold for 90 cents each while gourmet rolls sold for $1.45 each. How many of each type of roll were sold?

The two equations generated from these words were:

$$P + G = 1500 \qquad (1)$$

$$0.9P + 1.45G = 1460 \qquad (2)$$

Multiply equation (1) by 0.9 $\quad 0.9P + 0.9G = 1350 \qquad (3)$

Subtract equation (3) from equation (2)

$$\begin{array}{l} 0.9P + 1.45G = 1460 \\ \underline{0.9P + 0.9G = 1350} \\ 0P + 0.55G = 110 \end{array}$$

Equation is $0.55G = 110$

$G = 200$

To find the value of P substitute $G = 200$ into $P + G = 1500$

$$\begin{aligned} P + G &= 1500 \\ P + 200 &= 1500 \\ P &= 1300 \end{aligned}$$

Solution is $G = 200$ and $P = 1300$

Check:

In equation 1 LHS $= 200 + 1300 = 1500 =$ RHS

In equation 2 LHS $= 0.9 \times 1300 + 1.45 \times 200 = 1460 =$ RHS

So at the end of the day the shop sold 200 gourmet rolls and 1300 plain rolls.

Activity 3.19

1. Solve the following simultaneous equations by substitution or elimination.

(a) $y - x = 3$ (1)
$2x + y = 0$ (2)

(b) $2p + 3q = 0$ (1)
$p + q - 2 = 0$ (2)

(c) $3x - 2y = 4$ (1)
$5x + 4y = 3$ (2)

(d) $3x + 2y = -4$ (1)
$9x + 8y = -11$ (2)

(e) $6y - 5x = 18$ (1)
$12x - 9y = 0$ (2)

(f) $x + y = 2$ (1)
$2x - y = -1$ (2)

(g) $8x = 5y$ (1)
$13x = 8y + 1$ (2)

(h) $\frac{1}{4}x - \frac{1}{5}y = -1$ (1)
$3x + \frac{1}{2}y = 17$ (2)

2. Solve the word problems in activity 3.18 (a) – (c) for which you have already generated the pairs of equations.

Something to talk about…

Simultaneous equations are all around us when we look. To help increase your understanding of these types of equations why not get a friend or work colleague to make some up for you or send some to the discussion group so we can all have a go.

That's the end of this module. You will have experienced a lot of new algebraic techniques to arm you for your future studies in this course and later.

But before you are really finished you should do a number of things

1. Have a close look at your action plan for study. Are you on schedule? Or do you need to restructure your action plan or contact your tutor to discuss any delays or concerns?

2. Make a summary of the important points in this module noting your strengths and weaknesses. Add any new words to your personal glossary. This will help with future revision.

3. Practice some real world problems by having a go at 'A taste of things to come'

4. Check your skill level by attempting the post-test.

3.4 A taste of things to come

(There are questions you might encounter in your future studies. If you find them difficult proceed direct to post-test.)

1. When running a business the aim is to have a profit. This will occur when the company's total revenue exceeds its total costs. The total costs can be considered to be made up of the fixed costs (e.g. leasing charges, insurance, depreciation) and variable costs (e.g. labour, materials). When the total revenue equals the total costs, this is said to be the break even point.

   After analyzing her company's cost structure, a brush manufacturer found that her fixed costs were \$800 300 per annum with a constant variable cost of \$6.45 per unit. Each brush is marketed to sell for \$14 per unit.

   (a) Letting x be the number of units to be produced and sold, write an equation for Total Cost (TC) and an equation for Total Revenue (TR).

   (b) Solve these equations to find the break even point.

   (c) As Profit = Total Revenue – Total Cost, check your solution in (b).

2. How many litres of a 5% alcohol solution must be added to 10 litres of 40% alcohol solution to make a 25% alcohol solution?

3. In Australia, most mortgages are repaid in monthly instalments. The financial institution calculates your monthly repayment using the following formula:

$$R = \frac{IP}{1-(1+I)^{-N}}$$

In this formula:

R is the monthly repayment in dollars.

P is the principal (the amount borrowed) in dollars.

I is the monthly compound interest rate as a decimal.

N is the number of months of repayment.

The monthly interest rate as a decimal can be calculated as:

$$I = \frac{i}{12 \times 100}$$

where *i* is the annual compound interest rate as a percentage. In home loans the interest is usually compounded monthly.

(a) If you wanted to borrow \$120 000 at an interest rate of 12% for a period of 15 years. What amount will you have to repay each month?

(b) How much in total will you repay on your loan?

(c) What would happen to the monthly repayments if you took the loan for half this period of time, 7.5 years? How much in total will you repay on the loan?

(d) Make the principal amount, *P*, the subject of the equation. If you only had \$1 000 per month to repay the mortgage, what is the largest amount you could borrow under the conditions in question (a)?

(e) Imagine you are buying your own house. What factors would you consider to ensure that you are paying your mortgage as efficiently as possible?

3.5 Post-test

1. Simplify the following.

   (a) $3x^2 - 2x - 5x^2 + 7x$

   (b) $-\frac{5q}{12r^2} \div \frac{25q^2}{24r}$

   (c) $\frac{x}{x+3} - \frac{5+x}{x+2}$

2. Factorize the following.

   (a) $5q^2 - q$

   (b) $x^2 + 5x - 14$

3. Expand

   (a) $(x-2)(x-3)$

   (b) $(3+2x)(4x-1)$

4. Apply the rules of powers to simplify the following.

   (a) $\frac{36x^5y^2}{2x^6y}$

5. Solve the following equations and verify your solution is correct.

   (a) $16 = -2 + \frac{3x}{4}$

   (b) $5 - 3(b+1) = 6b$

   (c) $\frac{5x+4}{2} = -\frac{3x}{5}$

6. Solve the following systems of equations.

   $4x + 5y = 310$ (1)

   $5x + 4y = 320$ (2)

7. Rearrange to make the nominated variable the subject of the formula.

   (a) $v^2 = u^2 + 2as$ (a)

   (b) $S = \frac{1}{2}\left(\frac{Ee}{m}\right)\left(\frac{l}{v}\right)^2$ (l)

8. Solve the following inequations for x.

   (a) $-\frac{x}{4} + 2 < 6$

   (b) $15 \le 4x + 3 \le 23$

   (c) $|10 - 4x| \ge 6$

9. A man invested part of his income at 10% and the rest at 12%. The income from both investments totalled \$3 975. If he interchanged investments, his income would have totalled \$3 890. How much did he have in each investment?

10. What are the dimensions of a rectangle if its perimeter is 40 m and its area is 96 m^2?

3.6 Solutions

Solutions to activities

Activity 3.1

1.

(a) $CO = SV \times HR$

(b) $AD = C + I + G + X - M$

(c) $T = 120 \times \frac{V}{d}$ or $T = \frac{120V}{d}$

(d) $APC = \frac{C}{DI}$

(e) $F = 32 + 1.8C$

(f) $TVC = AVC \times Q$

(g) $P = \frac{Fs}{t}$

(h) $P = R - C$

(i) $F = \frac{Gm_1m_2}{d^2}$

(j) $HR = 220 - A$

Activity 3.2

1.

(a) $5y - 3y + 2$
$= 2y + 2$

(b) $8a + 4b - 2b + 3a$
$= 8a + 3a + 4b - 2b$
$= 11a + 2b$

(c) $p - 6p + 2p - 4n$
$= -3p - 4n$

(d) $-5x^2 - x + 2x^2 + 4x$
$= -5x^2 + 2x^2 - x + 4x$
$= -3x^2 + 3x$

(e) $-a^3 - 3a^2 + a^3$
$= -a^3 + a^3 - 3a^2$
$= -3a^2$

(f) $6ab + {}^-8a + 2b + 3ab$
$= 6ab + 3ab + {}^-8a + 2b$
$= 9ab + {}^-8a + 2b$

(g) $4qr + 6st - 2qr + 5st$
$= 4qr - 2qr + 6st + 5st$
$= 2qr + 11st$

(h) $-2a^2b + 3bc - 4ba^2$
$= -2a^2b - 4ba^2 + 3bc$
$= -6a^2b + 3bc$

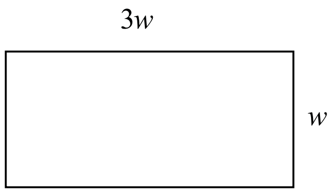
(i) $a^2 - 2a(a+1)$
$= a^2 - 2a \times a - 2a \times 1$
$= a^2 - 2a^2 - 2a$
$= -a^2 - 2a$

(j) $2(p-3) - (2p + p^2)$
$= 2p - 6 - 2p - p^2$
$= 2p - 2p - 6 - p^2$
$= -6 - p^2$

(k) $x - 3x(x+2)$
$= x - 3x^2 - 3x \times 2$
$= x - 3x^2 - 6x$
$= x - 6x - 3x^2$
$= -5x - 3x^2$

2.

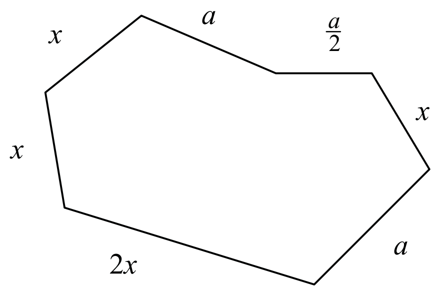
(a)



Perimeter is the distance around the edge of the shape i.e. the sum of the lengths of the sides. If perimeter is called *P* then,

$$P = 3w + w + 3w + w$$
$$P = 8w$$

(b)



If perimeter is *P* then,

$$P = x + x + 2x + a + x + \frac{a}{2} + a$$
$$P = x + x + 2x + x + a + \frac{a}{2} + a$$
$$P = 5x + \frac{5}{2}a$$

Activity 3.3

1.

(a) $5x + 5y$
$= 5 \times x + 5 \times y$
$= 5(x + y)$

(b) $ab + 8b$
$= a \times b + 8 \times b$
$= b(a + 8)$

(c) $n^2 - 5n$
$= n \times n - 5 \times n$
$= n(n - 5)$

(d) $-2y + -8$
$= -2 \times y + -2 \times 4$
$= -2(y + 4)$

(e) $-3qr - 9qt$
$= -3 \times q \times r - 3 \times 3 \times q \times t$
$= -3q(r + 3t)$

(f) $12pq - 28p^2$
$= 3 \times 4 \times p \times q - 4 \times 7 \times p \times p$
$= 4p(3q - 7p)$

2.

(a) $st^2 - t$
$= s \times t \times t - 1 \times t$
$= t(st - 1)$

(b) $7x^2y + 21xy^3 + 14xy^2$
$= 7 \times x \times x \times y + 3 \times 7 \times x \times y \times y \times y + 2 \times 7 \times x \times y \times y$
$= 7xy(x + 3y^2 + 2y)$

(c) $-2a^2b^3 + 5a^2b^2 + 3a^3b^2$
$= -2 \times a \times a \times b \times b \times b + 5 \times a \times a \times b \times b + 3 \times a \times a \times a \times b \times b$
$= a^2b^2(-2b + 5 + 3a)$

(d) $-2z^3 + 8z^2 - 4z$
$= -2 \times z \times z \times z + 2 \times 4 \times z \times z - 2 \times 2 \times z$
$= -2z(z^2 - 4z + 2)$

Activity 3.4

1.

(a) $5k(k + 3)$
$= 5k \times k + 5k \times 3$
$= 5k^2 + 15k$

(b) $-(2x - 5)$
$= -2x + 5$

(c) $(x + 4)(x + 3)$
$= x(x + 3) + 4(x + 3)$
$= x^2 + 3x + 4x + 4 \times 3$
$= x^2 + 7x + 12$

(d) $(y-2)(y+2)$
$= y(y+2)-2(y+2)$
$= y^2+2y-2y-2\times 2$
$= y^2-4$

(e) $(2z-1)(-6z+4)$
$=2z(-6z+4)-1(-6z+4)$
$=2z\times -6z+2z\times 4-1\times -6z-1\times 4$
$=-12z^2+8z+6z-4$
$=-12z^2+14z-4$

2.

(a) $(3y+4)^2$
$= (3y+4)(3y+4)$
$= 3y(3y+4)+4(3y+4)$
$= 9y^2+12y+12y+16$
$= 9y^2+24y+16$

(b) $(3a-1)(2a-1)$
$= 3a(2a-1)-1(2a-1)$
$= 6a^2-3a-2a-1\times -1$
$= 6a^2-5a+1$

(c) $(2+3x)(4x+1)$
$= 2(4x+1)+3x(4x+1)$
$= 8x+2+12x^2+3x$
$=12x^2+11x+2$

(d) $(-4q+2)(-3q+3)$
$=-4q(-3q+3)+2(-3q+3)$
$=-4q\times -3q-4q\times 3+2\times -3q+2\times 3$
$=12q^2-12q-6q+6$
$=12q^2-18q+6$

(e) $(-2g-3)(-6g-5)$
$=-2g(-6g-5)-3(-6g-5)$
$=-2g\times -6g-2g\times -5-3\times -6g-3\times -5$
$=12g^2+10g+18g+15$
$=12g^2+28g+15$

Activity 3.5

1.

