Module C3

Functions and relations

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Introduction

Mathematics means many things to many people. Some believe it is about numbers or measurements, others shapes. It is defined in the *Macquarie Dictionary* (1999) as:

> *The science dealing with the measurement, properties and relations of quantities, including arithmetic, geometry, algebra etc.*

But really it is about describing nature. A character in Pearl Buck's (1892–1973) novel *The Goddess Abides* describes it well,

> *She had not understood mathematics until he had explained to her that it was the symbolic language of relationships. 'And relationships,' he had told her, 'contained the essential meaning of life.'*

This module of mathematics is about just that. It is about relationships, including a specific type of relationship, the function. As we work through the module we will build on your previous knowledge of relationships and functions developed either in *Mathematics Tertiary Preparation Level B* or elsewhere. More formally, on completion of this module you should be able to:

- demonstrate an understanding of the concept of a function;
- demonstrate an understanding of the concept of continuity of a function;
- use functional notation;
- recognize, describe, sketch and use polynomial, exponential, logarithmic, rational functions and functions with a positive integral domain;
- demonstrate an understanding of the inverse of polynomial, exponential and logarithmic functions;
- recognize relations that are not functions;
- investigate non-continuous functions; especially those over the integral domain (sequences and series);
- understand the concept of a limit; and
- recognize and use arithmetic and geometric sequences and series.

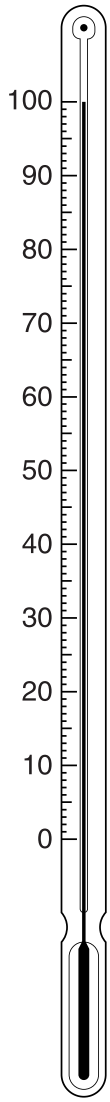
3.1 What are functions?

The weather is not thought of as being predictable, yet if we think about the science of meteorology it is based on very predictable relationships between different variables. The measurement of temperature using thermometers is one example of such a predictable phenomena.

Recall at this stage that a mathematical relationship is called a relation and is defined as a set of ordered pairs. We can detail a set of ordered pairs in one of four ways:

- a set of paired numbers or characteristics
- a mathematical formula
- a graph
- a written description

Thermometers consist of a fine glass tube filled with mercury, a liquid metal that expands on heating and contracts when it becomes cooler, and so moves up and down the tube. The distance travelled by the mercury up or down the tube indicates the value of the temperature. Thermometers come in all shapes and sizes so the relationship between height of mercury and the temperature will be different for different size thermometers.



Using this knowledge we could generate a relationship for the thermometer above, from which we could predict temperature from the height of mercury in the thermometer. This is exactly what both Gabriel Fahrenheit and Anders Celsius did in the 1700s when they developed their two temperature scales. Fahrenheit developed a scale between 32° and 212°, while Celsius's scale went between 0° and 100°. You could look at your own thermometer at home to see what this relationship is (it is different for different thermometers), but if we use the thermometer above we would find the following.

Height of mercury (*h*) in mm	**Temperature (*t*) in °C**
0	0
10	10
20	20
30	30
40	40
50	50
60	60
70	70
80	80
90	90
100	100

We might guess correctly that this relationship is linear and of the form, and would look like the graph below.

Figure 3.1: Graph of temperature and height of mercury

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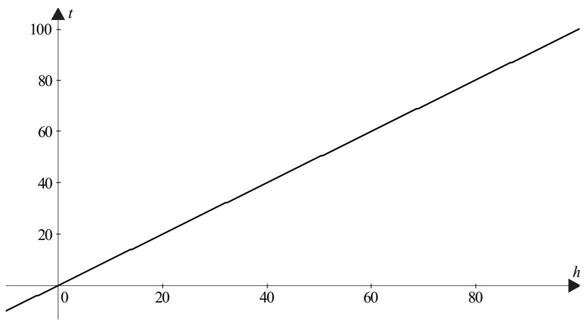


In this relation the measurement of temperature is dependent on the height of mercury, so we call height the independent variable (or input) and temperature the dependent (or output) variable. There is only a single output for each input, so it's easy to predict a unique temperature from the height of mercury. This type of relation is called a **function**. An easy way to see if a relation is a function is to run a vertical line down the graph: if it only ever touches the relation at most at one point then the relation is a function.

A function is a relation that takes certain numbers as input and assigns to each a definite output number. The set of all input numbers is called the domain and the set of the resulting output numbers is called the range.

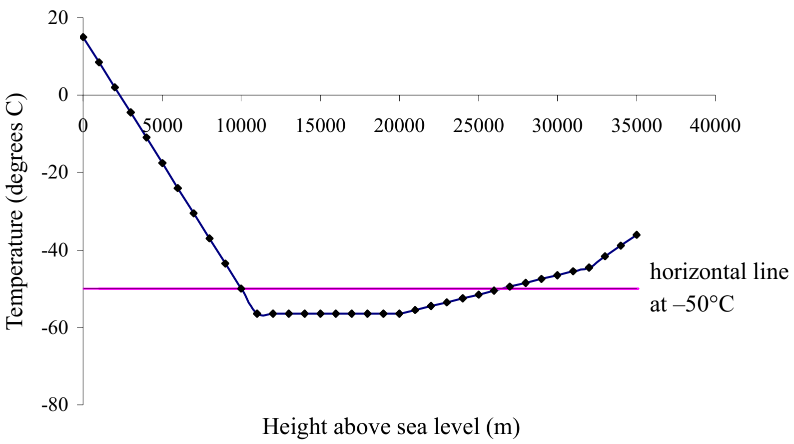
In the function above there is only one possible output number for each input number so we call it a **one-to-one** function. We could represent the set of ordered pairs in a one-to-one function by this diagram called a **mapping**. For the example above, using only some of the points, the mapping would like this.

20 mm → 20°C
30 mm → 30°C
40 mm → 40°C
50 mm → 50°C
100 mm → 100°C

The domain of this function is given by $0 \le h \le 100$, while the range is $0 \le t \le 100$. Of course other thermometers may have different domains and ranges depending on their requirements and design.

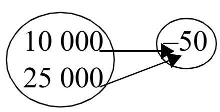
Below is another weather phenomenon in which height and temperature are related. This time standard atmospheric data recorded from weather balloons indicate that there is a relationship between height above sea level (m) and air temperature (°C).

Figure 3.2: Function predicting temperature (degrees C) from height above sea level (m)



In this case temperature can be predicted from height above sea level, so height is the independent (or input) variable and temperature is the dependent (or output) variable. For each value of the independent variable there is one value of the dependent variable, so the relationship is a function. If you ran a vertical line at any point along the relation it would only touch at one point, confirming that it is a function.

However, notice the difference between these two types of functions. In the first case, the one-to-one function, there was a only one output for each input, while if you examine the second function around the temperature of –50°C, you can see that this temperature occurs at 10 000 m and again at 25 000 m. There are two input numbers for the same output number. This type of function is called a many-to-one function. Note that you can always pick a **many-to-one function** because if you graph the function and then draw horizontal lines across the graph they will cut the graph in more than one place. We could represent the set of ordered pairs in the many-to-one function above by a mapping, but this would take up a lot of space. Let's just do it for a couple of points.

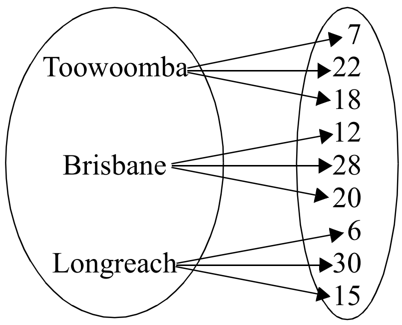


The domain of this function was $0 \le height \le 35000$ and the range was $-56.5 \le temperature \le 15$.

Let's look at one more example of a weather phenomenon. Temperatures at three different towns were recorded at dawn, noon and dusk and displayed by the local newspaper in the table below.

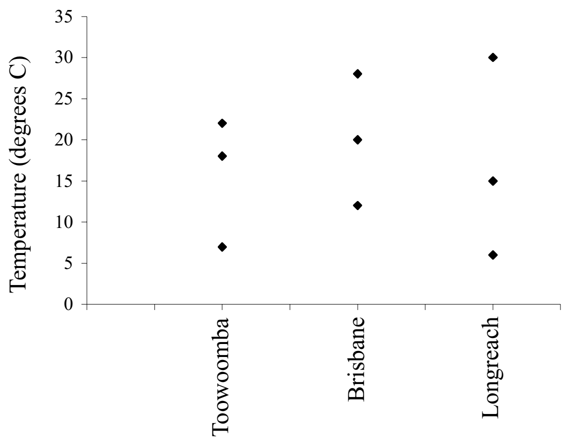
Town	Temperature (°C)
Toowoomba	7
	22
	18
Brisbane	12
	28
	20
Longreach	6
	30
	15

If we graph the temperatures it is clear that this type of relation is very different from the two above. Without the information which tells us which temperatures were recorded at dawn, noon and dusk we cannot predict with any certainty the temperature of a town. Each town gives us three alternatives. In this case the independent (or input) variable is 'town' and the dependent (or output) variable is temperature, but there are three output values for each input value. This type of relation is called one-to-many and is not a function. If we represented this relation by a mapping we would get something like this.



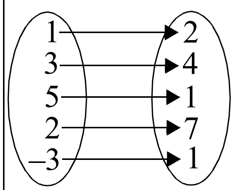
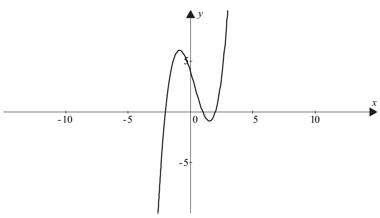
It is easy to see if a relation is one-to-many and thus not a function by drawing a graph of the set of ordered pairs and drawing vertical lines down the graph. If they cut the graph in more than one place then it is one-to-many and is not a function. It is of course still a relation. The graph of the temperatures below demonstrates this clearly.

Figure 3.3: Temperatures recorded in Toowoomba, Brisbane and Longreach



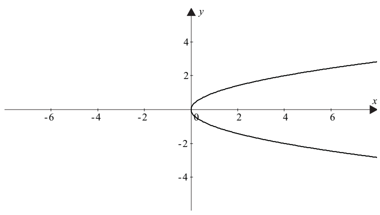
Example

Examine the following relations and decide if they are functions. If they are functions describe the type of function. Give reasons for your choice.

Relation	If function, what type?	Reason
(1,2), (3,4), (5,–1), (2,7), (–3,1)	One-to-one function	For each value of the independent variable (the first member of the ordered pair), there is only one dependent value and vice versa. We could draw a graph of this function and find that a vertical line touches at most one point. We could draw a mapping  etc.
	Many-to-one function	For each value of the independent variable (horizontal axis) there is only one value of the dependent variable, so it is a function (vertical line confirms this). However, for each value of the dependent variable there are at most three values of the independent variable. At times a horizontal line cuts the graph in three places. Thus the function is many-to-one.
$P = \pm\sqrt{1-q^2}, -1 \le q \le 1$	A one-to-many relation, thus not a function	For most values of q (the input variable) there are two values of P (the output variable), thus it is not a function.
Biologists were concerned about the amount of sodium in the blood of wombats. They measured the amount of sodium in the blood every hour for four hours and found that it was increasing rapidly every hour.	One-to-one function	Only five readings were taken and the sodium level is increasing rapidly, that means that every sodium reading must be greater than the previous reading, so for every hour there is only one value of sodium…it must be a one-to-one function. Draw a mapping to check.

Activity 3.1

Examine the following relations and determine whether they are functions. If they are functions, describe the type of function and give reasons for your choice.

Relation	If function, what type?	Reason
$h = t^2 - 3$		
		
(1,–2), (2,3), (3,0), (1,–4), (–2,3)		
Telstra currently charges a network access fee of $11.65 each month and then $0.25 for each local call. What is the relation between the total monthly charge and the number of local calls made each month?		
(2,3), (5,1), (–1,3), (7,0), (–3,1)		

Something to talk about…

Use Graphmatica* to sketch the graph of $x^2 + y^2 = 1$. Is this graph a function? Is the graph a function when the range is restricted to $0 \le y \le 1$? How do the domain and range of a relation effect whether or not the relation is also a function? Think of some examples to explain the effect. Talk with your friends or work colleagues about the examples and include something on the discussion list.

*If you are having trouble using Graphmatica go to the instructions in the introductory book.

3.2 Function toolbox

So we now know what a function is and that they can be described using different methods (numbers, words, formula, graphs). We also know that they can come in many different shapes and forms. However, before we can go further we need to stop and assess what tools we will need to understand particular functions in more detail. Let's look at some of our tools for functional analysis now, Specific examples of each tool will follow in the section Families of Functions.

3.2.1 Functional notation

Let's look at the following function from electronics.

The current through an electrical circuit is a function of the voltage.

If we call Current (*I*) and Voltage (*V*) then more briefly we would say

$$\text{Current is a function of voltage}$$
$$\Downarrow$$
$$I \text{ is a function of } V$$
$$\Downarrow$$
$$I = f(V)$$

If the electronics example was represented by the equation $I = 0.001V^2 + 0.010V$, where I is measured in amperes and V in volts, then to find the values of the function when $V = 0.385$ volts, we could calculate

$$I = f(0.385) = 0.001 \times (0.385)^2 + 0.010 \times 0.385 \approx 0.004$$
$$\Downarrow$$
$$\text{in words we would say}$$
$$\Downarrow$$

'the value of the function f at 0.385 is approximately 0.004'.

The letter *f* was chosen to represent function for obvious reasons, but if we have different functions using the same independent variable we can choose any letter or symbol we like e.g. $g(x)$ or $h(x)$.