(a) $x^2 + 6x + 5$
$= x^2 + (1+5)x + 1 \times 5$
$= x^2 + 1x + 5x + 5$
$= x(x+1) + 5(x+1)$
$= (x+5)(x+1)$

(b) $y^2 + 8y + 12$
$= y^2 + (2+6)y + 2 \times 6$
$= y^2 + 2y + 6y + 12$
$= y(y+2) + 6(y+2)$
$= (y+6)(y+2)$

(c) $x^2 - 11x - 42$
$= x^2 + (-14 + 3)x - 14 \times 3$
$= x^2 - 14x + 3x - 14 \times 3$
$= x(x - 14) + 3(x - 14)$
$= (x - 14)\ (x + 3)$

(d) $a^2 - 7a + 12$
$= a^2 + (^-3 - 4)a + -3 \times -4$
$= a^2 - 3a - 4a + 12$
$= a(a-3) - 4(a-3)$
$= (a-4)(a-3)$

(e) $z^2 + 2z - 15$
$= z^2 + (5-3)z - 3 \times 5$
$= z^2 + 5z - 3z - 15$
$= z(z+5) - 3(z+5)$
$= (z-3)(z+5)$

2.

(a) $b^2 + 13b + 30$
$= b^2 + (3+10)b + 3 \times 10$
$= b^2 + 3b + 10b + 30$
$= b(b+3) + 10(b+3)$
$= (b+3)(b+10)$

(b) $x^2 - 12x - 28$
$= x^2 + (2 - 14)x - 14 \times 2$
$= x^2 + 2x - 14x - 28$
$= x(x + 2) - 14(x + 2)$
$= (x - 14)(x + 2)$

(c) $c^2 - 15c - 16$
$= c^2 + (1 - 16)c - 16 \times 1$
$= c^2 + c - 16c - 16$
$= c(c + 1) - 16(c + 1)$
$= (c - 16)(c + 1)$

(d) $x^2 + 14x + 33$
$= x^2 + (3 + 11)x + 3 \times 11$
$= x^2 + 3x + 11x + 33$
$= x(x + 3) + 11(x + 3)$
$= (x + 11)(x + 3)$

(e) $m^2 - 20m + 64$
$= m^2 + (-4 - 16)m - 4 \times -16$
$= m^2 - 4m - 16m + 64$
$= m(m - 4) - 16(m - 4)$
$= (m - 16)(m - 4)$

(f) $x^2 + 9x - 36$
$= x^2 + (12 - 3)x - 3 \times 12$
$= x^2 + 12x - 3x - 36$
$= x(x + 12) - 3(x + 12)$
$= (x - 3)(x + 12)$

(g) $y^2 - 13y + 30$
$= y^2 + (-3 - 10)y - 3 \times -10$
$= y^2 - 3y - 10y + 30$
$= y(y - 3) - 10(y - 3)$
$= (y - 10)(y - 3)$

(h) $a^2 - a - 42$
$= a^2 + (6 - 7)a - 7 \times 6$
$= a^2 + 6a - 7a - 42$
$= a(a + 6) - 7(a + 6)$
$= (a - 7)(a + 6)$

(i) $v^2 + 2v + 1$
$= v^2 + (1+1)v + 1 \times 1$
$= v^2 + v + v + 1$
$= v(v+1) + 1(v+1)$
$= (v+1)(v+1)$

(j) $q^2 - 15q + 26$
$= q^2 + (-2-13)q - 2 \times -13$
$= q^2 - 2q - 13q + 26$
$= q(q-2) - 13(q-2)$
$= (q-13)(q-2)$

Activity 3.6

1.

(a) $2a^2 + 3a + 1$
$= 2a^2 + (2+1)a + 1$
$= 2a^2 + 2a + a + 1$
$= 2a(a+1) + 1(a+1)$
$= (2a+1)(a+1)$

(b) $3x^2 - 11x - 4$
$= 3x^2 + (1-12)x - 4$
$= 3x^2 + x - 12x - 4$
$= x(3x+1) - 4(3x+1)$
$= (x-4)(3x+1)$

2.

(a) $10a^2 - 11a - 6$
$= 10a^2 + (4-15)a - 6$
$= 10a^2 + 4a - 15a - 6$
$= 2a(5a+2) - 3(5a+2)$
$= (2a-3)(5a+2)$

(b) $6x^2 - 19x + 13$
$= 6x^2 + (-6-13)x + 13$
$= 6x^2 - 6x - 13x + 13$
$= 6x(x-1) - 13(x-1)$
$= (x-1)(6x-13)$

(c) $9x^2 + 30x + 25$

$= 9x^2 + (15 + 15)x + 25$

$= 9x^2 + 15x + 15x + 25$

$= 3x(3x + 5) + 5(3x + 5)$

$= (3x + 5)(3x + 5)$

(d) $3q^2 - 2q - 1$

$= 3q^2 + (1 - 3)q - 1$

$= 3q^2 + q - 3q - 1$

$= q(3q + 1) - 1(3q + 1)$

$= (q - 1)(3q + 1)$

Activity 3.7

1.

(a) $\frac{6c}{9d}$

$= \frac{\cancel{6}^{2} c}{\cancel{9}_{3} d}$

$= \frac{2c}{3d}$

(b) $\frac{16qr}{4qs}$

$= \frac{\cancel{16}^{4} \cancel{q} r}{\cancel{4}^{1} \cancel{q} s}$

$= \frac{4r}{s}$

(c) $\frac{28xy}{7xyz}$

$= \frac{4 \times \cancel{7} \times \cancel{x} \times \cancel{y}}{\cancel{7} \times \cancel{x} \times \cancel{y} \times z}$

$= \frac{4}{z}$

2.

(a) $\dfrac{3p}{7p} \times \dfrac{2}{3}$

$= \dfrac{3 \times p}{7 \times p} \times \dfrac{2}{3}$

$= \dfrac{2}{7}$

(b) $\dfrac{3x}{y} \div \dfrac{-3}{4y}$

$= \dfrac{3x}{y} \times \dfrac{-4y}{3}$

$= \dfrac{3 \times x}{y} \times \dfrac{-4 \times y}{3}$

$= \dfrac{-4x}{1}$

$= -4x$

(c) $\dfrac{-2q}{7qr} \times \dfrac{-r}{12}$

$= \dfrac{-2 \times q}{7 \times q \times r} \times \dfrac{-r}{2 \times 6}$

$= \dfrac{1}{42}$

(d) $\dfrac{-5xy}{3a} \div \dfrac{25ax}{18}$

$= \dfrac{-5 \times x \times y}{3 \times a} \times \dfrac{3 \times 6}{5 \times 5 \times a \times x}$

$= \dfrac{-6y}{5a^2}$

(e) $\dfrac{3p}{5p+2} \times \dfrac{5p+2}{9}$

$= \dfrac{3 \times p}{(5p+2)} \times \dfrac{(5p+2)}{3 \times 3}$

$= \dfrac{p}{3}$

3.

(a) Area of rectangle $= l \times w$

$$= \frac{2 \times x}{3 \times 3} \times \frac{3 \times y}{2 \times 2}$$

$$= \frac{xy}{6}$$

(b) Area of triangle $= \frac{b \times h}{2}$

$$= \frac{\frac{6ac}{25} \times \frac{5ab}{9}}{2}$$

$$= (\frac{3 \times 2 \times a \times c}{5 \times 5} \times \frac{5 \times a \times b}{3 \times 3}) \div 2$$

$$= \frac{2a^2bc}{15} \times \frac{1}{2}$$

$$= \frac{a^2bc}{15}$$

Activity 3.8

1.

(a) $\frac{2x}{5} + \frac{4x}{7}$

$$= \frac{2x}{5} \times \frac{7}{7} + \frac{4x}{7} \times \frac{5}{5}$$

$$= \frac{14x}{35} + \frac{20x}{35}$$

$$= \frac{14x + 20x}{35}$$

$$= \frac{34x}{35}$$

(b) $\frac{2}{5x} + \frac{4}{7x}$

$$= \frac{2}{5x} \times \frac{7}{7} + \frac{4}{7x} \times \frac{5}{5}$$

$$= \frac{14}{35x} + \frac{20}{35x}$$

$$= \frac{14 + 20}{35x}$$

$$= \frac{34}{35x}$$

(c) $\frac{1}{ab}-\frac{a}{bc}$

$=\frac{1}{ab}\times\frac{c}{c}-\frac{a}{bc}\times\frac{a}{a}$

$=\frac{c}{abc}-\frac{a^2}{abc}$

$=\frac{c-a^2}{abc}$

(d) $\frac{2a-b}{b}-\frac{4b}{c}$

$=\frac{(2a-b)}{b}\times\frac{c}{c}-\frac{4b}{c}\times\frac{b}{b}$

$=\frac{2ac-bc}{bc}-\frac{4b^2}{cb}$

$=\frac{-4b^2+2ac-bc}{bc}$

$=\frac{2ac-4b^2-bc}{bc}$

2.

(a) $\frac{2}{x}+\frac{5}{x^2}$

$=\frac{2}{x}\times\frac{x}{x}+\frac{5}{x^2}$

$=\frac{2x}{x^2}+\frac{5}{x^2}$

$=\frac{2x+5}{x^2}$

(b) $3-\frac{2a}{a+2}$

$=\frac{3}{1}\times\frac{(a+2)}{(a+2)}-\frac{2a}{a+2}$

$=\frac{3(a+2)}{(a+2)}-\frac{2a}{a+2}$

$=\frac{3a+6}{a+2}-\frac{2a}{a+2}$

$=\frac{3a+6-2a}{a+2}$

$=\frac{a+6}{a+2}$

(c) $\dfrac{x}{2x-3}+\dfrac{1}{x+2}$

$=\dfrac{x}{(2x-3)}\times\dfrac{(x+2)}{(x+2)}+\dfrac{1}{x+2}\times\dfrac{(2x-3)}{(2x-3)}$

$=\dfrac{x(x+2)}{(2x-3)(x+2)}+\dfrac{(2x-3)}{(x+2)(2x-3)}$

$=\dfrac{x^2+2x}{(2x-3)(x+2)}+\dfrac{2x-3}{(x+2)(2x-3)}$

$=\dfrac{x^2+2x+2x-3}{(2x-3)(x+2)}$

$=\dfrac{x^2+4x-3}{(2x-3)(x+2)}$

(d) $\dfrac{x+4}{x-4}-\dfrac{x-4}{x+4}$

$=\dfrac{(x+4)}{(x-4)}\times\dfrac{(x+4)}{(x+4)}-\dfrac{(x-4)}{(x+4)}\times\dfrac{(x-4)}{(x-4)}$

$=\dfrac{(x+4)(x+4)}{(x-4)(x+4)}-\dfrac{(x-4)(x-4)}{(x+4)(x-4)}$

$=\dfrac{x(x+4)+4(x+4)}{x^2-16}-\dfrac{x(x-4)-4(x-4)}{x^2-16}$

$=\dfrac{x^2+4x+4x+16}{x^2-16}-\dfrac{x^2-4x-4x+16}{x^2-16}$

$=\dfrac{x^2+8x+16}{x^2-16}-\dfrac{x^2-8x+16}{x^2-16}$

$=\dfrac{x^2+8x+16-(x^2-8x+16)}{x^2-16}$

$=\dfrac{x^2+8x+16-x^2+8x-16}{x^2-16}$

$=\dfrac{16x}{x^2-16}$

3.

(i) $\dfrac{1}{x-3}-\dfrac{x}{x^2-9}=\dfrac{1}{4-3}-\dfrac{4}{4^2-9}=\dfrac{1}{1}-\dfrac{4}{16-9}=1-\dfrac{4}{7}=\dfrac{3}{7}$

(ii) $\dfrac{3}{x^2-9}=\dfrac{3}{4^2-9}=\dfrac{3}{16-9}=\dfrac{3}{7}$

The answers are the same as they are equivalent expressions.

4.

(a) Correct solution:

$$\frac{5}{2x+3}-\frac{x+1}{2x+3}$$
$$=\frac{5-(x+1)}{2x+3}$$
$$=\frac{5-x-1}{2x+3}$$
$$=\frac{4-x}{2x+3}$$

The negative sign should be applied to both terms, not just the x in the $(x+1)$ part of the expression.

(b) Correct solution

$$\frac{1}{3x}+\frac{4}{3x}=\frac{5}{3x}$$

Because the terms have the same denominator, the numerators are added together.

(c) Correct solution

$$\frac{x+1}{x-2}+\frac{5}{x+1}$$
$$=\frac{x+1}{x-2}\times\frac{(x+1)}{(x+1)}+\frac{5}{(x+1)}\times\frac{(x-2)}{(x-2)}$$
$$=\frac{x(x+1)+1(x+1)}{(x-2)(x+1)}+\frac{5(x-2)}{(x+1)(x-2)}$$
$$=\frac{x^2+2x+1}{(x+1)(x-2)}+\frac{5x-10}{(x+1)(x-2)}$$
$$=\frac{x^2+7x-9}{(x+1)(x-2)}$$

It is only possible to cancel the $(x+1)$ term in this way when multiplying expressions.

Activity 3.9

1.

(a) $7 \times 7 \times 7$
$= 7^3$

(b) $\sqrt[4]{16}$
$= 16^{\frac{1}{4}}$

(c) $\dfrac{1}{4 \times 4 \times 4 \times 4}$
$= 4^{-4}$

(d) $-3 \times 3 \times 3 \times 3 \times 3$
$= -3^5$

(e) $(-3) \times (-3) \times (-3) \times (-3) \times (-3)$
$= (-3)^5$

(f) $\dfrac{1}{3 \times 3 \times 3 \times 3 \times 3}$
$= 3^{-5}$

(g) $\sqrt{16} \times \sqrt{16} \times \sqrt{16}$
$= 16^{\frac{3}{2}}$

(h) $-\dfrac{1}{2 \times 2 \times 2 \times 2}$
$= -2^{-4}$

2.

(a) 8^3 eight to the power three or eight cubed or three numbers which are all eight multiplied together

(b) -4^2 the negative of four squared

(c) $(-4)^2$ negative four to the power two or negative four squared or two numbers which are both negative four multiplied together

(d) $512^{\frac{1}{3}}$ the cubed root of 512

(e) 7^{-4} the reciprocal of seven to the power four or one over seven to the power four

Activity 3.10

1.

(a) $x^{\frac{3}{5}} \times x^{\frac{4}{5}}$

$= x^{\frac{3}{5}+\frac{4}{5}}$

$= x^{\frac{7}{5}}$

(b) $(x^4)^{\frac{n}{2}}$

$= x^{4\times\frac{n}{2}}$

$= x^{2n}$

(c) $(a^{\frac{3}{m}} b^{\frac{2}{n}})^{mn}$

$= a^{\frac{3}{m}\times mn} b^{\frac{2}{n}\times mn}$

$= a^{3n} b^{2m}$

(d) $\dfrac{x^2}{x^{\frac{1}{2}}}$

$= x^{2-\frac{1}{2}}$

$= x^{\frac{3}{2}}$

(e) $(a^2 b)^0$

$= 1$

2.

(a) $a^{-3} \times a^{-5}$

$= a^{-3+-5}$

$= a^{-8}$

$= \dfrac{1}{a^8}$

(b) $\dfrac{1}{z^{-n}}$

$= \dfrac{1}{\frac{1}{z^n}}$

$= z^n$

(c) $-x^{-n}$

$= -\frac{1}{x^n}$

(d) $\frac{a^{-3}}{a}$

$= a^{-3-1}$

$= a^{-4}$

$= \frac{1}{a^4}$

(e) $x^n \times x^{-2n}$

$= x^{n+-2n}$

$= x^{-n}$

$= \frac{1}{x^n}$

3.

(a) $\frac{8x^8y^2}{x^2y^4}$

$= 8x^{8-2}y^{2-4}$

$= 8x^6y^{-2}$

$= \frac{8x^6}{y^2}$

(b) $\frac{-36x^6y^{-1}}{-3x^{-3}y^2}$

$= 12x^{6--3}y^{-1-2}$

$= 12x^9y^{-3}$

$= \frac{12x^9}{y^3}$

(c) $\frac{16(s^2t)^2}{4s^{-3}t^2}$

$= \frac{16s^{2\times2}t^2}{4s^{-3}t^2}$

$= 4s^{4--3}t^{2-2}$

$= 4s^7$

(d) $(5^3x^{-2}y^{-4})^0$

$= 1$

(e) $\left(a^{\frac{5}{2}} \div b^{\frac{2}{3}}\right)^6$

$= a^{\frac{5}{2}\times 6} \div b^{\frac{2}{3}\times 6}$

$= a^{15} \div b^4$

$= \dfrac{a^{15}}{b^4}$

(f) $\left(\dfrac{a^3 b^{-2}}{b^3 c^{-1}}\right)^{-3}$

$= \left(a^3 b^{-2-3} c^1\right)^{-3}$

$= \left(a^3 b^{-5} c^1\right)^{-3}$

$= a^{3\times -3} b^{-5\times -3} c^{1\times -3}$

$= a^{-9} b^{15} c^{-3}$

$= \dfrac{b^{15}}{a^9 c^3}$

(g) $(16x^4 y^2)^{\frac{1}{4}} (x^3 y^6 z^9)^{\frac{1}{6}}$

$= (16^{\frac{1}{4}} x^{4\times\frac{1}{4}} y^{2\times\frac{1}{4}})(x^{3\times\frac{1}{6}} y^{6\times\frac{1}{6}} z^{9\times\frac{1}{6}})$

$= 2xy^{\frac{1}{2}} x^{\frac{1}{2}} y^1 z^{\frac{3}{2}}$

$= 2x^{\frac{3}{2}} y^{\frac{3}{2}} z^{\frac{3}{2}}$

(h) $\dfrac{(36a^4 b^2)^{\frac{3}{2}}}{(64a^6 b^3)^{\frac{2}{3}}}$

$= \dfrac{36^{\frac{3}{2}} a^{4\times\frac{3}{2}} b^{2\times\frac{3}{2}}}{64^{\frac{2}{3}} a^{6\times\frac{2}{3}} b^{3\times\frac{2}{3}}}$

$= \dfrac{216a^6 b^3}{16a^4 b^2}$

$= \dfrac{27a^{6-4} b^{3-2}}{2}$

$= \dfrac{27a^2 b}{2}$

Activity 3.11

1.

(a)
$$11m - 4 = 2 - 3m$$
$$11m + 3m - 4 = 2 - 3m + 3m$$
$$14m - 4 = 2$$
$$14m - 4 + 4 = 2 + 4$$
$$14m = 6$$
$$\frac{14m}{14} = \frac{6}{14}$$
$$m = \frac{3}{7}$$

Check: $LHS = 11(\frac{3}{7}) - 4 = \frac{5}{7}, RHS = 2 - 3(\frac{3}{7}) = \frac{5}{7} = LHS$

(b)
$$38 = 14 - \frac{6x}{11}$$
$$38 - 14 = 14 - 14 - \frac{6x}{11}$$
$$24 = -\frac{6x}{11}$$
$$24 \times 11 = -\frac{6x}{11} \times 11$$
$$264 = -6x$$
$$\frac{264}{-6} = \frac{-6x}{-6}$$
$$-44 = x$$
$$x = -44$$

Check: $RHS = 14 - \frac{6(-44)}{11} = 14 - -24 = 38 = LHS$

(c)
$$3(2x - 1) = x + 4$$
$$6x - 3 = x + 4$$
$$6x - 3 + 3 = x + 4 + 3$$
$$6x = x + 7$$
$$6x - x = x - x + 7$$
$$5x = 7$$
$$\frac{5x}{5} = \frac{7}{5}$$
$$x = \frac{7}{5}$$

Check: $LHS = 3(2(\frac{7}{5}) - 1) = 3(\frac{14}{5} - 1) = 3 \times \frac{9}{5} = \frac{27}{5} = 5\frac{2}{5}, RHS = \frac{7}{5} + 4 = 5\frac{2}{5} = LHS$

(d) $5b = 4 - 2(b + 1)$

$$5b = 4 - 2b - 2$$
$$5b = 2 - 2b$$
$$5b + 2b = 2 - 2b + 2b$$
$$7b = 2$$
$$\frac{7b}{7} = \frac{2}{7}$$
$$b = \frac{2}{7}$$

Check: $LHS = 5(\frac{2}{7}) = \frac{10}{7}, RHS = 4 - 2(\frac{2}{7} + 1) = 4 - \frac{4}{7} - 2 = \frac{10}{7} = LHS$

(e) $\frac{3r - 4}{10} = 3$

$$10 \times \frac{3r - 4}{10} = 3 \times 10$$
$$3r - 4 = 30$$
$$3r - 4 + 4 = 30 + 4$$
$$3r = 34$$
$$r = \frac{34}{3} = 11\frac{1}{3}$$

Check: $LHS = \frac{3(11\frac{1}{3}) - 4}{10} = \frac{34 - 4}{10} = \frac{30}{10} = 3 = RHS$

2.

(i) Complete this solution.

$$3x - 4 = 3(2x - 2)$$
$$3x - 4 = 6x - 6$$
$$3x - 3x - 4 = 6x - 3x - 6$$
$$-4 = 3x - 6$$
$$-4 + 6 = 3x - 6 + 6$$
$$2 = 3x$$
$$\frac{2}{3} = \frac{3x}{3}$$
$$x = \frac{2}{3}$$

(ii) Complete the solution.

$$\frac{x}{3}-\frac{x}{3}-\frac{4}{9}=\frac{2x}{3}-\frac{x}{3}-\frac{2}{3}$$
$$-\frac{4}{9}=\frac{x}{3}-\frac{2}{3}$$
$$-\frac{4}{9}+\frac{2}{3}=\frac{x}{3}$$
$$-\frac{4}{9}+\frac{6}{9}=\frac{x}{3}$$
$$3\times\frac{2}{9}=\frac{x}{3}\times 3$$
$$x=\frac{2}{3}$$

3.

(a) $\dfrac{4x+3}{10}=-\dfrac{2x}{5}$

$$10\times\frac{4x+3}{10}=-\frac{2x}{5}\times 10$$
$$4x+3=-2x\times 2$$
$$4x-4x+3=-4x-4x$$
$$3=-8x$$
$$-\frac{3}{8}=\frac{-8x}{-8}$$
$$x=-\frac{3}{8}$$

(b) $\dfrac{x}{3}+\dfrac{x+1}{4}=2$

$$\frac{x}{3}\times 12+\frac{x+1}{4}\times 12=2\times 12$$
$$4x+(x+1)3=24$$
$$4x+3x+3=24$$
$$7x+3-3=24-3$$
$$7x=21$$
$$\frac{7x}{7}=\frac{21}{7}$$
$$x=3$$

(c) $\frac{3x-1}{4}-\frac{5-2x}{6}=1-\frac{3x-5}{2}$

$$\frac{3x-1}{4}\times 12-\frac{5-2x}{6}\times 12=1\times 12-\frac{3x-5}{2}\times 12$$
$$(3x-1)\times 3-(5-2x)\times 2=12-(3x-5)\times 6$$
$$9x-3-10+4x=12-18x+30$$
$$13x-13=42-18x$$
$$13x-13+13=42+13-18x$$
$$13x=55-18x$$
$$13x+18x=55-18x+18x$$
$$\frac{31x}{31}=\frac{55}{31}$$
$$x=\frac{55}{31}$$

(d) $\frac{5}{x}+\frac{3}{2x}=2$

$$\frac{5}{x}\times 2x+\frac{3}{2x}\times 2x=2\times 2x$$
$$10+3=4x$$
$$13=4x$$
$$\frac{13}{4}=\frac{4x}{4}$$
$$x=\frac{13}{4}$$

(e) $\frac{x-2}{x+3}=\frac{3}{5}$

$$\frac{x-2}{x+3}\times 5(x+3)=\frac{3}{5}\times 5(x+3)$$
$$(x-2)5=3(x+3)$$
$$5x-10=3x+9$$
$$5x-3x-10=3x-3x+9$$
$$2x-10=9$$
$$2x-10+10=9+10$$
$$2x=19$$
$$\frac{2x}{2}=\frac{19}{2}$$
$$x=\frac{19}{2}=9\frac{1}{2}$$

4.

(a) $\frac{2x-1}{3}-5=\frac{x}{6}$

$$\frac{2x-1}{3}\times 6-5\times 6=\frac{x}{6}\times 6$$
$$(2x-1)\times 2-30=x$$
$$4x-2-30=x$$
$$4x-32=x$$
$$4x-32+32=x+32$$
$$4x=x+32$$
$$4x-x=x-x+32$$
$$3x=32$$
$$\frac{3x}{3}=\frac{32}{3}$$
$$x=10\frac{2}{3}$$

(b) $\frac{8}{x}=\frac{2}{3}$

$$\frac{8}{x}\times 3x=\frac{2}{3}\times 3x$$
$$8\times 3=2x$$
$$24=2x$$
$$\frac{24}{2}=\frac{2x}{2}$$
$$x=12$$

(c) $\frac{2x-1}{8}=\frac{3x+1}{4}$

$$\frac{2x-1}{8}\times 8=\frac{3x+1}{4}\times 8$$
$$2x-1=(3x+1)\times 2$$
$$2x-1=6x+2$$
$$2x-2x-1=6x-2x+2$$
$$-1=4x+2$$
$$-1-2=4x+2-2$$
$$-3=4x$$
$$-\frac{3}{4}=\frac{4x}{4}$$
$$x=-\frac{3}{4}$$

(d) $\frac{5}{x-4}=\frac{2}{x-2}$

$$\begin{aligned}
\frac{5}{x-4}\times(x-4)(x-2) &= \frac{2}{x-2}\times(x-4)(x-2)\\
5(x-2) &= 2(x-4)\\
5x-10 &= 2x-8\\
5x-2x-10 &= 2x-2x-8\\
3x-10 &= -8\\
3x-10+10 &= -8+10\\
3x &= 2\\
\frac{3x}{3} &= \frac{2}{3}\\
x &= \frac{2}{3}
\end{aligned}$$

(e) $\frac{1}{x+2}+\frac{1}{x-3}=\frac{1}{(x+2)(x-3)}$

$$\begin{aligned}
\frac{1}{x+2}\times(x+2)(x-3)+\frac{1}{x-3}\times(x+2)(x-3) &= \frac{1}{(x+2)(x-3)}\times(x+2)(x-3)\\
1\times(x-3)+1\times(x+2) &= 1\\
x-3+x+2 &= 1\\
2x-1 &= 1\\
2x-1+1 &= 1+1\\
2x &= 2\\
\frac{2x}{2} &= \frac{2}{2}\\
x &= 1
\end{aligned}$$

5.