You should have come across this type of notation previously in TPP7182 *Mathematics Tertiary Preparation Level B* or elsewhere. Here are some examples to practice using the notation to evaluate functions. **Remember do not confuse the brackets in this notation with multiplication.**

Example

If $f(a) = a^2 - 2a + 1$, find $f(-1),\ f(a^2),\ f(a+1),\ f(a+h)$

$$\begin{aligned} f(-1) &= (-1)^2 - 2 \times -1 + 1 \\ &= 1 + 2 + 1 \\ &= 4 \end{aligned}$$

$$\begin{aligned} f(a^2) &= (a^2)^2 - 2(a^2) + 1 \\ &= a^4 - 2a^2 + 1 \end{aligned}$$

$$\begin{aligned} f(a+1) &= (a+1)^2 - 2(a+1) + 1 \\ &= a^2 + 2a + 1 - 2a - 2 + 1 \\ &= a^2 \end{aligned}$$

$$\begin{aligned} f(a+h) &= (a+h)^2 - 2(a+h) + 1 \\ &= a^2 + 2ah + h^2 - 2a - 2h + 1 \end{aligned}$$

Example

If we are given a function $T = f(c)$ which is used to predict the time in minutes taken to mow a lawn from the capacity (c) of the engine in cubic centimetres, explain the meaning of the expressions $f(c+5)$ and $f(c)+5$.

The expressions both look similar because they both involve adding 5. However, in $f(c+5)$ we are increasing the capacity of the lawn mower engine by 5 cm^3 and then calculating the predicted time to mow the lawn. $f(c) + 5$ involves adding 5 minutes to the time taken to mow the lawn given a certain capacity (c) of the lawn mower.

Example

Two students were asked to evaluate $k(x+h)$, for the function $k(x) = x^2 + 2x + 1$. Discuss the correctness of the solutions presented below.

Student 1	Student 2
$k(x+h) = x^2 + 2x + 1 + h$	$\begin{aligned} k(x+h) &= (x+h)^2 + 2(x+h) + 1 \\ &= x^2 + h^2 + 2x + 2h + 1 \end{aligned}$

Student 1 has not substituted into the function properly. Instead of replacing each x by $x + h$ they have just added h to the end of the function.

Student 2 has made the substitution into the function correctly replacing every value of x by the value $x + h$, however the error has occurred because they have expanded $(x+h)^2$ incorrectly. $(x+h)^2$ is actually $(x+h)(x+h) = x^2 + xh + xh + h^2 = x^2 + 2xh + h^2$. The correct solution should be:

$$\begin{aligned} k(x+h) &= (x+h)^2 + 2(x+h) + 1 \\ &= x^2 + 2xh + h^2 + 2x + 2h + 1 \end{aligned}$$

Example

If $f(x) = x+1$ and $g(x) = x^2 - 1$, find the following functional combinations, $f(x) + g(x)$, $f(x) - g(x)$, $f(x) \times g(x)$ and $\dfrac{g(x)}{f(x)}$.

$$\begin{aligned} f(x)+g(x) &= x+1+x^2-1 \\ &= x+x^2 \text{ or } x(x+1) \end{aligned}$$

$$\begin{aligned} f(x)-g(x) &= x+1-(x^2-1) \\ &= x+1-x^2+1 \\ &= x-x^2+2 \text{ or } -(x^2-x-2) \text{ or } -(x-2)(x+1) \end{aligned}$$

$$\begin{aligned} f(x)\times g(x) &= (x+1)(x^2-1) \\ &\text{or } (x+1)(x-1)(x+1) \\ &\text{or } (x+1)^2(x-1) \end{aligned}$$

It is important not to confuse the brackets in the function notation with multiplication or division. We cannot multiply the left hand side to get *fg* and x^2, because the expression is not *g* times *x* but rather reads the value of the function at *x*.

$$\begin{aligned} \frac{g(x)}{f(x)} &= \frac{x^2-1}{x+1} \\ &= \frac{(x+1)(x-1)}{x+1} \\ &= x-1 \end{aligned}$$

It is important not to confuse the brackets in the function notation with multiplication or division. We cannot cancel out in the first line, because the expression is not *g* divided by *f* but rather reads the value of the function at *x* in each case.

Example

When $f(x) = x+1$ and $g(x) = x^2 - 1$, find the value of $f(g(x))$.

This type of example is called a **function of a function**. Wherever you see *x* in the *f* function you replace it by the function *g*(*x*).

$$\begin{aligned} f(x) &= x+1 \\ f(g(x)) &= f(x^2-1) \\ f(g(x)) &= (x^2-1)+1 \\ f(g(x)) &= x^2 \end{aligned}$$

Insert the function *g*(*x*) into the *f*(*x*) function

Evaluate the *f*(*x*) function, in this case wherever you see *x* replace it by $x^2 - 1$.

Example

If $f(x) = x^2$ and $g(x) = \sqrt{x}$, by finding the values of $f(g(x))$ and $g(f(x))$ show that in this special case $f(g(x)) = g(f(x)) = x$.

The left hand side (LHS) of the expression is

$$\begin{aligned} f(g(x)) &= f(\sqrt{x}) \\ &= (\sqrt{x})^2 \\ &= x \end{aligned}$$

The right hand side of the expression is

$$\begin{aligned} g(f(x)) &= g(x^2) \\ &= \sqrt{x^2} \\ &= x \end{aligned}$$

As the left hand side equals the right hand side and both are equal to x, then we have shown that $f(g(x)) = g(f(x)) = x$.

Activity 3.2

1. Write in words the meaning of the following mathematical expressions:

   (a) $f(12) = 3$

   (b) $h(-2) = 12.75$

2. If $p(t) = 3t^2 - 2$ find the value of the following:

   (a) $p(0.5)$

   (b) $p(m+2)$

3. If we coil a wire around a piece of metal and then put a current in the wire the metal becomes magnetic. Assume that the following function $m = B(n)$ measures the strength of the magnetic field produced when the coil of wire has n turns. Explain the meaning of the following expressions:

   (a) $B(2n)$

   (b) $B(n) + 2$

4. If $p(m) = 2m - 3$ and $h(m) = 4m^2 - 9$ find the following functional combinations:

   (a) $h(m) - p(m)$

   (b) $p(m) \times h(m)$

   (c) $\dfrac{p(m)}{h(m)}$

   (d) $p(h(m))$

5. If $f(x) = 2x^2, g(x) = x^2 + 2$, evaluate $f(g(x))$ and $g(f(x))$.

3.2.2 Zero conditions of a function

Another tool that is important for describing functions involves finding the zero conditions for the function. This means

- finding the value of the function when the independent variable is zero; and
- determining what values of the independent variable causes the function to be equal to zero.

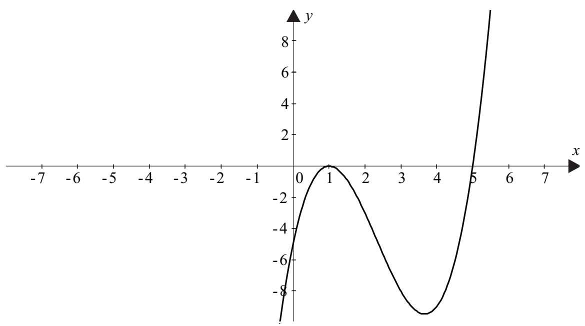
In function notation this means doing two things: evaluating $f(0)$ and solving $f(x)=0$, for all values of x. In graphical terms this means finding where the function cuts the vertical and horizontal axes.

We will return to this tool soon when we examine families of functions in detail.

Example

The following function is displayed on the graph below. From the graph determine the solution(s) to $f(x)=0$ and evaluate $f(0)$.

Figure 3.4: Graph of a polynomial function

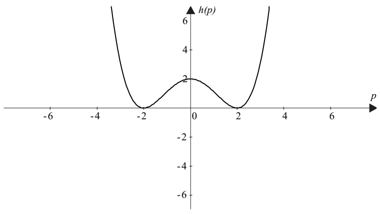


From the graph we can see that the horizontal intercepts are $x=1$ and $x=5$, thus these are the solutions to the equation $f(x)=0$.

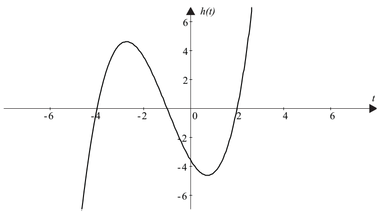
The vertical intercept is $f(x)=-5$, so this is the value of $f(0)$.

Activity 3.3

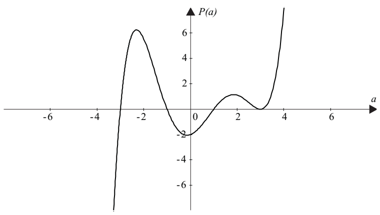
For each of the following functions determine the zero conditions.

1. 

Find the value of $h(0)$ and the solutions to $h(p) = 0$.

2. 

Find the value of $h(0)$ and the solutions to $h(t) = 0$.

3. 

Find the value of $P(0)$ and the solutions to $P(a) = 0$.

3.2.3 Average rate of change of a function

When investigating functions it is possible to measure how one variable changes with respect to the second or third variable. You might have heard the words speed, velocity, acceleration, birth rate, death rate, production rate – these are all words used to describe how one variable changes with respect to another.

The quantity that we use to measure how one variable changes with respect to another is called the rate of change. In straight lines it is easy to find and is called the gradient or slope of the straight line.

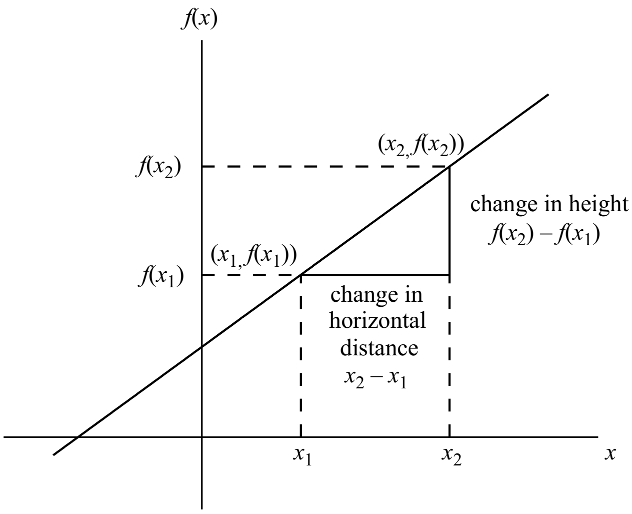
Recall that the gradient puts a value on the steepness of a straight line by comparing the change in height with the change in horizontal distance.

In function notation, if we have the linear function $y = f(x)$, which has two points with horizontal values of x_1 and x_2, the rate of change m is

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}} = \frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

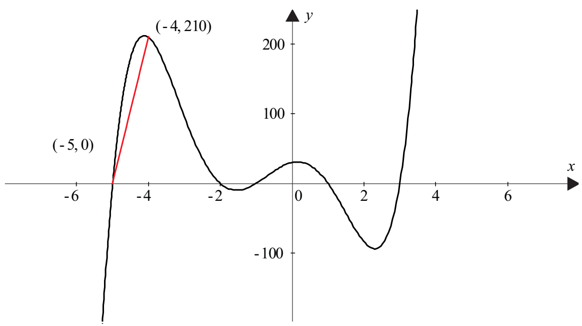
(Note: Δ is the Greek symbol delta used to mean a change, so that Δx means a change in x.)

Figure 3.5: Gradient of a straight line



It's easy to determine the gradient or rate of change of a function if it is a linear function because linear functions always have a constant gradient or rate of change. Curved functions are not so easy because the rate of change of one variable with respect to the other is always changing. However we can approximate the process by finding the gradient between two points of interest on the graph. We call this the **average rate of change** of the curve. Using the figure below, we have approximated the average rate of change of the curve between the points $x = -5$ and $x = -4$, by finding the gradient of the straight line connecting these two points.

Figure 3.6: Average rate of change of a curve



The average rate of change for the curve shown between the points (–5, 0) and (–4, 210) will be

$$\begin{aligned} m = \frac{\Delta f}{\Delta x} &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ &= \frac{210 - 0}{-4 - -5} \\ &= 210 \end{aligned}$$

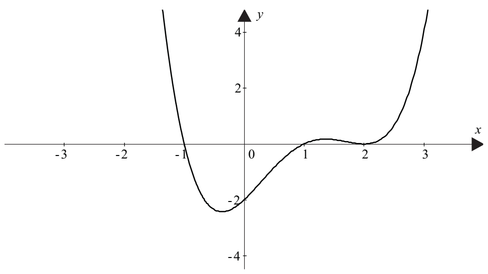
This means that if we know any two points on a curve we can use them to calculate the average rate of change of the function, by calculating the gradient of the straight line connecting the points.

Examples of this tool will be detailed when we examine Families of Functions.

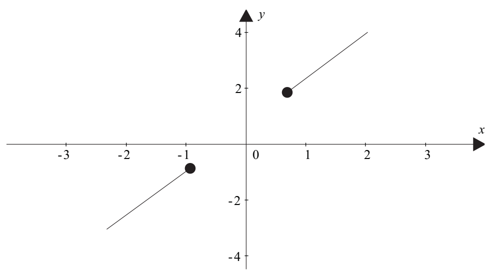
3.2.4 Continuity

Let's look at the functions described by the graphs below.

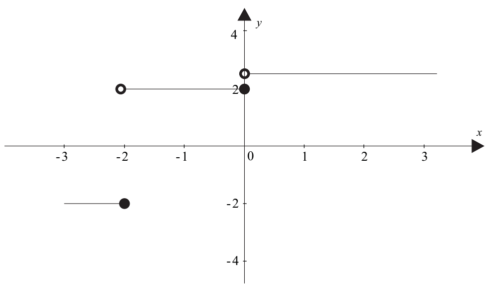
Figure 3.7: Some examples of different types on continuity



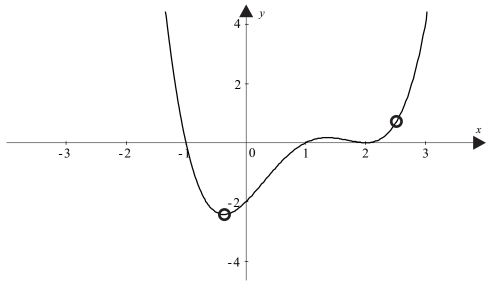
Graph A
Domain is all real values of x.