(a) Let x be the unknown number.
$3+x=15-x$

(b) Let x be the length of the longer piece of rope in centimetres.
$x+(x-15)=85$

(c) Let r be the number of cans of red paint.
$r+(r-4)=18$

(d) Let x be the number of pens bought.
$x+\frac{x}{5}=24$

(e) Let w be the width of the rectangle.
Recall, perimeter of a rectangle = $2l+2w$
$96=2\times25+2w$
$96=50+2w$

(f) Given x is the original speed in km/h
Recall, distance = speed × time
$x \times 0.5 + (x+20) \times 1 = 240$

(g) Let x be the greater amount of money invested in dollars.
$0.08x + 0.12(x-150) = 60$

(h) Let f be the amount the first person receives.
$$f + \frac{f}{2} + \frac{f}{3} = 22000$$

6.

(a)
$$\begin{aligned} 3 + x &= 15 - x \\ x + x &= 15 - 3 \\ 2x &= 12 \\ x &= 6 \end{aligned}$$

Unknown number is 6.

(b)
$$\begin{aligned} x + (x - 15) &= 85 \\ x + x - 15 &= 85 \\ 2x &= 100 \\ x &= 50 \end{aligned}$$

Length of large piece of rope is 50cm.

(c)
$$\begin{aligned} r + (r - 4) &= 18 \\ 2r &= 18 + 4 \\ r &= 11 \end{aligned}$$

Numbered red paint can is 11.

(d)
$$\begin{aligned} x + \frac{x}{5} &= 24 \\ 5x + x &= 24 \times 5 \\ 6x &= 120 \\ x &= 20 \end{aligned}$$

Number of pens is 20.

(e)
$$\begin{aligned} 96 &= 50 + 2w \\ 2w &= 46 \\ w &= 23 \end{aligned}$$

Width of rectangle is 23cms.

7.

(a) Let x be the number of metres of fencing purchased.

$$3x = 186$$

$$\frac{3x}{3} = \frac{186}{3}$$

$$x = 62$$

Therefore 62 metres of fencing was purchased.

Check: $LHS = 3(62) = 186 = RHS$

(b) Let w be the original width of the frame in metres.

$$A = lw$$

Original area is $A = 3w \times w$

Adjusted frame area $A = (3w-2)\times(w+4)$

$$A = 3w(w+4) - 2(w+4)$$

$$A = 3w^2 + 12w - 2w - 8$$

$$A = 3w^2 + 10w - 8$$

These two areas are equal therefore,

$$3w^2 = 3w^2 + 10w - 8$$

$$3w^2 - 3w^2 = 3w^2 - 3w^2 + 10w - 8$$

$$0 = 10w - 8$$

$$0 + 8 = 10w - 8 + 8$$

$$8 = 10w$$

$$\frac{8}{10} = \frac{10w}{10}$$

$$w = 0.8$$

Therefore the original width of the frame is 0.8 metres and the length is 2.4 metres.

Check: $LHS = 3(0.8)^2 = 1.92, RHS = 3(0.8)^2 + 10(0.8) - 8 = 1.92 = LHS$

Activity 3.12

1.

(a) $V = LWH$

$$\frac{V}{LW} = \frac{LWH}{LW}$$

$$H = \frac{V}{LW}$$

(b) $v^2 = u^2 + 2as$

$$v^2 - 2as = u^2 + 2as - 2as$$

$$v^2 - 2as = u^2$$

$$\pm\sqrt{v^2 - 2as} = \pm\sqrt{u^2}$$

$$u = \pm\sqrt{v^2 - 2as}$$

(c) $v = \frac{4}{3}\pi r^3$

$$\frac{3}{4}v = \frac{3}{4} \times \frac{4}{3}\pi r^3$$

$$\frac{3}{4}v = \pi r^3$$

$$\frac{3}{4}\frac{v}{\pi} = \frac{\pi r^3}{\pi}$$

$$\frac{3v}{4\pi} = r^3$$

$$\sqrt[3]{\frac{3v}{4\pi}} = \sqrt[3]{r^3}$$

$$r = \sqrt[3]{\frac{3v}{4\pi}}$$

(d) $v = u + at$

$$v - u = u - u + at$$

$$v - u = at$$

$$\frac{v-u}{a} = \frac{at}{a}$$

$$t = \frac{v-u}{a}$$

(e) $s = ut + \frac{1}{2}at^2$

$$s - ut = ut - ut + \frac{1}{2}at^2$$

$$s - ut = \frac{1}{2}at^2$$

$$2(s - ut) = 2 \times \frac{1}{2}at^2$$

$$2(s - ut) = at^2$$

$$\frac{2(s-ut)}{t^2} = \frac{at^2}{t^2}$$

$$a = \frac{2(s-ut)}{t^2}$$

(f) $s=\frac{1}{2}\left(\frac{Ee}{m}\right)\left(\frac{l}{v}\right)^2$

$$2\times s=2\times\frac{1}{2}\left(\frac{Ee}{m}\right)\left(\frac{l}{v}\right)^2$$

$$2s=\left(\frac{Ee}{m}\right)\left(\frac{l}{v}\right)^2$$

$$2s\times\left(\frac{v}{l}\right)^2=\left(\frac{Ee}{m}\right)\left(\frac{l}{v}\right)^2\times\left(\frac{v}{l}\right)^2$$

$$2s\left(\frac{v}{l}\right)^2=\frac{Ee}{m}$$

$$2s\left(\frac{v}{l}\right)^2\times\frac{m}{E}=\frac{Ee}{m}\times\frac{m}{E}$$

$$e=\frac{2sm}{E}\left(\frac{v}{l}\right)^2$$

(g) $P=(1+r)^n$

$$\sqrt[n]{P}=\sqrt[n]{(1+r)^n}$$

$$\sqrt[n]{P}=1+r$$

$$\sqrt[n]{P}-1=1-1+r$$

$$r=\sqrt[n]{P}-1$$

(h) $\frac{1}{\lambda}=R\left(\frac{1}{m^2}-\frac{1}{n^2}\right)$

$$\lambda\times\frac{1}{\lambda}=R\left(\frac{1}{m^2}\times\frac{n^2}{n^2}-\frac{1}{n^2}\times\frac{m^2}{m^2}\right)\times\lambda$$

$$1=R\left(\frac{n^2}{m^2n^2}-\frac{m^2}{m^2n^2}\right)\lambda$$

$$\frac{1}{R}=\frac{R}{R}\left(\frac{n^2-m^2}{m^2n^2}\right)\lambda$$

$$\frac{1}{R}=\frac{(n^2-m^2)}{(m^2n^2)}\times\lambda$$

$$\frac{1}{R}\times\frac{(m^2n^2)}{(n^2-m^2)}=\frac{(n^2-m^2)}{(m^2n^2)}\times\frac{(m^2n^2)}{(n^2-m^2)}\times\lambda$$

$$\lambda=\frac{m^2n^2}{R(n^2-m^2)}$$

(i) $T = 2\pi\sqrt{\frac{m}{k}}$

$$\frac{T}{2\pi} = \frac{2\pi}{2\pi}\sqrt{\frac{m}{k}}$$

$$\frac{T}{2\pi} = \sqrt{\frac{m}{k}}$$

$$\left(\frac{T}{2\pi}\right)^2 = \left(\sqrt{\frac{m}{k}}\right)^2$$

$$\left(\frac{T}{2\pi}\right)^2 = \frac{m}{k}$$

$$k \times \left(\frac{T}{2\pi}\right)^2 = \frac{m}{k} \times k$$

$$k\left(\frac{T}{2\pi}\right)^2 = m$$

$$k\left(\frac{T}{2\pi}\right)^2 \times \left(\frac{2\pi}{T}\right)^2 = m\left(\frac{2\pi}{T}\right)^2$$

$$k = m\left(\frac{2\pi}{T}\right)^2$$

(j) $m=\dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$

$$m\times\sqrt{1-\frac{v^2}{c^2}}=\frac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}\times\sqrt{1-\frac{v^2}{c^2}}$$

$$m\sqrt{1-\frac{v^2}{c^2}}=m_0$$

$$\frac{m}{m}\sqrt{1-\frac{v^2}{c^2}}=\frac{m_0}{m}$$

$$\sqrt{1-\frac{v^2}{c^2}}=\frac{m_0}{m}$$

$$\left(\sqrt{1-\frac{v^2}{c^2}}\right)^2=\left(\frac{m_0}{m}\right)^2$$

$$1-\frac{v^2}{c^2}=\left(\frac{m_0}{m}\right)^2$$

$$1-\frac{v^2}{c^2}+\frac{v^2}{c^2}=\left(\frac{m_0}{m}\right)^2+\frac{v^2}{c^2}$$

$$1=\left(\frac{m_0}{m}\right)^2+\frac{v^2}{c^2}$$

$$1-\left(\frac{m_0}{m}\right)^2=\left(\frac{m_0}{m}\right)^2-\left(\frac{m_0}{m}\right)^2+\frac{v^2}{c^2}$$

$$1-\left(\frac{m_0}{m}\right)^2=\frac{v^2}{c^2}$$

$$\left[1-\left(\frac{m_0}{m}\right)^2\right]\times c^2=\frac{v^2}{c^2}\times c^2$$

$$\left[1-\left(\frac{m_0}{m}\right)^2\right]c^2=v^2$$

$$\pm\sqrt{c^2\left[1-\left(\frac{m_0}{m}\right)^2\right]}=\pm\sqrt{v^2}$$

$$v=\pm\sqrt{c^2\left[1-\left(\frac{m_0}{m}\right)^2\right]}$$

2.

(a)

$$A = P(1-i)^n$$

$$\frac{A}{P} = \frac{P}{P}(1-i)^n$$

$$\frac{A}{P} = (1-i)^n$$

$$\sqrt[n]{\frac{A}{P}} = \sqrt[n]{(1-i)^n}$$

$$\sqrt[n]{\frac{A}{P}} = 1-i$$

$$\sqrt[n]{\frac{A}{P}} + i = 1 - i + i$$

$$\sqrt[n]{\frac{A}{P}} + i = 1$$

$$\sqrt[n]{\frac{A}{P}} - \sqrt[n]{\frac{A}{P}} + i = 1 - \sqrt[n]{\frac{A}{P}}$$

$$i = 1 - \sqrt[n]{\frac{A}{P}}$$

(b) Substituting the known variables,

$$i = 1 - \sqrt[n]{\frac{A}{P}}$$

$$i = 1 - \sqrt[3]{\frac{1500}{2800}}$$

$$i \approx 1 - 0.8122$$

$$i \approx 0.1878$$

Therefore the depreciation rate is approximately 18.78%.

Check: $RHS = P(1-i)^n \approx 2800(1-0.1878)^3 \approx 1500 \approx LHS$

3.

(i)

$$a = \frac{h\sqrt{a^2+b^2}}{b}$$

$$ab = \frac{h\sqrt{a^2+b^2}}{b} \times b$$

$$ab = h\sqrt{a^2+b^2}$$

$$\frac{ab}{\sqrt{a^2+b^2}} = h\frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}}$$

$$h = \frac{ab}{\sqrt{a^2+b^2}}$$

(ii) Substituting the known variables,

$$h = \frac{ab}{\sqrt{a^2 + b^2}}$$

$$h = \frac{12 \times 5}{\sqrt{12^2 + 5^2}}$$

$$h = \frac{60}{\sqrt{169}}$$

$$h = \frac{60}{13}$$

$$h = 4\frac{8}{13}$$

Check: $RHS = \dfrac{h\sqrt{a^2 + b^2}}{b} = \dfrac{4\frac{8}{13}\sqrt{12^2 + 5^2}}{5} = 12 = LHS$

4.

(i)

$$N = \frac{147}{11}\sqrt{\frac{5}{r}}$$

$$N \times \frac{11}{147} = \frac{147}{11} \times \frac{11}{147}\sqrt{\frac{5}{r}}$$

$$\frac{11N}{147} = \sqrt{\frac{5}{r}}$$

$$\left(\frac{11N}{147}\right)^2 = \left(\sqrt{\frac{5}{r}}\right)^2$$

$$\left(\frac{11N}{147}\right)^2 = \frac{5}{r}$$

$$r \times \left(\frac{11N}{147}\right)^2 = \frac{5}{r} \times r$$

$$r\left(\frac{11N}{147}\right)^2 = 5$$

$$r\left(\frac{11N}{147}\right)^2 \times \left(\frac{147}{11N}\right)^2 = 5\left(\frac{147}{11N}\right)^2$$

$$r = 5\left(\frac{147}{11N}\right)^2$$

(ii) Substituting known variables,

$$r = 5\left(\frac{147}{11N}\right)^2$$
$$r = 5\left(\frac{147}{11\times 3}\right)^2$$
$$r \approx 99.2149$$

Check: $RHS = \frac{147}{11}\sqrt{\frac{5}{r}} \approx \frac{147}{11}\sqrt{\frac{5}{99.2149}} \approx 3 \approx LHS$

5.

(i) $$F = \frac{mv^2}{r}$$
$$F \times r = \frac{mv^2}{r} \times r$$
$$Fr = mv^2$$
$$\frac{Fr}{F} = \frac{mv^2}{F}$$
$$r = \frac{mv^2}{F}$$

(ii) Substituting known values,

$$r = \frac{mv^2}{F}$$
$$r = \frac{0.05 \times 8^2}{10}$$
$$r = 0.32$$

Therefore the string is 0.32 m long.

Check: $RHS = \frac{mv^2}{r} = \frac{0.05 \times 8^2}{0.32} = 10 = LHS$

Activity 3.13

1.

(a) $\frac{x}{3}+2<5$

$$\frac{x}{3}+2-2<5-2$$

$$\frac{x}{3}<3$$

$$\frac{x}{3}\times 3<3\times 3$$

$$x<9$$

(b) $22\le 5x-3\le 32$

$$22+3\le 5x-3+3\le 32+3$$

$$25\le 5x\le 35$$

$$\frac{25}{5}\le\frac{5x}{5}\le\frac{35}{5}$$

$$5\le x\le 7$$

(c) $\frac{3x}{5}-\frac{2x}{3}>-7$

$$15\times\frac{3x}{5}-15\times\frac{2x}{3}>-7\times 15$$

$$3\times 3x-5\times 2x>-105$$

$$9x-10x>-105$$

$$-x>-105$$

$$-1\times -x<-105\times -1$$

$$x<105$$

(d) $3x-2(2x-7)\le 2(3+x)-4$

$$3x-4x+14\le 6+2x-4$$

$$-x+14\le 2+2x$$

$$-x+x+14\le 2+2x+x$$

$$14\le 2+3x$$

$$14-2\le 2-2+3x$$

$$12\le 3x$$

$$\frac{12}{3}\le\frac{3x}{3}$$

$$x\ge 4$$

(e) $-3 \le \frac{2x-1}{3} < 3$

$$-3\times 3 \le \frac{2x-1}{3}\times 3 < 3\times 3$$

$$-9 \le 2x-1 < 9$$

$$-9+1 \le 2x-1+1 < 9+1$$

$$-8 \le 2x < 10$$

$$-\frac{8}{2} \le \frac{2x}{2} < \frac{10}{2}$$

$$-4 \le x < 5$$

2.

(a) Solve $6-3(2y-3)=4(2y-2)$

$$6-6y+9=8y-8$$

$$15-6y=8y-8$$

$$15-6y+6y=8y+6y-8$$

$$15=14y-8$$

$$15+8=14y$$

$$23=14y$$

$$\frac{23}{14}=\frac{14y}{14}$$

$$y=1\frac{9}{14}$$

(b) Solve $6-3(2y-3)\le 4(2y-2)$

$$6-6y+9 \le 8y-8$$

$$15-6y \le 8y-8$$

$$15-6y+6y \le 8y+6y-8$$

$$15 \le 14y-8$$

$$15+8 \le 14y-8+8$$

$$23 \le 14y$$

$$\frac{23}{14} \le \frac{14y}{14}$$

$$y \ge 1\frac{9}{14}$$

(c) Solution (b) includes $1\frac{9}{14}$ and all values greater than this.

3.

(a) Let w be the number of women on the team.

$$9 \le w - 5 + w \le 15$$
$$9 \le 2w - 5 \le 15$$
$$9 + 5 \le 2w - 5 + 5 \le 15 + 5$$
$$14 \le 2w \le 20$$
$$\frac{14}{2} \le \frac{2w}{2} \le \frac{20}{2}$$
$$7 \le w \le 10$$

Therefore, there may be 7, 8, 9 or 10 women on the team.

(b) Let x be the value of the smaller integer.

$$x + x + 1 < 13$$
$$2x + 1 < 13$$
$$2x + 1 - 1 < 13 - 1$$
$$2x < 12$$
$$\frac{2x}{2} < \frac{12}{2}$$
$$x < 6$$

Therefore the small integer may have a value of 1, 2, 3, 4 or 5.

(c) Let s be the number of sausage rolls heated.

$$60 \le s + s - 10 \le 100$$
$$60 \le 2s - 10 \le 100$$
$$60 + 10 \le 2s - 10 + 10 \le 100 + 10$$
$$70 \le 2s \le 110$$
$$\frac{70}{2} \le \frac{2s}{2} \le \frac{110}{2}$$
$$35 \le s \le 55$$

Therefore the possible number of sausage rolls that should be heated ranges from 35 to 55.

Activity 3.14

1.

(a) $|x| \le 3$

$$-3 \le x \le 3$$

(b) $|2x| > 8$

$$2x > 8 \qquad\qquad 2x < -8$$
$$\frac{2x}{2} > \frac{8}{2} \quad \text{or} \quad \frac{2x}{2} < \frac{-8}{2}$$
$$x > 4 \qquad\qquad x < -4$$

(c) $|x-4| \le 1$

$$-1 \le x-4 \le 1$$
$$3 \le x \le 5$$

(d) $|8-4x| > 0$

$$8-4x > 0 \qquad\qquad 8-4x < 0$$
$$8-4x+4x > 0+4x \qquad\qquad 8-4x+4x < 0+4x$$
$$8 > 4x \qquad\qquad 8 < 4x$$
$$\frac{8}{4} > \frac{4x}{4} \qquad \text{or} \qquad \frac{8}{4} < \frac{4x}{4}$$
$$2 > x \qquad\qquad 2 < x$$
$$x < 2 \qquad\qquad x > 2$$

(e) $|3x-4| < 8$

$$-8 < 3x-4 < 8$$
$$-8+4 < 3x-4+4 < 8+4$$
$$-4 < 3x < 12$$
$$-\frac{4}{3} < \frac{3x}{3} < \frac{12}{3}$$
$$-\frac{4}{3} < x < 4$$

(f) $|2-(x+1)| \le 3$

$$-3 \le 2-(x+1) \le 3$$
$$-3 \le 2-x-1 \le 3$$
$$-3 \le 1-x \le 3$$
$$-3-1 \le 1-1-x \le 3-1$$
$$-4 \le -x \le 2$$
$$-4 \times -1 \ge -x \times -1 \ge 2 \times -1$$
$$4 \ge x \ge -2$$
$$-2 \le x \le 4$$

(g) $|x-1| > 2$

$$x-1 > 2 \qquad\qquad x-1 < -2$$
$$x-1+1 > 2+1 \qquad \text{or} \qquad x-1+1 < -2+1$$
$$x > 3 \qquad\qquad x < -1$$

Activity 3.15

1.

(a) $x^2 + 5x + 4 = 0$

$x^2 + (1+4)x + 4 = 0$

$x^2 + x + 4x + 4 = 0$

$x(x+1) + 4(x+1) = 0$

$(x+4)(x+1) = 0$

$x + 4 = 0, x + 1 = 0$

$x = -4$, or $x = -1$

Check: $x = -4, LHS = (-4)^2 + 5 \times (-4) + 4 = 0 = RHS$

$x = -1, LHS = (-1)^2 + 5 \times (-1) + 4 = 0 = RHS$

Solutions are $x = -4$ or $x = -1$.

(b) $x^2 - 7x - 30 = 0$

$x^2 + (3-10)x - 30 = 0$

$x^2 + 3x - 10x - 30 = 0$

$x(x+3) - 10(x+3) = 0$

$(x-10)(x+3) = 0$

$x - 10 = 0, x + 3 = 0$

$x = 10$, or $x = -3$

Check: $x = 10, LHS = (10)^2 - 7(10) - 30 = 0 = RHS$

$x = -3, LHS = (-3)^2 - 7(-3) - 30 = 0 = RHS$

Solutions are $x = 10$ or $x = -3$

(c) $v^2 = 4(v + 24)$

$v^2 = 4v + 96$

$v^2 - 4v - 96 = 0$

$v^2 + (8-12)v - 96 = 0$

$v^2 + 8v - 12v - 96 = 0$

$v(v+8) - 12(v+8) = 0$

$(v-12)(v+8) = 0$

$v - 12 = 0, v + 8 = 0$

$v = 12$, or $v = -8$

Check: $v = 12, RHS = 4(12 + 24) = 144 = LHS$

$v = -8, RHS = 4(-8 + 24) = 64 = LHS$

(d) $x^2 = \frac{3-5x}{2}$

$$2 \times x^2 = \frac{3-5x}{2} \times 2$$

$$2x^2 = 3 - 5x$$

$$2x^2 + 5x - 3 = 0$$

$$2x^2 + (6-1)x - 3 = 0$$

$$2x^2 + 6x - x - 3 = 0$$

$$2x(x+3) - 1(x+3) = 0$$

$$(2x-1)(x+3) = 0$$

$$2x - 1 = 0, x + 3 = 0$$

$$x = \frac{1}{2}, \text{ or } x = -3$$

Check: $x = \frac{1}{2}, RHS = \frac{3-5(\frac{1}{2})}{2} = \frac{\frac{1}{2}}{2} = \frac{1}{4} = LHS$

$$x = -3, RHS = \frac{3-5(-3)}{2} = \frac{18}{2} = 9 = LHS$$

Solutions are $x = \frac{1}{2}$ or $x = -3$

(e)

$$(6x+2)(x-4) = 2 - 11x$$

$$6x(x-4) + 2(x-4) = 2 - 11x$$

$$6x^2 - 24x + 2x - 8 = 2 - 11x$$

$$6x^2 - 22x - 8 = 2 - 11x$$

$$6x^2 - 22x + 11x - 8 - 2 = 2 - 2 - 11x + 11x$$

$$6x^2 - 11x - 10 = 0$$

$$6x^2 + (-15 + 4)x - 10 = 0$$

$$6x^2 - 15x + 4x - 10 = 0$$

$$3x(2x-5) + 2(2x-5) = 0$$

$$(3x+2)(2x-5) = 0$$

$$3x + 2 = 0, 2x - 5 = 0$$

$$x = \frac{-2}{3} \text{ or } x = \frac{5}{2}$$

Check:

$$x = \frac{-2}{3}, LHS = \left(6 < \frac{-2}{3}\right) + 2)\left(\frac{-2}{3} - 4\right)$$

$$= (-4 + 2)\left(-4\frac{2}{3}\right) = 9\frac{1}{3}$$

$$RHS = 2 - 11\left(\frac{-2}{3}\right) = 9\frac{1}{3} = LHS$$

$$x = \frac{5}{2}, LHS = \left(6\frac{5}{2} + 2\right)\left(\frac{5}{2} - 4\right)$$

$$= 17 \times -1\frac{1}{2} = -25\frac{1}{2}$$

$$RHS = 2 - 11\left(\frac{5}{2}\right) = -25\frac{1}{2} = LHS$$

Solutions are $x = \frac{-2}{3}$ or $x = \frac{5}{2}$

2.

(a) $x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0, x - 3 = 0$

$x = 0$, or $x = 3$

Check: $x = 0, LHS = 0 - 0 = 0 = RHS$

$x = 3, LHS = 3^2 - 3 \times 3 = 0 = RHS$

(b) $6x^2 = 24x$

$x^2 = 4x$

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0, x - 4 = 0$

$x = 0$, or $x = 4$

Check: $x = 0, LHS = 6(0)^2 = 0, RHS = 24(0) = 0 = LHS$

$x = 4, LHS = 6(4)^2 = 96, RHS = 24(4) = 96 = LHS$

(c) $9x^2 = 25$

$$x^2 = \frac{25}{9}$$

$$\sqrt{x^2} = \pm\sqrt{\frac{25}{9}}$$

$$x = \pm\frac{5}{3}$$

Check: $x = \frac{5}{3}, LHS = 9\left(\frac{5}{3}\right)^2 = 25 = RHS$

$x = -\frac{5}{3}, LHS = 9\left(\frac{-5}{3}\right)^2 = 25 = RHS$

3.

(a)

$$h = 64t - 12t^2$$

$$80 = 64t - 12t^2$$

$$12t^2 - 64t + 80 = 0$$

$$3t^2 - 16t + 20 = 0$$

$$3t^2 + (-6 - 10)t + 20 = 0$$

$$3t^2 - 6t - 10t + 20 = 0$$

$$3t(t - 2) - 10(t - 2) = 0$$

$$(3t - 10)(t - 2) = 0$$

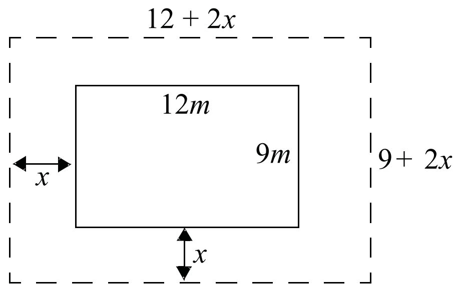
$3t - 10 = 0, t - 2 = 0$

$t = 3\frac{1}{3}$, or $t = 2$

Check: $t = 3\frac{1}{3}, RHS = 64(3\frac{1}{3}) - 12(3\frac{1}{3})^2 = 213\frac{1}{3} - 133\frac{1}{3} = 80 = LHS$

$t = 2, RHS = 64(2) - 12(2)^2 = 128 - 48 = 80 = LHS$

(b) Let x be the width of the path.
Recall Area of Rectangle = length × width
So,



$$A_{large} - A_{small} = 100m^2$$
$$(12+2x)(9+2x) - 12\times 9 = 100$$
$$12(9+2x) + 2x(9+2x) - 108 = 100$$
$$108 + 24x + 18x + 4x^2 - 108 = 100$$
$$42x + 4x^2 = 100$$
$$4x^2 + 42x - 100 = 0$$
$$2x^2 + 21x - 50 = 0$$
$$2x^2 + (25-4)x - 50 = 0$$
$$2x^2 + 25x - 4x - 50 = 0$$
$$x(2x+25) - 2(2x+25) = 0$$
$$(2x+25)(x-2) = 0$$
$$x = 2 \text{ or } x = \frac{-25}{2}$$

The only reasonable solution is the width is 2 m
Check: $x = 2, LHS = (12 + 2(2))(9 + 2(2)) - 12\times 9 = 16\times 13 - 108 = 100 = RHS$
Therefore the width of the path is 2 m.