Graph C
Domain is all real x except, $-1 < x < \frac{1}{2}$



Graph B
Domain is all real values of x.



Graph D
Domain is all real x except, $x = -0.5$ and $x = 2.5$

Graph A is the only continuous function in the group. We can say it is continuous because if you try to run your pen along the function you can do so without lifting the pen from the page. Try to do that with the other functions and you will find you cannot. Graph A is a **continuous** function while all the other graphs are **discontinuous** functions. Mathematicians have a specific definition for continuity but we will leave that for another course.

Let's continue to compare the continuity of the functions given. Notice that although Graph A and Graph B both have **unrestricted domains** (exist for all real values), only Graph A is continuous. Just because the domain of a function is defined for all real values of x (say) it does not mean that the function will be continuous. Breaks in continuity can take many forms. Graph B has a jump, Graph C has a break while Graph D has holes. Later in the unit in the topic on rational functions we will examine another type of break in continuity.

3.2.5 Inverse of a function

Although finding the inverse of a function is not directly a tool to help us understand the characteristics or shape of a function it does help us understand more deeply the nature of a function, hence the reason for including it here. But first recall what the inverse of a function means.

Let's return to the thermometer situation, but this time with a large wall thermometer. In most instances we would want to find the temperature from the height in the thermometer and use the table below.

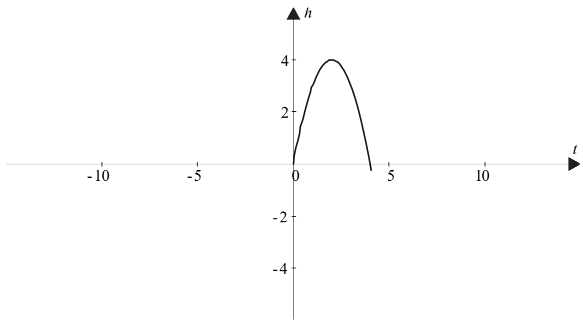
Height of mercury (*H*) in mm	**Temperature (*t*) in °C ($t = f(H)$)**
0	0
20	10
40	20
60	30
80	40
1000	50
120	60
140	70
160	80
180	90
200	100

So we would have $f(20) = 10$ and $f(160) = 80$. If we wanted to change our perspective and predict the height on the thermometer from the temperature (the original designers might have done just that) we would interchange the output and input variables, so that temperature would be the input variable and height the output variable. This function would no longer be f but the **inverse** of f which we would call f^{-1}. In this case we have $f^{-1}(10) = 20$ and $f^{-1}(80) = 160$ so that $f^{-1}(10) = 20$ is the inverse of $f(20) = 10$, and $f^{-1}(80) = 160$ is the inverse of $f(160) = 80$. In general $H = f^{-1}(t)$ is the inverse of $t = f(H)$.

If you were to graph both the original function and its inverse you would find that they are both functions, with the original function being a one-to-one function. Note however, that not all functions have an inverse function i.e. they are not **invertible**.

Consider what happens when we throw a stone. The stone will rise and then fall following a parabolic path.

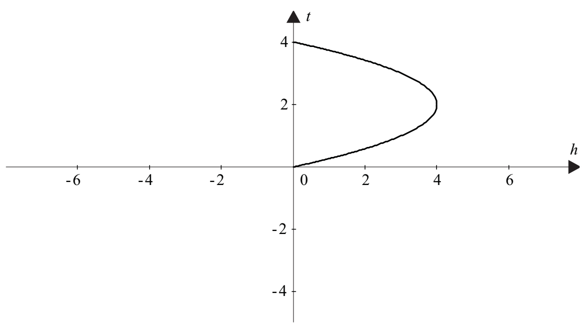
Figure 3.8: Path of thrown stone



The stone is 1 metre in the air at 0.2 of a second and again at 3 seconds, depending on whether it is going up or down. As a set of points this would be (0.2,1) and (3,1). If we tried to find the inverse of this function we would determine the time from the height. But in this case we have two alternatives to choose from at the height of 1 metre the stone could have been in the air for either 0.2 seconds or 3 seconds. This means the function is a many-to-one function and that the inverse function of the original parabolic function does not exist, since it is a one-to-many.

The inverse of the stone throwing function would actually look like this and is clearly not a function.

Figure 3.9: Inverse of stone throwing function



In general functions must be one-to-one for an inverse function to exist.

3.3 Families of functions

Although functions come in all shapes and sizes many of them can be grouped into similar types, you would have come across a number of these types before. For example, linear, quadratic, trigonometric, exponential or logarithmic functions are all functions you might have encountered previously. Let's now have a look at some of these in more detail and see if there are further differences or similarities between them.

3.3.1 Polynomial functions

Wave your hand in the air like you are starting to draw the ABC TV symbol and you have probably drawn a type of polynomial function. Polynomial functions are used in a wide range of situations. Look at these below.

The total cost, $C(x)$ in dollars, of producing x numbers of goods, after a very unstable season.	$C(x) = 0.03x^3 + 1.85x^2 + 35.57x$
The flight path of a piece of fruit in the air where $P(x)$ is the height in metres and x is the time in seconds.	$P(x) = -4.9x^2$
The bending moment, $M(x)$, of a beam supported at one end, x metres from the support.	$M(x) = -12.5x^2 + 3750x$
The number of eggs, $E(x)$, produced by a certain species of lizard as a function of lizard body weight in grams (x).	$E(x) = 9.04x^3 - 32.07x^2 + 37.55x - 10.48$
The relationship between height of mercury on a thermometer in mm ($H(x)$) and the room temperature (x) in degrees Centigrade.	$H(x) = x$
The height of mountains $T(x)$ in metres as a function of distance, x, in kilometres from a certain reference point about 250 m above sea level.	$T(x) = -0.002x^6 + 0.19x^5 - 6.41x^4 + 100.43x^3 - 728.33x^2 + 2020x$

Let's look at the algebraic expressions associated with each of the functions above. Do you notice any pattern in their form?

Write now in your own words the similarities and differences between the expressions represented in the functions above.

___

___

___

___

In summary you might have said something like this.

- The indices of the independent variable are all positive whole numbers or zero.
- The coefficients of the variables or their powers can be either positive, negative real numbers or zero.
- The expressions are a sum of a number of terms which involve different whole number powers of the independent variable.

The functions we have described are **polynomial functions** (pronounced pol-ee-no-mee-al). The **degree** of a polynomial is the highest power of the variable. Let's define a polynomial more precisely.

> **A polynomial of degree *n* has the form**
>
> $$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + ... + a_2 x^2 + a_1 x^1 + a_0 x^0$$
>
> **where *n* is an integer greater than or equal to zero and** $a_0, a_1, ...a_{n-1}, a_n$ **are any real numbers.**

Recall that $x^0 = 1$, by definition, so the last term would be just a_0. Do not confuse the notation used in the polynomial definition. If a small number is written above the variable as in x^2 then we would calculate the square of x, but if a small number is written below the variable as in a_2, then this means that this is just the second number of this type, no further calculations are involved. It is just a way of keeping track of similar constants in a large expression.

Example

Which of the following expressions represent polynomial expressions and what is their degree?

$$P(x) = 2x^2 + x - 1$$

$$q = 1 - \sqrt{x} + x^2 + x$$

$$f(c) = \frac{c^3}{3} + c$$

$$y = x^5 - 2x^3 + x^{-1}$$

$$g(c) = c^4 + c^3 + c^3 + c + 1$$

Expression	Is it a polynomial?	Degree
$P(x) = 2x^2 + x - 1$	Yes, all indices are whole numbers	2
$q = 1 - \sqrt{x} + x^2 + x$	No, $\sqrt{x}$ is actually $x^{\frac{1}{2}}$, so the power is not a whole number	Not applicable
$f(c) = \frac{c^3}{3} + c$	Yes all indices are whole numbers. Note that $\frac{c^3}{3} = \frac{1}{3}c^3$, and fractional coefficients are allowed within the definition	3
$y = x^5 - 2x^3 + x^{-1}$	This is not a polynomial because one of the indices is negative 1.	Not applicable
$g(c) = c^4 + c^3 + c^3 + c + 1$	This is a polynomial function because the variable has only whole numbers as indices	4

Activity 3.4

Which of the following expressions represent polynomials and what is their degree?

Expression	Is it a polynomial?	Degree
1. $h = 12t - 2t^2 + \frac{3}{t}$		
2. $r(x) = 12x + 3x^5 - 2x^2 + 4$		
3. $P(r) = 5r^4 - 2r^3 + 12$		
4. $y = 12x - 5x^4 + 2x^{0.5}$		
5. $F(w) = 5w + 3 - \frac{2}{w} \times w^2$		

Now that you can tell a polynomial function from other functions, let's have a look at the characteristics of polynomial functions and their graphs in more detail. Some of these functions will be very familiar to you. Let's experiment a bit first though.

Something to talk about…

The following are all polynomial equations. Sketch them using Graphmatica, thinking about their shape and their equations. Use a scale between ±4 on the horizontal axis and ±8 on the vertical axis. The aim is for you to get a feel for the functions, don't try to do perfect graphs. We will come back to that later.

$y = x$

$y = x^2$

$y = x^3$

$y = -x^3$

$y = x^3 - 6x^2 + 11x - 6$

$y = -(x^3 - 6x^2 + 11x - 6)$

$y = x^4 - 5x^3 + 5x^2 + 5x - 6$

$y = -(x^4 - 5x^3 + 5x^2 + 5x - 6)$

What did you discover about the relationship between the expressions and the shapes of their graphs? Share your ideas with the discussion group. Can you make up any more examples? You might have to change the grid range on Graphmatica.

Now that you have a general feel of polynomial functions let's examine some specific polynomial functions in detail.

The constant function

These are functions of the form $y = 2$ or $f(x) = -3$. Do you believe they are actually polynomial functions? Compare them with the definition of a polynomial function.

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + ... + a_2x^2 + a_1x^1 + a_0x^0$$

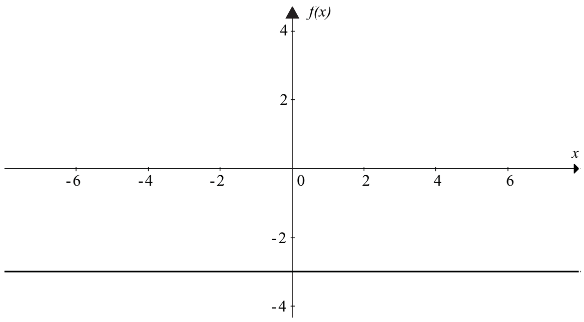
In the case of the constant function the highest power of the independent variable is zero, so the function $f(x) = -3$, could really be written as $f(x) = -3x^0$.

The constant function is the simplest of the polynomial functions. Consider the function $f(x) = -3$ and think about the following questions.

- What will be the shape of the graph?
- Where will it cut the vertical axis?
- Where will it cut the horizontal axis?
- What is the rate of change of the dependent variable with respect to the independent variable?
- Is it a continuous function?
- Will it have an inverse function?

Sketch the graph of $f(x) = -3$ for a domain of all real values of x on Graphmatica and think about its characteristics. The graph you sketch will look like this.

Figure 3.10: A Constant Function, $f(x) = -3$



Now we can answer the above questions in turn.

- It is a straight line parallel to the x-axis.
- It will have an intercept of –3 with the vertical axis, because when we put $x = 0$, the value of the function will be –3.
- It will **not** have an intercept with the horizontal axis because if we try to evaluate $f(x) = 0$, we would get the nonsense expression of 0 = –3.
- The rate of change of the function over its entire domain will be zero. We could confirm this by calculating m of the straight line connecting the points (1, –3) and (2, –3)

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}} = \frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{-3 - -3}{2 - 1} = 0$$

  You might recall that any line parallel to the horizontal axis will have a rate of change or gradient of zero. This is an important concept that we will return to later in the module on calculus.

- It will be a continuous function because it has no breaks or jumps in the domain given.
- It will not have an inverse because it is a many-to-one function.

The linear function

The next polynomial function you should be very familiar with is the linear function e.g. $f(x) = 2x + 1$. This function represents a straight line and is a polynomial function of degree 1. (Recall $x^1 = x$). The function's general form is $f(x) = mx + c$, where m is the gradient and c is the vertical intercept.

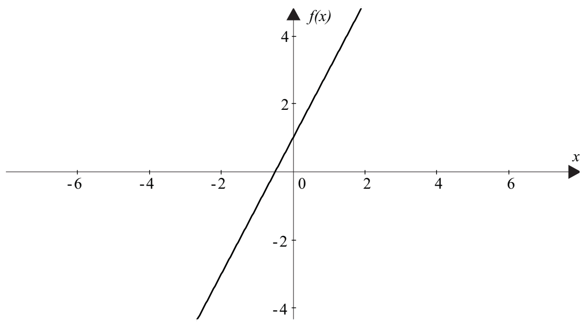
Consider the function $f(x) = 2x + 1$ and ask yourself the following questions.

- What will be the shape of the graph?
- Where will it cut the vertical axis?
- Where will it cut the horizontal axis?
- What is the rate of change of the dependent variable with respect to the independent variable?

- Is it a continuous function?
- Will it have an inverse function?

Sketch the graph of $f(x) = 2x + 1$ for a domain of all real values of x on Graphmatica, think about its characteristics.

Figure 3.11: A Straight Line, $f(x) = 2x + 1$



- It will be a straight line.
- It will have an intercept of 1 with the vertical axis, because when we put $x = 0$, the value of the function will be 1.
- It will have an intercept with the horizontal axis of $-\frac{1}{2}$. This satisfies the equation $f(x) = 0$.
- The rate of change of the function over its entire domain will be 2. We could read this directly from the equation where $m = 2$ or we could confirm this by calculating, m between any two points on the graph say ($-\frac{1}{2}$, 0) and (0, 1)

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}} = \frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{1 - 0}{0 - -\frac{1}{2}} = 2$$

  Note that if you were to calculate the gradient for any two points anywhere on the straight line then the gradient would be the same. This is a characteristic of a linear function – a constant rate of change.

- It will be a continuous function because if has no breaks or jumps in the domain given.
- It will have an inverse function because it is a one-to-one function.

The quadratic function

The general form of the quadratic function, $f(x) = ax^2 + bx + c$, you have studied before fits well with the definition of the polynomial function of degree 2. Although it should be clear that the only term that is essential for it to be defined as a polynomial function of degree 2 is the term which has the x^2 component.

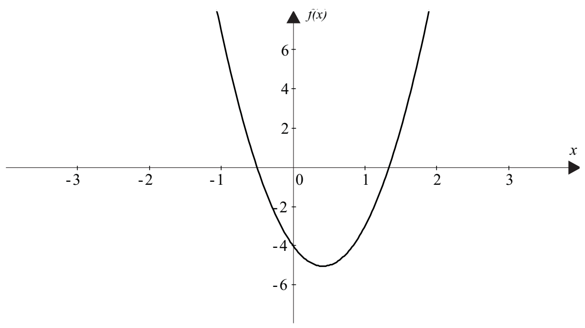
$F(x) = x^2 + 1$, $y = 1 - x - x^2$, $g(c) = 2c^2$ are all polynomial functions of degree two and hence are quadratic functions.

Consider the function $f(x) = 6x^2 - 5x - 4$ and ask yourself the following questions.

- What will be the shape of the graph?
- Where will it cut the vertical axis?
- Where will it cut the horizontal axis?
- What is the rate of change of the dependent variable with respect to the independent variable?
- Is it a continuous function?
- Will it have an inverse function?

Sketch the graph of $f(x) = 6x^2 - 5x - 4$ on Graphmatica for a domain of all real values of x.

Figure 3.12: A quadratic function, $f(x) = 6x^2 - 5x - 4$



- It will have a parabolic shape with a minimum turning point because the coefficient of the x^2 is positive.
- It will have an intercept of –4, with the vertical axis because when we put $x = 0$, the value of the function will be –4.
- To find the intercepts with the horizontal axis we have to solve the equation $f(x) = 0$. $f(x) = 6x^2 - 5x - 4 = 0$, To do this recall that we either need to factorize the expression or use the quadratic formula. Go to module 2 now if you are rusty on these topics. If we factorize the expression on the left hand side we get

$$6x^2 - 5x - 4 = 0$$
$$(3x - 4)(2x + 1) = 0$$
$$3x - 4 = 0, \text{ or } 2x + 1 = 0$$
$$x = \frac{4}{3}, x = -\frac{1}{2}$$

So the intercepts on the horizontal axis are $x = \frac{4}{3}, x = -\frac{1}{2}$.

- This function is a curved shape so the rate of change of the function over its domain will be different in different places. We can see this directly from the graph. On the left hand side the function is decreasing and we would expect a negative gradient. At the bottom of the graph the function does not change and we would expect the gradient to be zero, while on the right hand side the function is increasing and we would expect it to have a positive gradient. We could confirm this in an approximate way by finding the average rate of change between two points on either side of the turning point at the bottom of the graph.

  On the left hand side the average rate of change between $(-\frac{1}{2},0)$ and $(-\frac{3}{2},17)$ is

  $$m=\frac{\Delta f}{\Delta x}=\frac{f(x_2)-f(x_1)}{x_2-x_1}=\frac{0-17}{-\frac{1}{2}--\frac{3}{2}}=\frac{-17}{1}=-17$$

  On the right hand side the average rate of change between (1,–3) and (2,10) is

  $$m=\frac{\Delta f}{\Delta x}=\frac{f(x_2)-f(x_1)}{x_2-x_1}=\frac{-3-10}{1-2}=\frac{-13}{-1}=13$$

- It will be a continuous function because if has no breaks or jumps in the domain given.
- It will not have an inverse function because it is a many-to-one function.

Other polynomial functions

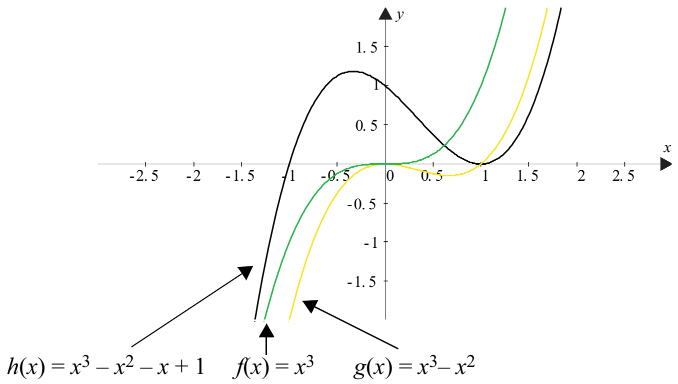
From the experiments you did on the previous pages it appears that as the degree of a polynomial function increases we might get more twists and turns to the function. Before we investigate this in detail let's again draw some graphs using Graphmatica but this time we will analyse them in more detail.

Using Graphmatica draw the cubic functions:

$$f(x)=y=x^3$$
$$g(x)=y=x^3-x^2$$
$$h(x)=y=x^3-x^2-x+1$$

What can you tell about the functions from their graphs? (Note it may be useful to use the coordinate cursor available under the point menu to find the coordinates of the intercepts and the turning points).

Figure 3.13: Graph of three cubic equations

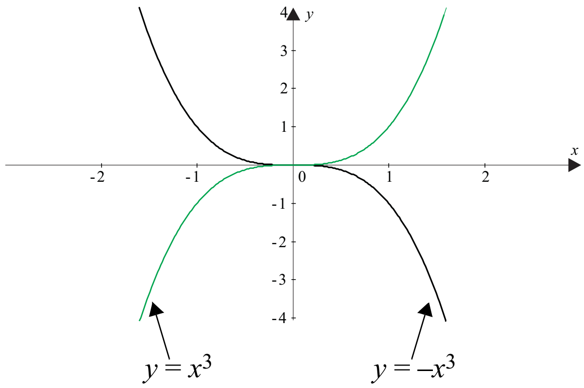


Characteristic	$y = f(x) = x^3$	$y = g(x) = x^3 - x^2$	$y = h(x) = x^3 - x^2 - x + 1$
Degree	3	3	3
Shape	Has no turning points (i.e. no local maximum or minimum points); enters from the bottom left hand corner and exits at the top right hand corner	Has two turning points; enters from the bottom left hand corner and exits at the top right hand corner	Has two turning points; enters from the bottom left hand corner and exits at the top right hand corner
Vertical intercepts	Vertical intercept is 0, the value of the constant in the function	Vertical intercept is 0, the value of the constant in the function	Vertical intercept is 1, the value of the constant in the function
Horizontal intercepts	From the graph this is 0	From the graph these are 0,1	From the graph these are –1,1 (all are factors of 1)
Rate of change	Changes continually from positive to zero to positive again (from left to right)	Changes continually from positive to zero to negative to zero to positive (from left to right)	Changes continually from positive to zero to negative to zero to positive (from left to right)
Continuity	Continuous over the domain	Continuous over the domain	Continuous over the domain
Inverse?	It possesses an inverse because it is a one-to-one function	It does not possess an inverse because it is a many-to-one function	It does not possess an inverse because it is a many-to-one function

Let's put a minus sign in front of the three **cubic functions** and see what happens to the functions.

Firstly, draw and compare $y = f(x) = x^3$ with $y = F(x) = -x^3$

Figure 3.14: $y = f(x) = x^3$ and $y = F(x) = -x^3$



What did you notice? You might have said that the graphs were identical except in two characteristics. In terms of the shape $F(x)$ enters from the top left hand corner and exists from the bottom right hand corner. The rate of change goes from negative to zero to negative again.

Try doing the same to the other two cubic functions yourself.

Alright, you now have a feel for what happens to the function. Let's have a look at some polynomial functions in more detail expanding our algebra along the way.

Technique 1: Understanding the shape of a graph from its equation

We know that the function, $f(x) = x^3$ will look like  while the function $F(x) = -x^3$ will look like , but how can we tell this from their equations. In this example the coefficient of x^3 the gives us the clue, but in more complex examples it is not always so transparent. Let's look at this simple case first.

Find some values of $f(x)$ when x gets very big.

$$f(100) = 1000\,000$$
$$f(1000) = 1\,000\,000\,000$$
$$f(10\,000) = 1\,000\,000\,000\,000$$

The bigger x gets, the bigger the value of the function gets. This is because the value of the function is dominated by the x^3 term. We might have expected this because any positive number cubed will produce another positive number.

What happens to $f(x)$ when x gets very small.

$$f(-100) = -1\,000\,000$$
$$f(-1000) = -1\,000\,000\,000$$
$$f(-10000) = -1\,000\,000\,000\,000$$

The smaller x gets, the smaller the value of the function gets. This is again because the function is dominated by the x^3 term. We might have expected this because any negative number cubed will produce another negative number.

What we are really considering here is the value of the function as x approaches positive **infinity** (∞) or negative infinity ($-\infty$). When we translate from words to symbols, we might say that

As x approaches infinity the value of the function approaches infinity

$$\Downarrow$$

$$\text{as } x \to \infty, f(x) \to \infty$$

$$\Downarrow$$

As x approaches negative infinity the value of the function approaches negative infinity

$$\Downarrow$$

$$\text{as } x \to -\infty, f(x) \to -\infty$$

Example

Consider the functions $h(x) = x^3 - x^2 - x + 1$ and $H(x) = -(x^3 - x^2 - x + 1)$, what happens when x gets very large or small (i.e. as x approaches $\pm\infty$)?

Consider $h(x) = x^3 - x^2 - x + 1$, when x gets very large (i.e. as $x \to \infty$). This function involves x to the power three so we might expect this to produce a large positive number, because the cubic power will dominate the value of the expression. Note that $h(10\,000) = 999\,899\,990\,001$ which confirms that the value is very large. So as $x \to \infty, h(x) \to \infty$.

When x gets very small (i.e. as $x \to -\infty$), the value of the x^3 term will dominate and the function will approach a very small number i.e. approach negative infinity (note $h(-10\,000) = -1\,000\,099\,989\,999$). So as $x \to -\infty, h(x) \to -\infty$.

Consider $H(x) = -(x^3 - x^2 - x + 1)$ when $x \to \infty$. Because of the negative sign in front of the entire function it will be opposite to $h(x)$ and approach negative infinity. When $x \to -\infty$ the function will approach positive infinity.

So as $x \to \infty, H(x) \to -\infty$ and **as** $x \to -\infty, H(x) \to \infty$.

(Note that calculating some large and small values of the function or sketching the graph should confirm these conclusions.)

Example

How does $g(x) = x^4 - 2x^3 - 13x^2 - 26x - 24$ behave as $x \to \pm\infty$?

To get a feel for the value of the function when x approaches positive and negative infinity, evaluate $g(x)$ for $x = 1000$ (say) and $x = -1000$.

$g(1\,000) = 997\,986\,973\,976$ and $g(-1\,000) = 1\,001\,987\,025\,976$

You might have expected this result because x^4 will dominate the value of the function and any number raised to the fourth power will produce another even number. So in conclusion,

$$\text{as } x \to \infty, g(x) \to \infty$$
$$\text{as } x \to -\infty, g(x) \to \infty$$

(check your answer by drawing the graphs on Graphmatica)

Example

Examine the polynomial function $k(x) = x^5 - 3x^4 - 11x^3 - 13x^2 + 2x + 24$, what happens when x gets very large or very small?

When x gets very large we are considering the value of the function as x approaches positive infinity. Because we are working with a function that involves x to the power five we might expect this to produce another large positive number. If we evaluate $f(1000)$ we find that it is equal to 996 988 987 002 024, which confirms our belief.

When the function gets very small we are considering the value of the function as x approaches negative infinity. Because we are working with a function that involves x to the power five we might expect this to produce another large negative number. If we evaluate $f(–1000)$ we find that it is equal to –1 002 989 013 001 980, which confirms our belief.

As $x \to \infty, k(x) \to \infty$
As $x \to -\infty, k(x) \to -\infty$

Activity 3.5

For each of the functions below, investigate what happens when x gets very large and very small, i.e. what happens at the function's **extremities**.

1. $h(x) = x^3 - 3x^2 - x + 3$
2. $y = -x^2 + 6x - 12$
3. $P(x) = -x^4 + 4x^3$
4. $f(x) = -x^3 + 2x^2 + 8x + 1$
5. $y = 3x^4 + 2x^3 - x^2 + x - 5$

Technique 2: Factorizing polynomial functions

In order for us to be able to find the horizontal intercepts of a polynomial equation we need to have a technique for solving these equations. When we work with quadratic functions (polynomial functions of degree 2) we used two methods, factorizing and the quadratic formula. Unfortunately finding the roots of general polynomial functions is not so easy…..there is not a last stop formula. However, we can solve some functions by using our knowledge of the meaning of a solution to an equation.

A bit of history…

A mathematician Girolamo Cardano published the solution of the cubic equations in 1545. This is considered the greatest achievement in algebra since the Babylonians (he must have thought it fairly important since he wrote at the end of the article 'Written in five years, may it last as many thousands').

After this mathematicians had been trying to find an algebraic solution for higher degree polynomials until a Norwegian mathematician Niels Hiel proved that it was impossible for fifth and higher degree equations. The consequence is that we use other methods to find the solutions.

Dunham, W 1991, *Journey through genius*, Penguin Books, NY.

Recall that if a value of the independent variable **satisfies** an equation then it is a solution to the equation. So if we make a guess at some possible solutions we can then check to see if we are correct. Let's try it for a simple quadratic equation.