Activity 3.16

1.

(a) $x^2 - 3x + 2 = 0$
a = 1, b = –3, c = 2
$b^2 - 4ac = (-3)^2 - 4\times 1\times 2 = 9 - 8 = 1 > 0$
Therefore, the equation will have two solutions.

(b) $x^2 + 10x + 25 = 0$
a = 1, b = 10, c = 25
$b^2 - 4ac = 10^2 - 4\times 1\times 25 = 100 - 100 = 0$
Since, $b^2 - 4ac = 0$ the equation will have one solution.

(c) $x^2 - 3x + 7 = 0$
a = 1, b = –3, c = 7
$b^2 - 4ac = (-3)^2 - 4\times 1\times 7 = 9 - 28 = -19 < 0$
Therefore, the equation will have no solutions.

(d) $4x^2 - 4x + 1 = 0$
a = 4, b = –4, c = 1
$b^2 - 4ac = (-4)^2 - 4\times 4\times 1 = 16 - 16 = 0$
Since, $b^2 - 4ac = 0$ the equation will have one solution.

(e) $x^2 + 1 = 0$

a = 1, b = 0, c = 1

$b^2 - 4ac = 0 - 4 \times 1 \times 1 = -4 < 0$

Therefore, the equation will have no solutions.

2.

(a) $4x^2 - 12x + 9 = 0$

a = 4, b = –12, c = 9

Using the formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \times 4 \times 9}}{2 \times 4}$$

$$x = \frac{12 \pm \sqrt{144 - 144}}{8}$$

$$x = \frac{12}{8} = \frac{3}{2}$$

Check: $LHS = 4(\frac{3}{2})^2 - 12(\frac{3}{2}) + 9 = 9 - 18 + 9 = 0 = RHS$

(b) $x^2 + 6x + 9 = 0$

a = 1, b = 6, c = 9

Using the formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 9}}{2 \times 1}$$

$$x = \frac{-6 \pm \sqrt{36 - 36}}{2}$$

$$x = \frac{-6}{2}$$

$$x = -3$$

Check: $LHS = (-3)^2 + 6 \times (-3) + 9 = 9 - 18 + 9 = 0 = RHS$

(c) $2q^2 + q - 6 = 0$

a = 2, b = 1, c = –6

Using the formula,

$$q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$q = \frac{-1 \pm \sqrt{1^2 - 4 \times 2 \times -6}}{2 \times 2}$$

$$q = \frac{-1 \pm \sqrt{49}}{4}$$

$$q = \frac{-1 \pm 7}{4}$$

$$q = -2, q = \frac{3}{2}$$

Check: $q = -2, LHS = 2(-2)^2 - 2 - 6 = 8 - 8 = 0 = RHS$

$$q = \frac{3}{2}, LHS = 2(\frac{3}{2})^2 + \frac{3}{2} - 6 = 4\frac{1}{2} + \frac{3}{2} - 6 = 0 = RHS$$

(d) $3x^2 - x - 2 = 0$

a = 3, b = –1, c = –2

Using the formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 3 \times -2}}{2 \times 3}$$

$$x = \frac{1 \pm \sqrt{1 + 24}}{6}$$

$$x = \frac{1 \pm 5}{6}$$

$$x = 1, x = -\frac{2}{3}$$

Check: $x = 1, LHS = 3(1)^2 - 1 - 2 = 3 - 3 = 0 = RHS$

$$x = -\frac{2}{3}, LHS = 3(-\frac{2}{3})^2 - \left(-\frac{2}{3}\right) - 2 = 2 - 2 = 0 = RHS$$

(e) $y^2 = 4y + 1$

Rearranging,

$y^2 - 4y - 1 = 0$

a = 1, b = –4, c = –1

Using the formula,

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 1 \times -1}}{2 \times 1}$$

$$y = \frac{4 \pm \sqrt{16 + 4}}{2}$$

$$y = \frac{4 \pm \sqrt{20}}{2}$$

$y \approx 4.2361, y \approx -0.2361$

Check:

$y \approx 4.2361, LHS \approx 4.2361^2 \approx 17.9445, RHS \approx 4 \times 4.2361 + 1 \approx 17.9444 \approx LHS$

$y \approx -0.2361, LHS \approx (-0.2361)^2 \approx 0.0557, RHS \approx 4 \times -0.2361 + 1 \approx 0.0556 \approx LHS$

(f) $3x^2 = 6x - 2$

Rearranging,

$3x^2 - 6x + 2 = 0$

a = 3, b = –6, c = 2

Using the formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 3 \times 2}}{2 \times 3}$$

$$x = \frac{6 \pm \sqrt{36 - 24}}{6}$$

$$x = \frac{6 \pm \sqrt{12}}{6}$$

$x \approx 1.5774, x \approx 0.4226$

Check:

$x \approx 1.5774, LHS \approx 3(1.5774)^2 \approx 7.4646, RHS \approx 6(1.5774) - 2 \approx 7.4644 \approx LHS$

$x \approx 0.4226, LHS \approx 3(0.4226)^2 \approx 0.5358, RHS \approx 6(0.4226) - 2 \approx 0.5356 \approx LHS$

(g) $9b^2 = 2 + 6b$

Rearranging,

$9b^2 - 6b - 2 = 0$

a = 9, b = –6, c = –2

Using the formula,

$$b = \frac{-\mathrm{b} \pm \sqrt{\mathrm{b}^2 - 4\mathrm{ac}}}{2\mathrm{a}}$$

$$b = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 9 \times -2}}{2 \times 9}$$

$$b = \frac{6 \pm \sqrt{36 + 72}}{18}$$

$$b = \frac{6 \pm \sqrt{108}}{18}$$

$b \approx 0.9107, b \approx -0.2440$

Check:

$b \approx 0.9107, LHS \approx 9(0.9107)^2 \approx 7.4644, RHS \approx 2 + 6(0.9107) \approx 7.4642 \approx LHS$

$b \approx -0.2440, LHS \approx 9(-0.2440)^2 \approx 0.5358, RHS \approx 2 + 6(-0.2440) \approx 0.5360 \approx LHS$

(h) $x^2 = \dfrac{3 - 5x}{2}$

Rearranging,

$$x^2 \times 2 = \frac{3 - 5x}{2} \times 2$$

$$2x^2 = 3 - 5x$$

$2x^2 + 5x - 3 = 0$

a = 2, b = 5, c = –3

Using the formula,

$$x = \frac{-\mathrm{b} \pm \sqrt{\mathrm{b}^2 - 4\mathrm{ac}}}{2\mathrm{a}}$$

$$x = \frac{-5 \pm \sqrt{5^2 - 4 \times 2 \times -3}}{2 \times 2}$$

$$x = \frac{-5 \pm \sqrt{25 + 24}}{4}$$

$$x = \frac{-5 \pm \sqrt{49}}{4}$$

$$x = \frac{-5 \pm 7}{4}$$

$$x = \frac{1}{2}, x = -3$$

Check: $x = \dfrac{1}{2}, LHS = (\dfrac{1}{2})^2 = 0.25, RHS = \dfrac{3 - 5(\frac{1}{2})}{2} = \dfrac{\frac{1}{2}}{2} = 0.25 = LHS$

$x = -3, LHS = (-3)^2 = 9, RHS = \dfrac{3 - 5(-3)}{2} = \dfrac{18}{2} = 9 = LHS$

Activity 3.17

1.

(a) Let *w* be the width of the rectangle.
Therefore the length of the rectangle = *w*+2
Substituting into the rule for area,

$A = lw$

$A = (w + 2)w$

Substituting known values,
$2 = (w + 2)w$

Rearranging,

$2 = w^2 + 2w$

$w^2 + 2w - 2 = 0$

a = 1, b = 2, c = –2

Substituting into the formula,

$w = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$w = \dfrac{-2 \pm \sqrt{2^2 - 4 \times 1 \times -2}}{2 \times 1}$

$w = \dfrac{-2 \pm \sqrt{4 + 8}}{2}$

$w = \dfrac{-2 \pm \sqrt{12}}{2}$

$w \approx 0.7321, w \approx -2.7321$

Therefore the only possible solution is the width of the rectangle is 0.7321m and length is 2.7321 m.
Check: $w \approx 0.7321, RHS \approx (0.7321 + 2)\,0.7321 \approx 2 = LHS$

(b) Let *x* be the cost of the dress.

$\dfrac{x}{100} \times x + x = 156$

Simplifying and rearranging,

$\dfrac{x^2}{100} + x = 156$

$\dfrac{x^2}{100} + x - 156 = 0$

$100(\dfrac{x^2}{100} + x - 156) = 0$

$x^2 + 100x - 15600 = 0$

a = 1, b = 100, c = −15600

Substituting into the formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-100 \pm \sqrt{100^2 - 4 \times 1 \times -15600}}{2 \times 1}$$

$$x = \frac{-100 \pm \sqrt{10000 + 62400}}{2}$$

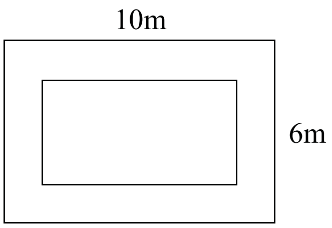
$$x = \frac{-100 \pm \sqrt{72400}}{2}$$

$x \approx 84.54, x \approx -184.54$

Therefore the only possible solution is that the dress cost \$84.54.

Check: $x \approx 84.54, LHS \approx \frac{84.54}{100} \times 84.54 + 84.54 \approx 156\ RHS$

(c)



Let x be width of walkway.
Length of available area is $10 - 2x$
Width of $6 - 2x$

So that available area is

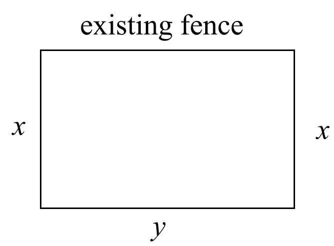
$$(10 - 2x)(6 - 2x) = 40$$
$$60 + 4x^2 - 12x - 20x = 40$$
$$4x^2 - 32x + 20 = 0$$
$$x^2 - 8x + 5 = 0$$
$$x = \frac{8 \pm \sqrt{(-8)^2 - 4 \times 1 \times 5}}{2}$$
$$x \approx \frac{8 \pm 6.63}{2}$$
$$x \approx 7.3,\ 0.69$$

Since 7.3m is not meaningful, the width must be 0.69 metres.

(d) Let x be the length of the rectangular field.
Using the perimeter, we know that

$$2x + y = 200$$

so $$y = 200 - 2x$$



Using the area of 5 000 sq.m. we know

$(200 - 2x) \times x = 5\ 000$

$200x - 2x^2 = 5\ 000$

$2x^2 - 200x + 5\ 000 = 0$

$x^2 - 100x + 2\ 500 = 0$

$$x = \frac{100 \pm \sqrt{100^2 - 4 \times 1 \times 2\ 500}}{2}$$

$x = 50$

This length of paddock is 50m, width is 100m.

(e) Let x be the length of the side of the rectangle
y be the width of the side of the rectangle.

Using the length of the wire
$2x + 2y = 40$
$2y = 40 - 2x$
$$y = \frac{40 - 2x}{2}$$

We know the area must be 100cm^2
so $xy = 100$

$$x \times \left(\frac{40 - 2x}{2}\right) = 100$$

$x(40 - 2x) = 200$

$40x - 2x^2 = 200$

$2x^2 - 40x + 200 = 0$

$x^2 - 20x + 100 = 0$

$$x = \frac{20 \pm \sqrt{20^2 - 4 \times 100}}{2}$$

$x = 10$

Dimensions of rectangle must be 10cm by 10cm (actually a square).

(f) Let x be the number of hours worked in a week before the new contract.
The hourly rate before the new contract plus \$2 gives the new hourly rate.

$$\frac{330}{x} + 2 = \frac{330 + 50}{x - 4}$$

Simplifying and rearranging,

$$x(x-4)\frac{330}{x} + x(x-4) \times 2 = \frac{380}{x-4} \times x(x-4)$$

$$330x - 1320 + 2x^2 - 8x = 380x$$

$$2x^2 - 58x - 1320 = 0$$

a = 2, b = –58, c = –1320

Substituting into the formula,

$$x = \frac{-\text{b} \pm \sqrt{\text{b}^2 - 4\text{ac}}}{2\text{a}}$$

$$x = \frac{-(-58) \pm \sqrt{(-58)^2 - 4 \times 2 \times -1320}}{2 \times 2}$$

$$x = \frac{58 \pm \sqrt{3364 + 10560}}{4}$$

$$x = \frac{58 \pm \sqrt{13924}}{4}$$

$$x = \frac{58 \pm 118}{4}$$

$x = 44, x = -15$

Therefore the only possible solution is that the original working week was 44 hours long.