Consider the function $f(x) = x^2 + x - 2$. The horizontal intercepts occur when $f(x) = 0$, so we have to solve the equation $x^2 + x - 2 = 0$. We would expect to get at most two solutions because to get the x^2 term we would have to have two factors of the form $x - a$ and $x - b$. We could guess that a possible solution might be ±2 or ±1. We picked these numbers because they are the factors of the constant term, –2.

If we evaluate the following, what happens?

$$\begin{aligned} f(1) &= 1^2 + 1 - 2 = 0 \\ f(-1) &= (-1)^2 - 1 - 2 = -2 \\ f(2) &= 2^2 + 2 - 2 = 4 \\ f(-2) &= (-2)^2 - 2 - 2 = 0 \end{aligned}$$

From this we can conclude that only 1 and –2, are solutions to the equation. This means that $x = 1$, and $x = -2$. If we rearrange these we get $x - 1 = 0$ and $x + 2 = 0$. These are the factors of $x^2 + x - 2$, this means that $x^2 + x - 2 = (x + 2)(x - 1)$.

The method we use here can be used to solve any polynomial equations. In general we might say the following.

If $P(x)$ is a polynomial function and $P(a) = 0$, then $x - a$, is a factor of $P(x)$ and $x = a$ is a solution to $P(x) = 0$.

Example

Find the solutions of $f(x) = x^3 + 2x^2 - 5x - 6 = 0$.

Try solutions that are factors of –6. Note we would expect to find at most three solutions, because to get the x^3 term in the expression we would have to have at most three factors of the form $x - a$, $x - b$ and $x - c$.

Possible solutions are ±2, ±1, ±3, ±6. By this guess and check method we get;

x	$f(x)$
1	–8
–1	0
2	0
–2	4
3	24
–3	0
6	252
–6	–120

The solutions are $x = -1$, $x = 2$, $x = -3$. Note that after you get three solutions there is no need to check further as this is the maximum number for a 3rd degree polynomial equation.

Example

What are the factors of the polynomial expression, $x^3 - 4x^2 - 11x + 30$?

Before we find the factors we need to find the possible solutions to the equation, $x^3 - 4x^2 - 11x + 30 = 0$.

Possibilities would be ±1, ±2, ±3, ±5, ±6, ±10, ±15, ±30. This is a lot of numbers to try, practicalities suggest that we would start with the smallest, easiest to calculate numbers first.

Possible solutions	Value of function
1	16
–1	36
2	0
–2	28
3	–12
–3	0
5	0

From these calculations we can see that the solutions are $x = 2$, $x = -3$, $x = 5$.

The factors of the expression would be $(x-2)$, $(x+3)$, $(x-5)$.

You could check the correctness of the answer by expanding the factors to get the original expression.

Activity 3.6

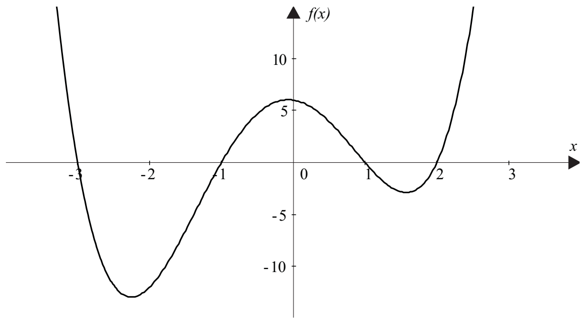
1. Factorize the following polynomial expressions:
   (a) $x^2 + 7x + 12$
   (b) $p^3 + 7p^2 + 12p$
   (c) $x^3 + x^2 - 25x - 25$
   (d) $a^3 + 2a^2 - a - 2$
   (e) $m^3 - 4m$
2. Find the solutions of the following polynomial equations:
   (a) $x^2 + 5x + 6 = 0$
   (b) $p^3 + 6p^2 - p - 30 = 0$
   (c) $3m^3 + 15m^2 + 18m = 0$
   (d) $x^3 - 2x^2 - 9x + 18 = 0$
   (e) $a^3 + 7a^2 - a - 7 = 0$

Let's now use some of these techniques to make some generalizations about polynomial functions.

However, before progressing further use Graphmatica to draw $f(x) = x^4 + x^3 - 7x^2 - x + 6$ for all real values of x, think about its characteristics and ask yourself the following questions.

- What will be the shape of the graph?
- Where will it cut the vertical axis?
- Where will it cut the horizontal axis?
- What is the rate of change of the dependent variable with respect to the independent variable?
- Is it a continuous function?
- What happens when the independent variable gets very large or very small?
- Will it have an inverse function?

Figure 3.15: $f(x) = x^4 + x^3 - 7x^2 - x + 6$



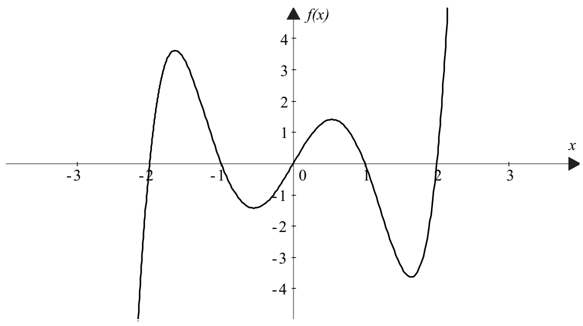
Now we can answer each of the above questions in turn.

- Functions to the power 2 had one turning point, those to the power 3 had at most two turning points so we would expect this function to have at most 3 turning points. The positive coefficient means that it will start in the top left corner of the graph and exits at the top right hand corner.
- It will cut the vertical axis at $f(x) = 6$, which is the value of the function at $x = 0$. Note that this is the constant on the right hand side of the function expression.
- It appears to cut the horizontal axis at 4 places $x = \pm 1$, $x = -3$, $x = 2$. These are the solutions to the equation $f(x) = 0$.
- The function appears to have 3 turning points where the rate of change would be zero or stationary. Between those points the rate of change goes from negative to positive, then to negative then finally positive. We could confirm this by evaluating the average gradient of the straight line connecting two points on the function in these regions.
- The function has no jumps or breaks and is continuous as are all polynomial functions with a domain of all real x.
- When x gets very large (approaches infinity), because the function is a polynomial of degree 4, it will approach positive infinity. When x approaches negative infinity the function will still approach positive infinity. As $x \to \pm\infty, f(x) \to \infty$.
- It is a many-to-one function so will not have an inverse function.

Now use Graphmatica again to draw $g(x) = x^5 - 5x^3 + 4x$ for all real values of x, think about its characteristics and ask yourself the following questions.

- What will be the shape of the graph?
- Where will it cut the vertical axis?
- Where will it cut the horizontal axis?
- What is the rate of change of the dependent variable with respect to the independent variable?
- Is it a continuous function?
- What happens when the independent variable gets very large or very small?
- Will it have an inverse function?

Figure 3.16: $g(x) = x^5 - 5x^3 + 4x$



You can now answer each of the above questions in turn.

- The degree of the function is 5 so we would expect it to have at most 4 turning points. The positive coefficient indicate that it will enter in the bottom left hand corner and exit at the top right hand corner.
- It will cut the vertical axis at $g(x) = 0$, which is the value of the function at $x = 0$. Note that this is the constant in the right hand side of the function expression.
- It appears to cut the horizontal axis at 5 places $x = \pm 1, x = 0, x = \pm 2$. These are the solutions to the equation $g(x) = 0$.
- The function appears to have 4 turning points where the rate of change would be zero or stationary. Between those points it changes from positive to negative to positive, then negative then finally positive. We could confirm this by evaluating the average gradient of the straight line connecting two points on the function in the different regions.
- The function has no jumps or breaks and is continuous as are all polynomial functions with a domain of all real x.
- When x approaches infinity, because the function is a polynomial of degree 5, we would expect it will approach positive infinity. When x approaches negative infinity the function will approach negative infinity. As $x \to \infty, g(x) \to \infty,\ x \to -\infty, g(x) \to -\infty$.
- The function is a many-to-one function and will not have an inverse.

Now you have examined a range of polynomials from the constant function through to the higher order polynomials above. Think carefully about what you have seen and write some generalizations about the behaviour of polynomial functions.

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In summary you might have said something like this:

- All polynomials will have one vertical intercept. This is calculated by finding the value of the function at $x = 0$.
- If a polynomial has an even degree of n, it will cut the horizontal axis between 0 and n times. If it has an odd degree than it will cut the horizontal axis between 1 and n times. The values of these intercepts are found by solving the equation $f(x) = 0$.
- The average rate of change of a polynomial function of degree n will vary continually within the function which can have up to $n - 1$ turning points.
- All polynomial functions are continuous in the real domain. They will approach $\pm\infty$ depending on the degree of the function and the coefficient of the variable with the highest power.
- Most polynomial functions are many-to-one functions and will not have inverse functions. Some polynomials such as linear functions and special cases of higher order functions such as $f(x) = x^3$ are one-to-one functions and will thus have an inverse function.

You may like to look over the polynomial functions and see if you can see these generalizations.

Something to talk about…

Using Graphmatica or a graphing calculator is a great tool to help understand polynomials. Have you found any short cuts or tricks to help with the process? Or are you having problems with it? Have you found any interesting graphs? Contact the discussion group and share your ideas with your fellow students. Might help you all when it comes to assignment and exam work!

Now that we have looked at polynomial functions in more detail and learnt a couple of techniques to help along the way, let's put all this together and investigate some real world situations using polynomial functions.

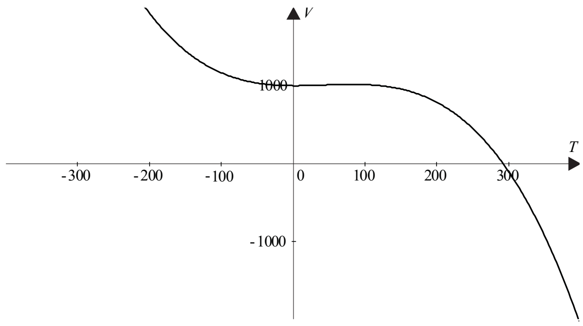
Example

The volume of 1 kg of water varies with temperature according to the formula:

$V = 1000 - 0.06T + 0.009T^2 - 0.00007T^3$, where V is measured in millilitres and T in °C over the domain $0 \le T \le 30$°C. Describe the behaviour of the volume of water using a graph and in words. Include in your description the average rate of change of volume between 0 and 15°C and between 15 and 30°C.

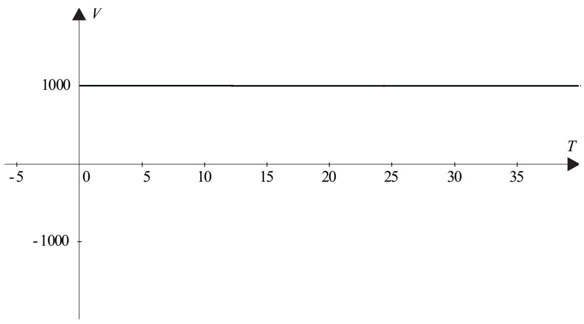
Using Graphmatica when the function is graphed over an unrestricted domain it looks like a cubic function we have graphed before, in that volume first decreases then remains stationary then decreases again as we have seen in the behaviour of $f(x) = -x^3$.

Figure 3.17: $V = 1000 - 0.06T + 0.009T^2 - 0.00007T^3$



However, if we zoom in to see what happens between 0 and 35, the graph looks like this:

Figure 3.18: Close up of $V = 1000 - 0.06T + 0.009T^2 - 0.00007T^3$ between 0 and 40 degrees



This appears linear and parallel to the horizontal axis. However, if we calculate the actual average rate of change between 0 and 15 and 15 and 30 we find something different.

We can examine the average rate of change by evaluating

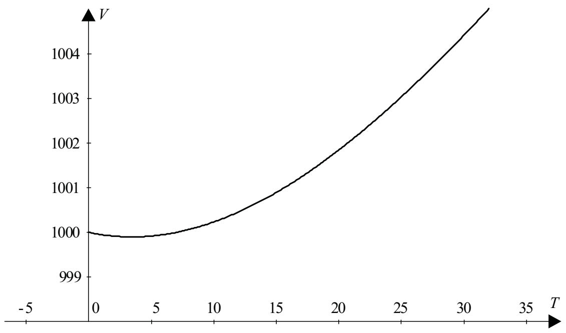
$$m = \frac{\text{change in height}}{\text{change in horizontal distance}} = \frac{\Delta V}{\Delta x} = \frac{V_{15} - V_0}{15 - 0} \approx \frac{1000.889 - 1000}{15} \approx 0.059$$

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}} = \frac{\Delta V}{\Delta x} = \frac{V_{30} - V_{15}}{30 - 15} \approx \frac{1004.41 - 1000.889}{15} \approx 0.23$$

Even though the graph in this domain looks to be linear, the rate of change has actually increased from 0.059 to 0.23 mL/°C, close to four times the original value.

If we zoom in further using Graphmatica (notice the scale on the vertical axis is now 1000 to 1004) we can see the behaviour of the function in more detail, which confirms the average rate of change calculations.

Figure 3.19: Close up of $V = 1000 - 0.06T + 0.009T^2 - 0.00007T^3$ between 1000 and 1004 mL



Example

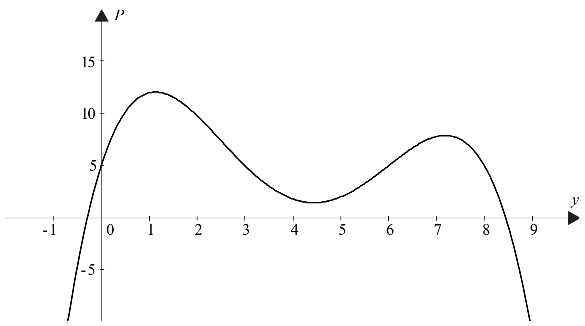
Population analysts are concerned about the difficulties of life in small country towns and have developed a polynomial model to help predict the number of people (in hundreds) in a typical town over 10 years. Their model was as follows:

$$P = -0.1y^4 + 1.7y^3 - 9y^2 + 14.4y + 5$$

Sketch a graph of the model, including a realistic domain for the function. What is the maximum population of the town in this model and what are the meanings of the vertical and horizontal intercepts?