Check: $x = 44, LHS = \frac{330}{44} + 2 = 9.5, RHS = \frac{330 + 50}{44 - 4} = 9.5 = LHS$

(g) Let s be the original speed of the car.

Recall, $\text{speed} = \frac{\text{distance}}{\text{time}}$, and therefore, $\text{time} = \frac{\text{distance}}{\text{speed}}$

The new time will be equal to the original time plus 20 minutes (1/3 of an hour).

$$\frac{312}{s-5} = \frac{312}{s} + \frac{1}{3}$$

Simplifying and rearranging,

$$3s(s-5)\frac{312}{s-5} = 3s(s-5)\frac{312}{s} + 3s(s-5)\frac{1}{3}$$

$$936s = 936s - 4680 + s^2 - 5s$$

$s^2 - 5s - 4680 = 0$

a = 1, b = –5, c = –4680

Substituting into the formula,

$$s = \frac{-\text{b} \pm \sqrt{\text{b}^2 - 4\text{ac}}}{2\text{a}}$$

$$s = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 1 \times -4680}}{2 \times 1}$$

$$s = \frac{5 \pm \sqrt{25 + 18720}}{2}$$

$$s = \frac{5 \pm \sqrt{18745}}{2}$$

$s \approx 70.96, s \approx -65.96$

Therefore the only possible solution is that the car was originally travelling at approximately 71 km/h.

Check: $s \approx 70.96, LHS \approx \frac{312}{70.96 - 5} \approx 4.73, RHS \approx \frac{312}{70.96} + \frac{1}{3} \approx 4.73 \approx LHS$

Activity 3.18

1.

(a)

$$v = u + at$$

when $v = 7$ and $t = 3$ $\qquad 7 = u + 3a$

when $v = 5$ and $t = 2$ $\qquad 5 = u + 2a$

(b) Let cost of a tyre be x, cost of a tube be y.

$$80x + 53y = 959$$
$$106x + 75y = 1\,285$$

(c) Let m be the mother's age in years
Let s be the son's age in years

$m + 6 = 3(s + 6)$ (1)

$m = 4s$ (2)

(Ans: $m = 48$, $s = 12$)

(d) Let c be the weekly wage of a carpenter.
Let a be the weekly wage of an apprentice.

$6c + 8a = 5160$ (1)

$8c + 6a = 5760$ (2)

(Ans: $c = 540$, $a = 240$)

(e) Let x be the number of ten cent coins.
Let y be the number of twenty cent coins.

$x + y = 40$ (1)

$0.1x + 0.2y = 5$ (2)

(Ans: $x = 30$, $y = 10$)

Activity 3.19

(a) $y - x = 3$ (1)

$2x + y = 0$ (2)

From equation (1) $y = 3 + x$ (3)

Substitute this value for y into equation (2)

$$2x + y = 0$$
$$2x + 3 + x = 0$$
$$3x + 3 = 0$$
$$3x = {}^{-}3$$
$$x = {}^{-}1$$

Substituting into equation (3)

$y = 3 + x$

$y = 3 + {}^{-}1$

$y = 2$

Therefore the solution is $x = -1$ and $y = 2$

Check: Equation (1) $LHS = y - x$ $= 2 - {}^{-}1$ $= 3$ $= RHS$

Equation (2) $LHS = 2x + y$ $= 2 \times {}^{-}1 + 2$ $= 0$ $= RHS$

(b) $2p + 3q = 0$ (1)

$p + q - 2 = 0$ (2)

From equation (2) $p = 2 - q$ (3)

Substitute the value for p into equation (1)

$$2p + 3q = 0$$
$$2(2 - q) + 3q = 0$$
$$4 - 2q + 3q = 0$$
$$4 + q = 0$$
$$q = -4$$

Substituting into equation (3)

$$p = 2 - q$$
$$p = 2 - (-4)$$
$$p = 6$$

Therefore the solution is $q = -4$ and $p = 6$

Check: Equation (1) $LHS = 2p + 3q$ $= 2 \times 6 + 3 \times -4$ $= 12 - 12$ $= 0$ $= RHS$

$LHS = p + q - 2$ $= 6 + (-4) - 2$ $= 0$ $= RHS$

(c) $3x - 2y = 4$ (1)

$5x + 4y = 3$ (2)

From equation (1) $y = \dfrac{3x - 4}{2}$ (3)

Substitute this value for y into equation (2)

$$5x + 4y = 3$$
$$5x + 4\left(\frac{3x - 4}{2}\right) = 3$$
$$5x + \frac{12x - 16}{2} = 3$$
$$5x + 6x - 8 = 3$$
$$11x = 11$$
$$x = 1$$

Substituting into equation (3)

$y = \dfrac{3x - 4}{2}$

$y = \dfrac{3 \times 1 - 4}{2}$

$y = \dfrac{^{-}1}{2}$

Therefore the solution is $x = 1$ and $y = \dfrac{-1}{2}$

Check: Equation (1)

$$\begin{aligned} LHS &= 3x - 2y \\ &= 3 \times 1 - 2 \times -\frac{1}{2} \\ &= 3 + 1 \\ &= 4 \\ &= RHS \end{aligned}$$

Equation (2)

$$\begin{aligned} LHS &= 5x + 4y \\ &= 5 \times 1 + 4 \times -\frac{1}{2} \\ &= 5 - 2 \\ &= 3 \\ &= RHS \end{aligned}$$

(d) $3x + 2y = -4$ (1)

$9x + 8y = -11$ (2)

From equation (1) $y = \dfrac{-4 - 3x}{2}$ (3)

Substituting this value for y into equation (2)

$$\begin{aligned} 9x + 8y &= -11 \\ 9x + 8\left(\frac{-4 - 3x}{2}\right) &= -11 \\ 9x + \frac{-32 - 24x}{2} &= -11 \\ 9x - 16 - 12x &= -11 \\ -3x &= 5 \\ x &= -\frac{5}{3} \end{aligned}$$

Substituting into equation (3)

$y = \dfrac{-4 - 3x}{2}$

$y = \dfrac{-4 - 3 \times (-\frac{5}{3})}{2}$

$y = \dfrac{-4 + 5}{2}$

$y = \dfrac{1}{2}$

Therefore the solution is $x = -\dfrac{5}{3}$ and $y = \dfrac{1}{2}$

Check: Equation (1) $LHS = 3x + 2y$
$= 3 \times (-\frac{5}{3}) + 2 \times \frac{1}{2}$
$= -5 + 1$
$= -4$
$= RHS$

Equation (2) $LHS = 9y + 8y$
$= 9 \times (-\frac{5}{3}) + 8 \times \frac{1}{2}$
$= -15 + 4$
$= -11$
$= RHS$

(e) $6y - 5x = 18$ (1)
$12x - 9y = 0$ (2)

Multiply equation (1) by 3 $3 \times 6y + 3 \times -5x = 3 \times 18$
$18y - 15x = 54$ (3)
Multiply equation (2) by 2 $2 \times 12x + 2 \times -9y = 0 \times 2$
$24x - 18y = 0$ (4)
Add equation (3) to equation (4)

$$\begin{array}{r} 18y - 15x = 54 \\ -18y + 24x = 0 \\ \hline 0y + 9x = 54 \end{array}$$

Solving
$9x = 54$
$x = 6$

Substituting into the original equation (1)

$$\begin{aligned} 6y - 5x &= 18 \\ 6y - 5 \times 6 &= 18 \\ 6y - 30 &= 18 \\ 6y &= 48 \\ y &= 8 \end{aligned}$$

Therefore the solution is $x = 6$ and $y = 8$
Check:
In equation 1 $LHS = 6(8) - 5(6) = 18 = RHS$
In equation 2 $LHS = 12(6) - 9(8) = 0 = RHS$

(f) $x + y = 2$ (1)

$2x - y = -1$ (2)

Add equation (1) to equation (2)

$$\begin{array}{l} x + y = 2 \\ \underline{2x - y = -1} \\ 3x + 0y = 1 \end{array}$$

Solving

$$3x = 1$$

$$x = \frac{1}{3}$$

Substituting into the original equation (1)

$$x + y = 2$$

$$\frac{1}{3} + y = 2$$

$$y = 1\frac{2}{3}$$

Therefore the solution is $x = \frac{1}{3}$ and $y = 1\frac{2}{3}$

Check:

In equation (1) $LHS = \frac{1}{3} + 1\frac{2}{3} = 2 = RHS$

In equation (2) $LHS = 2(\frac{1}{3}) - 1\frac{2}{3} = -1 = RHS$

(g) $8x = 5y$ (1)

$13x = 8y + 1$ (2)

Multiply equation (1) by 8 $\quad 8 \times 8x = 8 \times 5y$

$64x = 40y$ (3)

Multiply equation (2) by 5 $\quad 5 \times 13x = 5 \times 8y + 5 \times 1$

$65x = 40y + 5$ (4)

Subtracting equation (4) from equation (3)

$$\begin{array}{l} 64x = 40y \\ \underline{65x = 40y + 5} \\ -x = 0y - 5 \end{array}$$

Solving

$$-x = -5$$

$$x = 5$$

Substituting into equation (1)

$$8x = 5y$$
$$8 \times 5 = 5y$$
$$40 = 5y$$
$$y = 8$$

Therefore the solution is $x = 5$ and $y = 8$

Check:

In equation (1) $\quad LHS = 8 \times 5 = 40 \quad RHS = 5 \times 8 = 40 = LHS$

In equation (2) $\quad LHS = 13 \times 5 = 65 \quad RHS = 8 \times 8 + 1 = 65 = LHS$

(h) $\frac{1}{4}x - \frac{1}{5}y = -1 \qquad (1)$

$3x + \frac{1}{2}y = 17 \qquad (2)$

Multiply equation (1) by 5 $\quad 5 \times \frac{1}{4}x - 5 \times \frac{1}{5}y = 5 \times -1$

$$\frac{5}{4}x - 1y = -5 \qquad (3)$$

Multiply equation (2) by 2 $\quad 2 \times 3x + 2 \times \frac{1}{2}y = 2 \times 17$

$$6x + y = 34 \qquad (4)$$

Add equation (3) to equation (4)

$$\frac{5}{4}x - y = -5$$
$$\underline{6x + y = 34}$$
$$7\frac{1}{4}x + 0y = 29$$

Solving

$$7\frac{1}{4}x = 29$$
$$\frac{29}{4}x = 29$$
$$x = \frac{29 \times 4}{29}$$
$$x = 4$$

Substituting into equation (1)

$$\frac{1}{4}x - \frac{1}{5}y = -1$$

$$\frac{1}{4}(4) - \frac{1}{5}y = -1$$

$$1 - \frac{1}{5}y = -1$$

$$-\frac{1}{5}y = -2$$

$$y = -2 \times -5$$

$$y = 10$$

Therefore the solution is $x = 4$ and $y = 10$

Check:

In equation (1) $LHS = \frac{1}{4}(4) - \frac{1}{5}(10) = 1 - 2 = -1 = RHS$

In equation (2) $LHS = 3(4) + \frac{1}{2}(10) = 12 + 5 = 17 = RHS$

2.

(a)

$7 = u + 3a$ (1)

$5 = u + 2a$ (2)

By elimination

$(1) - (2)$ $a = 2$

$u = 1$

$a = 2m/s^2, \qquad u = 1\,m/s.$

(b)

$80x + 53y = 959$ (1)

$106x + 75y = 1\,285$ (2)

By elimination

$(1) \times 106 \Rightarrow$ $8480x + 5618y = 101654$ (3)

$(2) \times 80 \Rightarrow$ $8480x + 6000y = 102800$ (4)

$(3) - (4)$ $-382y = -1146$

$y = 3$

$$x = \frac{959 - 53 \times 3}{80} = 10$$

Cost of tyre is \$10; cost of tube is \$3
(must be manufactory costs, certainly not retail price today).

(c)

$$m + 6 = 3(s + 6) \qquad (1)$$

$$m = 4s \qquad (2)$$

(1) is $m + 6 = 3s + 18$

$m = 3s + 12$

Thus $4s = 3s + 12$

$s = 12$

Age of son is 12 years, age of mother is 48 years.

Solutions to a taste of things to come

1.

(a) $TC = \text{fixed costs} + \text{variable costs}$ (1)

$TC = 800300 + 6.45x$

$TR = 14x$ (2)

(b) The break even point occurs when $TC = TR$
Therefore,

$$TC = TR$$
$$800300 + 6.45x = 14x$$
$$800300 + 6.45x - 6.45x = 14x - 6.45x$$
$$800300 = 7.55x$$
$$\frac{800300}{7.55} = \frac{7.55x}{7.55}$$
$$x = 106000$$

Therefore, 106 000 brushes must be produced and sold to break even.

(c) $TC = 800300 + 6.45(106000) = 1484000$

$TR = 14(106000) = 1484000$

$\text{Profit} = TR - TC = 1484000 - 1484000 = 0$

Therefore as total costs equal total revenue this is the break even point.

2. Let x be the number of litres of 5% solution to be added
Then,

$$0.05x + 0.4 \times 10 = 0.25(10 + x)$$
$$0.05x + 4 = 2.5 + 0.25x$$
$$0.05x - 0.05x + 4 = 2.5 + 0.25x - 0.05x$$
$$4 = 2.5 + 0.2x$$
$$4 - 2.5 = 2.5 - 2.5 + 0.2x$$
$$1.5 = 0.2x$$
$$\frac{1.5}{0.2} = \frac{0.2x}{0.2}$$
$$x = 7.5$$

Therefore 7.5 litres of 5% solution must be added.