Using Graphmatica the graph of the function is below.

Figure 3.20: $P = -0.1y^4 + 1.7y^3 - 9y^2 + 14.4y + 5$



As populations cannot be negative a realistic domain would be between 0 and 8.4 years. The latter figure determined using the coordinate curser on Graphmatica.

The maximum population was about 1200 people after 1 year of operation. Another peak occurred at about 7.2 years after the initial census. The vertical intercept is the number of people originally in the town in this case 500 people. The horizontal intercepts represent time when the population of the town was zero. Within the realistic domain this occurred about 8.4 years after the initial census.

Activity 3.7

1. Graph the following polynomial functions:

   (a) $y = x^4$

   (b) $y = x^4 + 2x^3 - 2x + 1$

   (c) $y = -x^4 + x^2 + 2$

   What was the vertical intercept of each function? How can you use the original equation to find this?

   What happens at the extremities for each function?

   What effect does the negative coefficient of the highest power have on the shape of the function in the third question?

2. Graph the following polynomial functions:

   (a) $y = x^5$

   (b) $y = x^5 + x^4 - 2x^3 + 2x - 1$

   (c) $y = -\dfrac{x^5}{2} + 3x + 2$

   What was the vertical intercept of each function?

   What happens at the extremities for each function?

3. Extending what you have done in the last two questions, can you guess what the following polynomial function may look like: $y = -\dfrac{x^6}{4} + x^4 - 1$?

   Where should it cross the vertical axis?

   What effect should the negative coefficient of the highest power have on the shape of the graph?

   Since the function has an even degree, what will happen to it at its extremities?

   Use Graphmatica to check your hypotheses.

4. Consider the following polynomial function: $y = (x-2)(x+1)^2(x-1)$. Without fully expanding the function, calculate the following:

   (a) the degree of the function;

   (b) where the function will cut the vertical axis;

   (c) where the function will cut the horizontal axis.

5. Using the information from question 4, draw a neat sketch of the polynomial function: $y = (x-2)(x+1)^2(x-1)$ and then use Graphmatica to check your result.

6. Over a 14 hour period, scientists gathered temperatures during a day and found that the fluctuations in temperature could be described using the following polynomial function:

$$T = 0.006t^4 - 0.18t^3 + 1.4t^2 - 0.9t + 5$$

Where T is the temperature in °C and t the time in hours after the first measurement at 4 am.

(a) What should the temperature have been at 6 am?

(b) If we were to graph this function, what would be the most number of turning points that you would expect it to have, and where should it cut the vertical axis?

(c) Draw a graph of the function (think very carefully about what range and domain you should use).

(d) Find the times between which the temperature is rising (i.e. the rate of change is positive).

(e) Find the times between which the temperature is falling (i.e. the rate of change is negative).

Time to reassess…..

You are over half way through this module now. How is it going? Now is a good time to reassess your action plan for study. Are you on schedule or do you need to upgrade it? You may need to contact your tutor if you are having problems keeping to your plan. If you have forgotten how to manage your action plan return to module 1 where you will find details. The key to success in this unit is keeping on track and keeping in touch with your tutor.

3.3.2 Exponential and logarithmic functions

You will have come across exponential and logarithmic functions in your previous studies. Module 2 summarizes these briefly. Let's look at the exponential functions briefly again. Simple versions are of the form $y = 10^x$ or $y = e^x$. More complicated versions can include the following examples. You might try to graph some of them using Graphmatica, but you will have to change the grid range to get a good view of the function.

Population growth of rabbits where f is the number of rabbits and n the number of seasons.	$f(n) = 10^n$
The population growth of Australia, where P is the number of people in millions and t is time in years.	$P(t) = 19.5e^{0.0163t}$ *
The time for a cup of coffee to cool, where C is the temperature in centigrade and t is time in minutes.	$C = 79.345e^{-0.0166t}$ *
The amount of a drug in the body where A is the amount in milligrams and t is time since ingestion in hours.	$A = 25 \times 0.8^t$

The atmospheric pressure p in Pascals varies with the height (h) in kilometres above sea level. p_0 is the pressure at sea level.	$p = p_0 e^{-0.15h}$ *
The charge q on a discharging capacitor is related to the time t, where Q_0 is the initial charge, R the resistance in the circuit and C the capacitance	$q = Q_0 e^{-\frac{t}{CR}}$ *

*If you are uncertain about the meaning of e go to module 2 for an explanation.

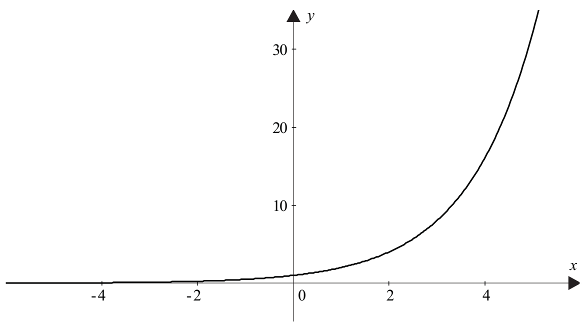
An important characteristic of exponential functions is the rapidity of their growth or decay. Examination of the average rate of change of an exponential function at different points of the function confirms this.

So if the rate of change of the function is so rapid what happens to functions when the value of the independent variable gets very large or very small. Think about two different functions now, $y = 2^x$ and $y = 2^{-x}$. Sketch both graphs on Graphmatica to get a feel for what is happening.

Let's think about the first function, $y = 2^x$. You might expect that when x gets very large i.e. when $x \to \infty$, then the value of the function, y, will get very large. Examination of the graph and calculation of some values of the function when x = 1000 (say), confirm this trend.

What happens when x gets very small i.e. $x \to -\infty$? It appears from the graph that the value of the function gets very close to 0. Let's try some values to confirm this.

Figure 3.21: $y = 2^x$



x	$y = 2^x$
0	1
-10	9.77×10^{-4}
-100	7.89×10^{-31}
-1000	9.33×10^{-302}

We can see that as x gets very small, the value of the function gets closer and closer to zero or symbolically as $x \to -\infty, y \to 0$. We say that the **limit** of the function, y, as x approaches negative infinity is zero or $\lim_{x \to -\infty} y = 0$. You might recognize this type of behaviour from your previous work on exponential functions. When a variable approaches a value, gets very close to it, but does not touch it, we can say that the value is an **asymptote** of the function. For the function $y = 2^x$, $y = 0$ or the x-axis is an asymptote.

Confirm for yourself that the asymptote for the function $y = 2^{-x}$ is also $y = 0$, because $\lim_{x \to \infty} y = 0$.

A further characteristic of exponential functions is the nature of the inverse of its function. All exponential functions will have an inverse function because they are one-to-one. You might recall that the inverse of an exponential function is a logarithmic function. To prove this consider $y = 10^x$. Recall from your previous studies or module 2 that to find the inverse of a function we would interchange the independent and dependant variables. In this case interchange the x and y variables, so that the function becomes $x = 10^y$. To make y the subject of the formula, as is usual in most functions, we rearrange the function using our knowledge of logarithms. Recall that $a = b^n$ is the same as $n = \log_b a$ (if you are unfamiliar with this return to the module 2 to refresh your knowledge of the definition of a logarithm and the associated logarithmic rules).

Using the definition of a logarithm	or	Using logarithmic rules
$x = 10^y$		$x = 10^y$
$y = \log_{10} x$		$\log_{10} x = \log_{10} 10^y$
		$\log_{10} x = y \log_{10} 10$
		$\log_{10} x = y$
		$y = \log_{10} x$
		Recall that $\log_{10} 10 = 1$

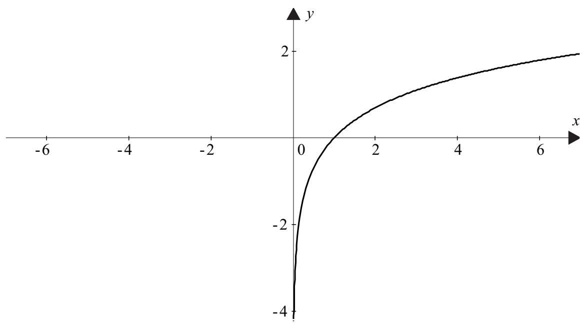
Before we complete some examples of the applications of logarithmic and exponential functions let's look at the logarithmic function in more detail.

Logarithmic functions have many applications in science, engineering and economics but by far their most frequent application is in scale development. The Richter scale for earthquakes, the decibel scale for sound intensity, the pH scale for acidity and the personal wealth scale are all based on logarithmic functions. The property of the logarithmic function that is made use of is the fact that as the value of the independent variable increases the rate of change of the function decreases. This means that large values of the independent variable result in only small changes in the value of the dependent variable. Let's look at the logarithmic function in more detail. Graph $y = \ln x$ on Graphmatica. (recall that $\ln x = \log_e x$) and ask yourself the following questions.

- What will be the shape of the graph?
- Where will it cut the vertical axis?
- Where will it cut the horizontal axis?
- What is the rate of change of the dependent variable with respect to the independent variable?
- Is it a continuous function?

- What happens when the variables get very large or small?
- Will it have an inverse function?

Figure 3.22: $y = \ln x$



You can now answer each question in turn.

- The graph will have a logarithmic form, and will be undefined for all values of x less than or equal to zero.
- To find where the function will cut the vertical axis put $x = 0$. If you try to calculate $y = \ln 0$ on your calculator an error message will appear. The function is not defined for $x = 0$, so the function does not cut the y-axis.
- To find where the function cuts the x-axis put $y = 0$ and solve

$0 = \ln x$	Use the definition of a logarithm to put in exponential form
$e^0 = x$	$0 = \ln x$
$x = 1$	Recall that $e^0 = 1$

  So the function cuts the x-axis at $x = 1$.

- As the value of the x variable increases the rate of change gets smaller. You can confirm this by evaluating the average rate of change connecting the straight line between two points.
- From the graph the function appears continuous over the domain $x > 0$.
- If we investigate the behaviour of the graph as $x \to +\infty$, we find the function appears unlimited or approaches infinity. Consider some points.

x	$y = \ln x$	
1	0.00	As x gets very large (approaches positive infinity) the y value also approaches infinity but very slowly. We write this as $x \to \infty, y \to \infty$.
10	2.30	
100	6.91	
10 000	9.21	
10 000 000	16.12	

Note that because logarithms are not defined for negative values we cannot find the values when $x \to -\infty$.

Because the function is undefined for $x \leq 0$, it is interesting to investigate what happens to the y value as x approaches zero. From the graph it appears that as x approaches zero then y gets very small i.e. approaches negative infinity.

x	$y = \ln x$	
1	0	As we found before, as x approaches zero then y approaches negative infinity or as $x \to 0, y \to -\infty$.
0.1	–2.30	
0.01	–4.61	
0.0001	–9.21	
0.0000001	–16.12	

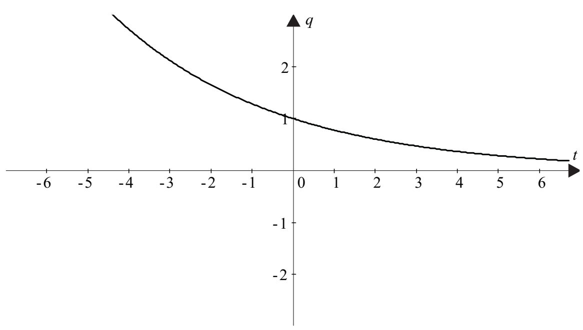
This behaviour means that $x = 0$ (the y axis) must be a **vertical asymptote**.

- Of course the function is one-to-one function and will have an inverse. As previously stated the inverse of $y = \ln x$ is $y = e^x$.

Example

A capacitor is part of an electrical circuit that stores electric charge. The quantity of the charge stored (q), measured in micro Coulombs (μC), decreases exponentially over time, t, (seconds) depending on the formula $q = e^{-\frac{t}{4}}$. Graph the function on Graphmatica and describe in your own words how the stored charge changes over time. Include in your description, values and meaning of the vertical and horizontal intercepts (if they occur), estimates of the average rate of change and what the horizontal asymptote tells us about the charge.

Figure 3.23: $q = e^{-\frac{t}{4}}$



The curve drawn above is over the entire real domain but because the function involves stored charge over time we would expect the domain of the function to be $t \geq 0$. The initial value of the function occurs when $t = 0$. If we substitute this into the formula the initial charge stored is 1 μCoulombs. If we examine the average change in the charge after 0.1 seconds and between 0.2 seconds and 0.3 seconds we find that the charge is decreasing at a decreasing rate.

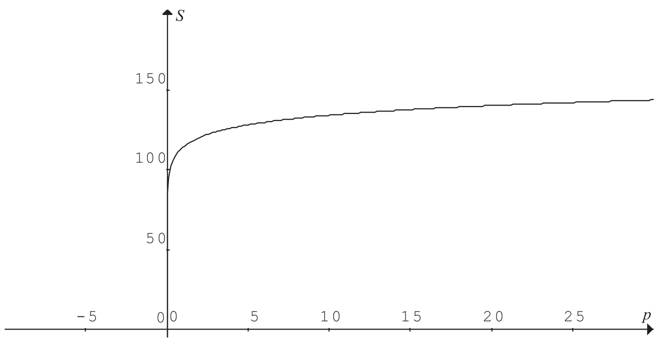
Time (seconds)	Stored Charge (μC)	Rate of change
0	1.000	
0.1	0.975	$\frac{0.975-1}{0.1-0} \approx -0.25$
0.2	0.951	
0.3	0.928	$\frac{0.928-0.951}{0.4-0.3} \approx -0.23$

This will continue on indefinitely because the function has an asymptote at $q = 0$, and as t increases so theoretically q will approach zero. However, in real terms the value of the charge would be unmeasurable after about 60 seconds.