Check: $x = 7.5, LHS = 0.05(7.5) + 4 = 4.375, RHS = 0.25(10 + 7.5) = 4.375 = LHS$

3. (a)

$$R = \frac{IP}{1-(1+I)^{-N}} \qquad I = \frac{i}{12\times 100}$$

$$R = \frac{0.01\times 120000}{1-(1+0.01)^{-180}} \qquad I = \frac{12}{12\times 100}$$

$$R \approx 1440.20 \qquad I = \frac{1}{100} = 0.01$$

Therefore the monthly repayments would be approximately \$1440.20.

(b) Total repayment = monthly repayment × number of months
Total repayment = $1440.20 \times (15\times 12)$
= 259236
Therefore the total repayment would be \$259 236. This is \$139 236 more than the original loan.

(c)
$$R = \frac{IP}{1-(1+I)^{-N}}$$
$$R = \frac{0.01\times 120000}{1-(1+0.01)^{-90}}$$
$$R \approx 2028.37$$

Therefore by halving the repayment time the monthly repayment is \$2028.37.

Total repayment = monthly repayment × number of months
Total repayment = 2028.37×90
= 182553.30
Therefore the total repayment is only \$182 553.30.

(d) Make the principal amount, P, the subject of the equation. If you only had \$1 000 per month to repay the mortgage, what is the largest amount you could borrow under the conditions in question 1?

$$R = \frac{IP}{1-(1+I)^{-N}}$$
$$R\times\left(1-(1+I)^{-N}\right) = \frac{IP}{1-(1+I)^{-N}}\times\left(1-(1+I)^{-N}\right)$$
$$R\left(1-(1+I)^{-N}\right) = IP$$
$$\frac{R}{I}\left(1-(1+I)^{-N}\right) = \frac{IP}{I}$$
$$P = \frac{R}{I}\left(1-(1+I)^{-N}\right)$$

If $R = 1000$ then,

$$P = \frac{R}{I}\left(1-(1+I)^{-N}\right)$$
$$P = \frac{1000}{0.01}\left(1-(1+0.01)^{-180}\right)$$
$$P \approx 83321.66$$

Therefore the maximum amount that could be borrowed is approximately \$83 300.

Solutions to post-test

1.

(a) $3x^2 - 2x - 5x^2 + 7x$
$= -2x^2 + 5x$

(b) $-\dfrac{5q}{12r^2} \div \dfrac{25q^2}{24r}$
$= -\dfrac{5q}{12r^2} \times \dfrac{24r}{25q^2}$
$= -\dfrac{5 \times q}{12 \times r \times r} \times \dfrac{12 \times 2 \times r}{5 \times 5 \times q \times q}$
$= -\dfrac{2}{5qr}$

(c) $\dfrac{x}{x+3} - \dfrac{5+x}{x+2}$
$= \dfrac{x}{(x+3)} \times \dfrac{(x+2)}{(x+2)} - \dfrac{5+x}{(x+2)} \times \dfrac{(x+3)}{(x+3)}$
$= \dfrac{x(x+2)}{(x+3)(x+2)} - \dfrac{(5+x)(x+3)}{(x+2)(x+3)}$
$= \dfrac{x^2 + 2x}{(x+3)(x+2)} - \dfrac{5(x+3) + x(x+3)}{(x+2)(x+3)}$
$= \dfrac{x^2 + 2x}{(x+3)(x+2)} - \dfrac{5x + 15 + x^2 + 3x}{(x+2)(x+3)}$
$= \dfrac{x^2 + 2x}{(x+3)(x+2)} - \dfrac{x^2 + 8x + 15}{(x+2)(x+3)}$
$= \dfrac{x^2 + 2x - x^2 - 8x - 15}{(x+3)(x+2)}$
$= \dfrac{-6x - 15}{(x+3)(x+2)}$
$= \dfrac{-3(2x+5)}{(x+3)(x+2)}$

2.

(a) $5q^2 - q$
$= 5 \times q \times q - 1 \times q$
$= q(5q - 1)$

(b) $x^2 + 5x - 14$

$= x^2 + (7-2)x - 14$

$= x^2 + 7x - 2x - 14$

$= x(x+7) - 2(x+7)$

$= (x+7)(x-2)$

3.

(a) $(x-2)(x-3)$

$= x(x-3) - 2(x-3)$

$= x^2 - 3x - 2x + 6$

$= x^2 - 5x + 6$

(b) $(3+2x)(4x-1)$

$= 3(4x-1) + 2x(4x-1)$

$= 12x - 3 + 8x^2 - 2x$

$= 8x^2 + 10x - 3$

4.

(a) $\dfrac{36x^5y^2}{2x^6y}$

$= 18x^{5-6}y^{2-1}$

$= 18x^{-1}y$

$= \dfrac{18y}{x}$

5.

(a) $16 = -2 + \dfrac{3x}{4}$

$16 + 2 = -2 + 2 + \dfrac{3x}{4}$

$18 = \dfrac{3x}{4}$

$18 \times 4 = \dfrac{3x}{4} \times 4$

$72 = 3x$

$\dfrac{72}{3} = \dfrac{3x}{3}$

$x = 24$

Check: $x = 24, RHS = -2 + \dfrac{3(24)}{4} = -2 + 18 = 16 = LHS$

(b) $5-3(b+1)=6b$

$$5-3b-3=6b$$
$$2-3b=6b$$
$$2-3b+3b=6b+3b$$
$$2=9b$$
$$\frac{2}{9}=\frac{9b}{9}$$
$$b=\frac{2}{9}$$

Check: $b=\frac{2}{9}, LHS=5-3(\frac{2}{9}+1)=5-3\frac{2}{3}=1\frac{1}{3}, RHS=6(\frac{2}{9})=1\frac{1}{3}=LHS$

(c) $\frac{5x+4}{2}=-\frac{3x}{5}$

$$10\times\frac{5x+4}{2}=-\frac{3x}{5}\times 10$$
$$5(5x+4)=-3x\times 2$$
$$25x+20=-6x$$
$$25x+6x+20=-6x+6x$$
$$31x+20=0$$
$$31x+20-20=0-20$$
$$31x=-20$$
$$\frac{31x}{31}=-\frac{20}{31}$$
$$x=-\frac{20}{31}$$
$$x\approx-0.6452$$

Check:

$$x\approx-0.6452, LHS\approx\frac{5(-0.6452)+4}{2}\approx 0.3870, RHS\approx-\frac{3(-0.6452)}{5}\approx 0.3871\approx LHS$$

6. $4x+5y=310$ (1)

$5x+4y=320$ (2)

Multiply equation (1) by 5

$20x+25y=1550$ (3)

Multiply equation (2) by 4

$20x+16y=1280$ (4)

Subtract equation (4) from equation (3)

$$\begin{array}{r} 20x+25y=1550 \\ \underline{20x+16y=1280} \\ 9y=270 \end{array}$$

Solving for *y*,

$$9y = 270$$

$$\frac{9y}{9} = \frac{270}{9}$$

$$y = 30$$

Substituting into equation (1)

$$4x + 5y = 310$$

$$4x + 5(30) = 310$$

$$4x + 150 - 150 = 310 - 150$$

$$4x = 160$$

$$\frac{4x}{4} = \frac{160}{4}$$

$$x = 40$$

Therefore the solution is $x = 40$ and $y = 30$

Check: $x = 40$, $y = 30$

Equation (1)

$LHS = 4(40) + 5(30) = 160 + 150 = 310 = RHS$

Equation (2)

$LHS = 5(40) + 4(30) = 200 + 120 = 320 = RHS$

7.

(a) $v^2 = u^2 + 2as$

$$v^2 - u^2 = u^2 - u^2 + 2as$$

$$v^2 - u^2 = 2as$$

$$\frac{v^2 - u^2}{2s} = \frac{2as}{2s}$$

$$a = \frac{v^2 - u^2}{2s}$$

(b) $S = \frac{1}{2}\left(\frac{Ee}{m}\right)\left(\frac{l}{v}\right)^2$

$$2 \times S = 2 \times \frac{1}{2}\left(\frac{Ee}{m}\right)\left(\frac{l}{v}\right)^2$$

$$2S = \left(\frac{Ee}{m}\right)\left(\frac{l}{v}\right)^2$$

$$2S \times \left(\frac{m}{Ee}\right) = \left(\frac{Ee}{m}\right)\left(\frac{m}{Ee}\right)\left(\frac{l}{v}\right)^2$$

$$\frac{2Sm}{Ee} = \left(\frac{l}{v}\right)^2$$

$$\pm\sqrt{\frac{2Sm}{Ee}} = \pm\sqrt{\left(\frac{l}{v}\right)^2}$$

$$\pm\sqrt{\frac{2Sm}{Ee}} = \frac{l}{v}$$

$$\pm\sqrt{\frac{2Sm}{Ee}} \times v = \frac{l}{v} \times v$$

$$l = \pm\sqrt{\frac{2Sm}{Ee}}v$$

8.

(a) $-\frac{x}{4} + 2 < 6$

$$-\frac{x}{4} + 2 - 2 < 6 - 2$$

$$-\frac{x}{4} < 4$$

$$-\frac{x}{4} \times -4 > 4 \times -4$$

$$x > -16$$

(b) $15 \le 4x + 3 \le 23$

$$15 - 3 \le 4x + 3 - 3 \le 23 - 3$$

$$12 \le 4x \le 20$$

$$\frac{12}{4} \le \frac{4x}{4} \le \frac{20}{4}$$

$$3 \le x \le 5$$

(c) $|10-4x| \geq 6$

$$10-4x \geq 6 \qquad\qquad 10-4x \leq -6$$

$$10-10-4x \geq 6-10 \qquad\qquad 10-10-4x \leq -6-10$$

$$-4x \geq -4 \qquad\qquad -4x \leq -16$$

$$\frac{-4x}{-4} \leq \frac{-4}{-4} \qquad\qquad \frac{-4x}{-4} \geq \frac{-16}{-4}$$

$$x \leq 1 \qquad \text{or} \qquad x \geq 4$$

9. Let x be one sum of money invested.
Let y be the other sum of money invested.

Then,

$0.10x + 0.12y = 3975$ (1)

$0.12x + 0.10y = 3890$ (2)

Multiply equation (1) by 6

$0.60x + 0.72y = 23850$] (3)

Multiply equation (2) by 5

$0.60x + 0.50y = 19450$ (4)

Subtract equation (4) from equation (3)

$$\begin{aligned} 0.60x + 0.72y &= 23850 \\ \underline{0.60x + 0.50y} &\underline{= 19450} \\ 0.22y &= 4400 \end{aligned}$$

Solving for y

$$0.22y = 4400$$

$$\frac{0.22y}{0.22} = \frac{4400}{0.22}$$

$$y = 20000$$

Substitute into equation (1)

$$\begin{aligned} 0.10x + 0.12y &= 3975 \\ 0.10x + 0.12(20000) &= 3975 \\ 0.10x + 2400 &= 3975 \\ 0.10x + 2400 - 2400 &= 3975 - 2400 \\ 0.10x &= 1575 \\ \frac{0.10x}{0.10} &= \frac{1575}{0.10} \\ x &= 15750 \end{aligned}$$

Therefore the amounts invested were \$20000 and \$15750.

Check:

Equation (1)

$LHS = 0.10(15750) + 0.12(20000) = 1575 + 2400 = 3975 = RHS$

Equation (2)

$LHS = 0.12(15750) + 0.10(20000) = 1890 + 2000 = 3890 = RHS$

10. Formula for the perimeter of a rectangle is, $P = 2l + 2w$
Therefore, $40 = 2l + 2w$

Formula for the area of a rectangle is, $A = lw$
Therefore, $96 = lw$

Rewriting in terms of the length, $l = \frac{96}{w}$

Substituting this into the perimeter formula,

$$40 = 2\left(\frac{96}{w}\right) + 2w$$

Rearranging,

$$40w = \frac{192}{w} \times w + 2w \times w$$

$$40w = 192 + 2w^2$$

$$2w^2 - 40w + 192 = 0$$

Dividing through by 2

$$w^2 - 20w + 96 = 0$$

Solving for w using the formula or by inspection,

$$w^2 + (-12 - 8)w + 96 = 0$$

$$w^2 - 12w - 8w + 96 = 0$$

$$w(w - 12) - 8(w - 12) = 0$$

$$(w - 8)(w - 12) = 0$$

$$w = 8, w = 12$$

Therefore, the width of the rectangle could be 8 or 12.
Substituting into the formula for perimeter or area,

when $w = 8$	when $w = 12$
$A = lw$	$A = lw$
$96 = l \times 8$	$96 = l \times 12$
$\frac{96}{8} = \frac{l \times 8}{8}$	$\frac{96}{12} = \frac{l \times 12}{12}$
$12 = l$	$8 = l$
$l = 12$	$l = 8$

Therefore, when the width of the rectangle is 8 m the length is 12 m and when the width is 12 m the length is 8 m.

Check:

$$w = 8, LHS = 2(8)^2 - 40(8) + 192 = 128 - 320 + 192 = 0 = RHS$$

$$w = 12, LHS = 2(12)^2 - 40(12) + 192 = 288 - 480 + 192 = 0 = RHS$$