Example

The sound pressure level (S) measured in decibels in a room is given by the formula $S = 20\log(50000p)$, where p is the sound pressure in Pascals. Sketch a graph of the function, determine and interpret the vertical and horizontal intercepts (if they exist) and the vertical asymptote.

Figure 3.24: $S = 20\log(50000p)$



To determine the vertical intercept $p = 0$ put in the formula

$$S = 20\log(50000p)$$
$$S = 20\log(50000 \times 0)$$
$$S \text{ is undefined}$$

The function does not intersect the vertical axis and has an asymptote in this position. This can be confirmed by evaluating the function for values of p close to zero.

The horizontal intercept is found by putting $S = 0$ in the formula

$$\begin{aligned} S &= 20\log(50000p) \\ 0 &= 20\log(50000 \times p) \\ 0 &= \log(50000 \times p) \\ 10^0 &= 50000 \times p \\ 1 &= 50000 \times p \\ p &= \frac{1}{50000} \\ p &= 0.00002 \end{aligned}$$

The horizontal intercept is equivalent to 0.00002 Pascals of sound pressure. This is the baseline level for the decibel scale for the sound level is zero when the pressure is 0.00002 Pascals. (This is usually chosen as the reference level as it is the lowest perceptible sound pressure for the human ear.)

Activity 3.8

1. It is estimated that the population of Australia has increased by a factor of 6 since 1894. A mathematical model which would resemble this growth is:

   $$P = 3.2e^{0.0169t}$$

   where: t is the time in years since 1894 and P the population in millions.

   Using the above mathematical model, find:

   (a) The population of Australia in 1894.

   (b) The expected population in the year 2010.

   (c) When the population of Australia had reached 10 million.

   (d) How quickly the population was changing in 1954 and 1955.

   (e) Explain the significance of the y-intercept.

2. Experts claim that the world wide web is currently growing exponentially at a rate of 20% every year (Lemay,1999). If this claim is true, then a mathematical model for the growth of the web might be:

   $$N = N_0(1.2)^t$$

   where: N is the number of users in millions.
   N_0 the number of current users.
   t the time in years from the year 1999.

   Use the model to find:

   (a) Calculate the average rate of growth in number of users between 1999 and 2004.

   (b) Calculate the average rate of change in the number of users between 2004 and 2009.

   (c) Write a sentence comparing the two average rates of growth.

3. Radioactive substances decay according to the exponential model:

   $$A = A_0 e^{-kt}$$

   where: A is the amount of the substance present after t years
   A_0 is the amount of the substance originally present.
   k is a constant which depends on the material itself.

   Radium (Ra_{226}) has a half life of 1620 years. That means it takes 1620 years for one gram of Radium to decay to one half a gram.

   (a) Use this information to calculate the value of k if we were to use the equation $A = A_0 e^{-kt}$ to model the decay of Radium.

   (b) If 10g of Radium were released inadvertently from a nuclear reactor, how long would it take for this amount to break down to 1g?

   (c) How quickly would the original 10g be decaying after 5 years?

4. A cancerous cell is known to double every 24 hours. A pathologist isolates 50 of these cancerous cells and allows them to multiply freely.

   (a) Construct a table showing the number of cancerous cells which should be present each day for a week.

   (b) Draw a graph showing this change in the number of cancerous cells over the week.

   (c) What mathematical function should describe this change in the number of cells?

5. The effectiveness of any voltage amplifier is dependent on the frequency of the voltage signals. This occurs in a region of frequencies called the midband. The voltage gain in this midband, called the midband gain, is given by the expression:

   $$G = 20\log\left(\frac{v_o}{v_i}\right)$$

   where: G is the midband gain in decibels
   v_o is the output voltage
   v_i the input voltage.

   A number of amplifiers were tested, each with a fixed input voltage of 50 mV. The resulting output voltages ranged from 50 mV to 200 mV.

   (a) Calculate the midband gain for an amplifier which produced an output voltage of 160 mV.

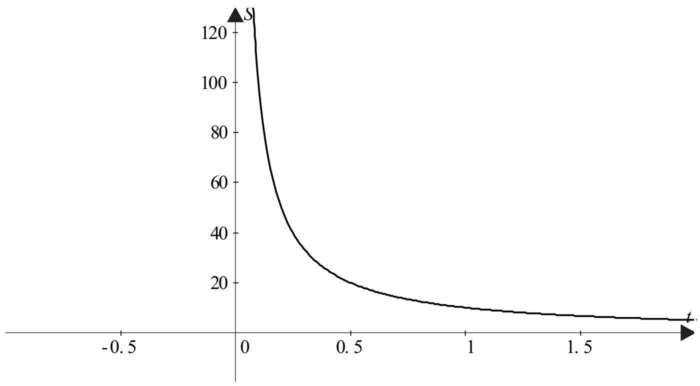
   (b) Draw a graph showing the midband gain against the output voltage for these amplifiers.

   (c) How quickly would the midband gain be changing when an amplifier was producing an output voltage between 100 mV and 105 mV?

3.3.3 Rational functions

In our investigation of functions so far we have looked at functions that you have probably seen in your past studies, now let's look at something completely different. We all know that when we drive anywhere the speed we travel at is dependent on the distance travelled and the time taken to get there. So if we were to travel 10 kilometres, we could determine the speed from the function, $S = \frac{10}{t}$. We say that the speed attained is **indirectly or inversely related** to the time taken. If we graph this function on Graphmatica over the domain $t \geq 0$ we get the following shape.

Figure 3.25: $S = \frac{10}{t}$



But what is happening with this function? As you go faster and faster you take less time to travel the 10 kilometres. That's probably self evident. But what are the limits on the function? For example, when you drive, can you ever travel so fast that it will take you zero time? Probably not in this universe. On the other hand will it take you so long to get there that you travel at zero speed (assuming that of course you do leave). In this function when time gets very large (i.e. $t \to \infty$) speed will get extremely small but can never actually reach zero. We say that the **limit** of the function, S, as time, t, approaches infinity is zero or in symbolic notation $\lim_{t \to \infty} S = 0$. We have seen this type of behaviour before in exponential and logarithmic functions and called the values that the variables approach but can never reach or touch, **asymptotes**. In this function then, we will have two asymptotes $t = 0$ and $S = 0$.

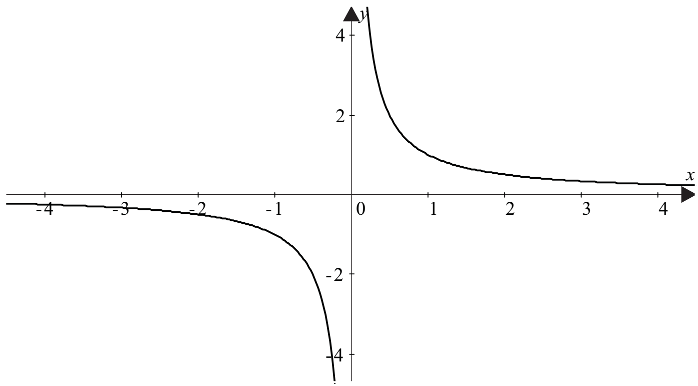
But there is much more to this function than what we have discussed above. You might have noticed when you were drawing the graph on Graphmatica that there appeared to be another part of the graph in the lower left hand corner of the window. Let's investigate this in more detail by examining the more general function, $y = \frac{1}{x}$, for all real values of x. Consider the function $y = \frac{1}{x}$ and ask yourself the following questions:

- What is the shape of the graph?
- Where will it cut the vertical axis?
- Where will it cut the horizontal axis?

- What is the rate of change of the dependent variable with respect to the independent variable?
- Is it a continuous function?
- What happens when the variables get very large or small?
- Will it have an inverse function?

Sketch the graph of $y = \frac{1}{x}$ for the domain of all real values of x on Graphmatica and think about its characteristics.

Figure 3.26: $y = \frac{1}{x}$



Well we were right. It certainly looks very different over the real domain from any graph we have examined before, in that it comes in two parts and so is not continuous. Let's go through the questions above one at a time so we can get an understanding of the behaviour of this type of function.

- The graph is a curve in two parts in diagonally opposite quadrants of the plane. It appears to have discontinuities along the axes, when examined over the real domain.
- To find where this function cuts the vertical axis we would put $x = 0$. If we do this we get $y = \frac{1}{0}$. We know from previous work that this is one of the operations that cannot be performed in real arithmetic and so the expression is undefined. This means that the function will never cut the y-axis. There will be a vertical asymptote at this point and the function is discontinuous.
- To find where this function cuts the horizontal axis we would put $y = 0$. If we do this in this function we get $0 = \frac{1}{x}$. Again in the real number system we cannot find a number such that when we divide 1 by it we will get zero. Again the function is undefined for $y = 0$. This means that the function will never cut the x-axis. There will be a horizontal asymptote at this point.
- The rate of change of the function like most curves changes continually, with no turning points as are evident in polynomial functions. In both parts of the function the rate of change is negative and the function decreasing. We can confirm this by estimating the

average rate of change using the gradient of a straight line connecting two points of the function.

$m = \dfrac{\text{change in height}}{\text{change in horizontal distance}} = \dfrac{\Delta y}{\Delta x}$. If we do this for the points,

$(-3,-\frac{1}{3}),(-2,-\frac{1}{2}),(3,\frac{1}{3}),(2,\frac{1}{2})$

Between points	Δy	Δx	m
$(-3,-\frac{1}{3}),(-2,-\frac{1}{2})$	$\frac{1}{6}$	-1	$-\frac{1}{6}$
$(3,\frac{1}{3}),(2,\frac{1}{2})$	$-\frac{1}{6}$	1	$-\frac{1}{6}$

- From the graph it appears that the function is not continuous over the real domain but has a break in the middle. This was confirmed when we attempted to evaluate the vertical and horizontal intercepts. The function is undefined for $x = 0$, $y = 0$.
- If we investigate the behaviour of the graph as $x \to \pm\infty$, we find the function approaches zero. But how does it do this? Consider some points.

x	$y = \frac{1}{x}$	
1	1	As x gets very large (approaches positive infinity) the y value approaches zero from the positive direction i.e. from above the x-axis. We write this as $\lim\limits_{x\to\infty} \frac{1}{x} = 0^+$
10	0.1	
100	0.01	
1000	0.001	
10000	0.0001	
–1	–1	As x gets very small (approaches negative infinity) the y value approaches zero from the negative direction i.e. from below the x-axis. We write this as $\lim\limits_{x\to-\infty} \frac{1}{x} = 0^-$
–10	–0.1	
–100	–0.01	
–1000	–0.001	
–10000	–0.0001	

This behaviour means that $y = 0$ (the x-axis) must be a **horizontal asymptote**.

To investigate what happens when y gets very large or small it is useful to investigate what is happening around the vertical asymptote i.e. what happens as x approaches zero.

Now what happens when y gets very large or very small.

x	$y=\frac{1}{x}$	
1	1	As x approaches 0 from the positive direction y gets very large and approaches positive infinity. Write this as $x \to 0^+$, $y \to +\infty$
0.1	10	
0.001	1000	
0.000001	1000000	
–1	–1	As x approaches 0 from the negative direction y gets very small and approaches negative infinity. Write this as $x \to 0^-$, $y \to -\infty$.
–0.1	–10	
–0.001	–1000	
–0.000001	–1000000	

This behaviour assures us that $x = 0$ (the y-axis) must be a **vertical asymptote**.

Note a short cut way to determine the position of the vertical asymptote is to think about where the function is undefined. In this case it will be undefined at $x = 0$, so this must be the vertical asymptote.

- The function is a one-to-one function over its domain so it will possess an inverse function. Note the domain of the function is all real x except $x = 0$.

This type of function is called a **rectangular hyperbola** and is a member of a group of functions called **rational functions**. A rational function has the following form.

> $f(x)=\frac{p(x)}{q(x)}$**, where $p(x)$ and $q(x)$ are polynomials and where x takes all real values except those which make $q(x)=0$.**

Note that functions of the form $y=\frac{3}{x+1}-2$ are also rational functions because they can be rearranged into the correct form.

$$y=\frac{3}{x+1}-2$$

$$y=\frac{3}{x+1}-2\times\frac{x+1}{x+1}$$

$$y=\frac{3-2(x+1)}{x+1}$$

$$y=\frac{1-2x}{x+1}$$

Put right hand side of the expression over a common denominator of $x + 1$.

These functions occur in nature in a variety of forms. Many of them can be described using the words **inversely or indirectly proportional** to a variable, just as speed was inversely proportional to time in our first example (we symbolize this by $S \propto \frac{1}{t}$). Other real world examples include the following.

- Fish tail beat frequency (f) is inversely proportional to length of fish (L) $f \propto \frac{1}{L}$.
- Newton's Law of Gravitation Force, F, for two different masses (m_1, m_2), $F = \frac{6.67 \times 10^{-11} m_1 m_2}{r^2}$ is a rational function with the independent variable, r, being the distance separating the masses.
- Boyle's Law for a gas at fixed temperature says that pressure (P) is inversely related to volume (V), $P = \frac{k}{V}$, where k is a constant.
- Pressure (P) is inversely related to the area (A) over which it is exerted, $P \propto \frac{1}{A}$.
- The resistance in an electrical circuit (R) of constant voltage is inversely related to the current (I) in the circuit, $R = \frac{V}{I}$.
- The total energy (E) of an electron is related to the square of its distance (r) from the nucleus, $E \propto \frac{1}{r^2}$.

So far we have looked at an example of a rational function that has asymptotes at the x and y-axes. Let's have a look at some examples where the asymptotes occur elsewhere.

Example

Examine the behaviour of the function, $y = \frac{2}{x} + 1$, include in your answer confirmation that this function is a rational function and the exact values of the vertical and horizontal intercepts, any asymptotes and a hand drawn sketch of the graph.

To show that this function is a rational function, we have to rewrite the given equation into a rational form.

$$y = \frac{2}{x} + 1$$
$$y = \frac{2}{x} + \frac{1 \times x}{x}$$
$$y = \frac{2 + x}{x}$$

The right hand side of the equation is now written in the form of one polynomial divided by a second polynomial so the function is a rational function.

To find the vertical intercept put $x = 0$.

$$y = \frac{2}{0} + 1$$

The function has no vertical intercept and is undefined at $x = 0$. This means that $x = 0$ must be the vertical asymptote.

To find the horizontal intercept, put $y = 0$.

$$\begin{aligned} 0 &= \frac{2}{x} + 1 \\ -1 &= \frac{2}{x} \\ -1(x) &= 2 \\ x &= -2 \end{aligned}$$

The horizontal intercept is $x = -2$.

To find the vertical asymptote of the function, consider where the domain of the function will be undefined. We know this from our previous attempts to calculate the vertical intercept, but to confirm the accuracy of it try a few points when y gets very small and large.

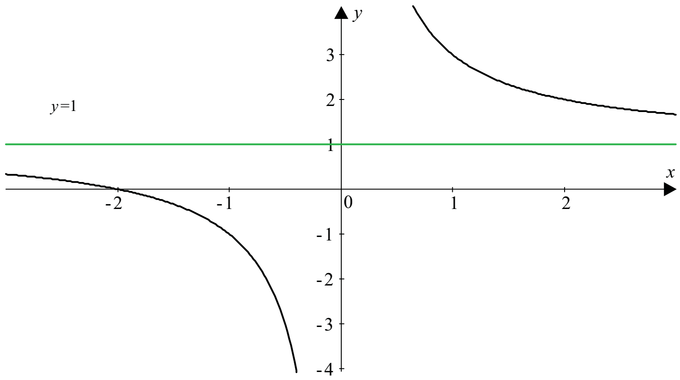
To find the horizontal asymptote investigate what happens to the function when x gets very large or small. We could try some points to see what happens.

x	$y = \frac{2}{x} + 1$	
10	1.2000	As x gets very large (approaches positive infinity) the y value approaches 1 from the positive direction i.e. from above the line $y - 1 = 0$. We write this as $\lim_{x \to \infty}(\frac{2}{x} + 1) = 1^+$.
100	1.0200	
1000	1.0020	
10000	1.0002	
–10	0.8000	As x gets very small (approaches negative infinity) the y value approaches 1 from the negative direction i.e. from below the line $y - 1 = 0$. We write this as $\lim_{x \to -\infty}(\frac{2}{x} + 1) = 1^-$
–100	0.9800	
–1000	0.9980	
–10000	0.9998	

It appears that as x gets very large or very small the value of the function approaches $y = 1$. This must be the horizontal asymptote.

Using this information and our understanding of the general shape of this type of rational function we can now sketch the curve.

Figure 3.27: $y = \frac{2}{x} + 1$, with asymptote at $y = 1$



Check your sketch by drawing it on Graphmatica or plot a few extra points to confirm the position of the 'arms' of the function.

Example

Examine the behaviour of the function, $y = \frac{2}{x-1} - 3$, include in your answer confirmation that this function is a rational function and the exact values of the vertical and horizontal intercepts, any asymptotes and a hand drawn sketch of the graph.

To show that this function is a rational function, we have to rewrite the given equation into a rational form.

$$y = \frac{2}{x-1} - 3$$
$$y = \frac{2}{x-1} - \frac{3(x-1)}{x-1}$$
$$y = \frac{2-3(x-1)}{x-1}$$
$$y = \frac{5-3x}{x-1}$$

The right hand side of the equation is now written in the form of one polynomial divided by a second polynomial so the function is a rational function.

To find the vertical intercept put $x = 0$.

$$y = \frac{2}{0-1} - 3$$
$$y = -5$$

The vertical intercept is $y = -5$. Note that this time the vertical intercept exists and the y-axis is not an asymptote.

To find the horizontal intercept, put $y = 0$.

$$\begin{aligned}0 &= \frac{2}{x-1} - 3\\ 3 &= \frac{2}{x-1}\\ 3(x-1) &= 2\\ 3x - 3 &= 2\\ x &= \frac{5}{3}\end{aligned}$$

The horizontal intercept is $x = \frac{5}{3}$.

To find the vertical asymptotes of the function, consider where the function will be undefined. From the equation $y = \frac{2}{x-1} - 3$ it is apparent that it is undefined when $x - 1 = 0$ or when $x = 1$. We can confirm this by examining the behaviour of the function around the asymptote.

The vertical asymptote is at $x = 1$.

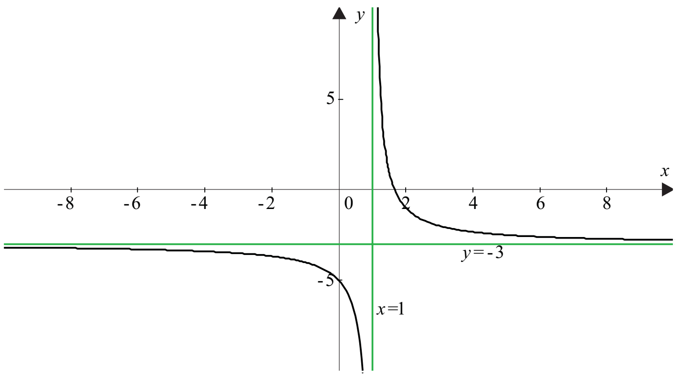
To find the horizontal asymptote investigate what happens to the function when x gets very large or small. We could try some points to see what happens.

x	$y = \frac{2}{x-1} - 3$	
10	–2.7778	As x gets very large (approaches positive infinity) the y value approaches –3 from the positive direction i.e. from above the line $y + 3 = 0$. We write this as $\lim\limits_{x\to\infty}(\frac{2}{x-1} - 3) = -3^{+}$
100	–2.9798	
1000	–2.9980	
10000	–2.9998	
–10	–3.1818	As x gets very small (approaches negative infinity) the y value approaches zero from the negative direction i.e. from below the line $y + 3 = 0$. We write this as $\lim\limits_{x\to-\infty}(\frac{2}{x-1} - 3) = -3^{-}$
–100	–3.0198	
–1000	–3.0020	
–10000	–3.0002	

It appears that as x gets very large or very small the value of the function approaches $y = -3$. This must be the horizontal asymptote.

Using this information and our understanding of the general shape of this type of rational function we can now sketch the curve.

Figure 3.28: $y = \dfrac{2}{x-1} - 3$, with asymptotes at $x = 1$ and $y = -3$



Check your sketch by drawing it on Graphmatica or plot a few extra points to confirm the position of the arms of the function.

So far to investigate the behaviour of any graph we have looked at a series of questions, we need to add one more question to include the concept of asymptotes.

- What is the shape of the graph?
- Where will it cut the vertical axis?
- Where will it cut the horizontal axis?
- What is the rate of change of the dependent variable with respect to the independent variable?
- Is it a continuous function?
- Will the function have any vertical or horizontal asymptotes?
- What happens when the variables get very large or small?
- Will it have an inverse function?

Something to talk about…

Will all rational functions be discontinuous? What about the function $y = \frac{1}{x}$ over the domain $1 < x < 10$? How will the domain effect the discontinuities of the function? Share you ideas with your colleagues or the discussion group.

Activity 3.9

1. Use Graphmatica to sketch the functions, $y = \frac{1}{x}$ and $y = -\frac{1}{x}$. Compare the behaviour and shape of these two functions.
2. In 1650, Robert Boyle discovered experimentally that if the temperature of a certain mass of gas is kept constant, then the product of its volume (V) and pressure (P) is constant. This is usually written: $PV =$ constant. In an experiment 14 mg of nitrogen gas is placed in a container at 27°C, and it is determined that for this particular gas $PV = 1.25$, (where the volume is measured in and the pressure in N/m^2). If the volume of the container is varied (in much the same way that the volume in a bicycle pump cylinder is varied) and the pressure measured, then the relationship between the two should be:

   $$P = \frac{1.25}{V}$$

   (a) Use Graphmatica to sketch the function $P = \frac{1.25}{V}$.

   (b) What restrictions should be placed on the domain if it is to accurately represent the above situation?

   (c) Use the graph to determine what will happen to the pressure of the gas when the volume is made very large?

3. Examine the behaviour of the function, $y = \frac{3}{x+1}$, by answering the following question.

   (a) Confirm that the function is a rational function.

   (b) Find the exact values of the vertical and horizontal intercepts.

   (c) Determine the location of the asymptotes.

   (d) Draw a neat sketch of the function and confirm your analyses using Graphmatica.

4. Examine the behaviour of the function, $y = \frac{-2}{x-3} + 2$, by answering the following questions.

   (a) Confirm that the function is a rational function.

   (b) Find the exact values of the vertical and horizontal intercepts.

   (c) Determine the location of the asymptotes.

   (d) Draw a neat sketch of the function and confirm your analyses using Graphmatica.

5. A variable resistor was placed into a direct current circuit. The resistance (R) was varied and the subsequent change in current (I) measured. The magnitude of the current (in amps) was found to be given by the function, $I = \dfrac{12}{R}$, where R was the resistance, measured in ohms.

   (a) Determine the average rate of change of current when the resistance changes from 1.9 to 2 ohms.

   (b) Determine the average rate of change of current when the resistance is changes from 14.9 to 15 ohms.

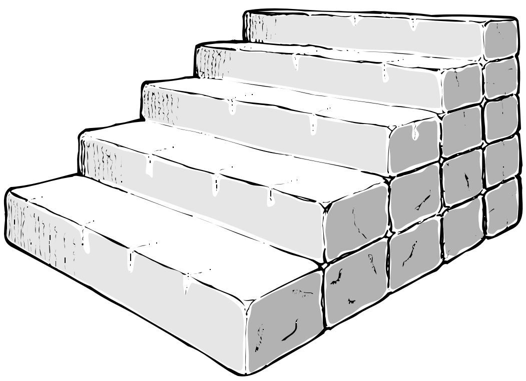
   (c) What domain restrictions should be placed on the above function?

   (d) Draw a neat sketch of the function and check your results using Graphmatica.

3.3.4 Functions over an integral domain

Functions come in all shapes and sizes. So far we have studied functions with a domain of all real numbers, except for rational functions in which specific values of the independent variable were undefined. In this section we will look at some special functions which have the set of positive integers as the domain.

If you were building a set of rustic stone stairs at home you would stack stones of similar size one on top of another to make a stair shape.



The total number of stones used is the sum of the number of stones on each level.

Number of stones is 1 + 2 + 3 + 4 + 5 = 15.

If we wanted to build higher steps, then the next level would have 6 stones and the level after that would have 7 stones. We can determine the number of stones to be used by adding one to each previous level. So to determine the total number of stones in the stairs we could write this as a function $f(n) = 1 + 2 + 3 + 4 + 5 + 6 + \ldots\ldots + n$, where n is the number of levels.

(Check for yourself that this satisfies the definition of a function stated earlier in this module.)

The numbers that make up this sum are called an **arithmetic sequence**, because each level has one more stone than the previous level.

An arithmetic sequence is a list of numbers in which the terms increase or decrease by adding a constant.

In the stone steps example the constant was one, but in other examples any real constant can be added to an initial term to generate an arithmetic sequence.

Here are some examples.

2, 4, 6, 8, 10, 12, 14,……	Add 2 to each term
10, 7, 4, 1, –2, –5, –8,……	Add –3 to each term
1, 2.2, 3.4, 4.6, 5.8,…….	Add 1.2 to each term
$3, 3+\sqrt{2}, 3+2\sqrt{2}, 3+3\sqrt{2},....$	Add $\sqrt{2}$ to each term

There are two things that are useful to know about arithmetic sequences; how can you calculate any term of the sequence and what is the sum of the sequence for any number of terms? Let's look at these two things now.

A bit of history… **For interest only**

Carl Gauss was a famous German mathematician who lived between 1777 and 1855. He was a child prodigy. '*One day, in order to keep the class occupied, the teacher had the students add up all the numbers from 1 to 100, with instructions that each should place his slate on a table as soon as he had completed the task. Almost immediately Carl placed his slate on the table saying "There it is". The teacher looked at him scornfully while others worked diligently. When the instructor looked at the results, the slate of Gauss was the only one with the correct answer, 5050, with no further calculations. The ten year old boy evidently had computed mentally the sum of the arithmetic progression, presumably using the formula* $\frac{m(m+1)}{2}$.' (Boyer 1991). How did he do that? Well nobody knows for certain but it is thought that he did the following.

$$
\begin{aligned}
S_{100} &= 1+2+3+4+........98+99+100 \\
&= \underbrace{(1+100)+(2+99)+(3+98)+......+(50+51)}_{50\text{pairs}} \\
&= \underbrace{101+101+101+.....+101}_{50\text{terms}} \\
&= 50\times 101 \\
&= 5050
\end{aligned}
$$

Do not learn this.

(Boyer, C & Merzbach, U 1991, *A history of mathematics*, Wiley & Sons, New York.)

If the first term of an arithmetic sequence is a_1 and the constant we would add is *d*, then the second term would be

No need to learn this proof

$$a_2 = a_1 + d$$
$$a_3 = a_2 + d = \underbrace{a_1 + d}_{a_2} + d = a_1 + 2d$$
$$a_4 = a_3 + d = \underbrace{a_1 + 2d}_{a_3} + d = a_1 + 3d$$
$$a_5 = a_4 + d = \underbrace{a_1 + 3d}_{a_4} + d = a_1 + 4d$$

You might have noticed the pattern that has developed.

The nth (a_n) term of an arithmetic sequence is given by, $a_n = a_1 + (n-1)d$, where a_1 is the first term and *d* the constant.

We can use a similar algebraic technique to find the sum of *n* terms of an arithmetic sequence.

Let the sum of the *n* terms above be, S_n, so that

$$S_n = a_1 + a_2 + a_3 + a_4 + \ldots\ldots\ldots\ldots\ldots\ldots + a_{n-1} + a_n$$
$$= (a_1 + a_n) + (a_2 + a_{n-1}) + (a_3 + a_{n-2}) + \ldots\ldots.$$

Pair together the first and last terms, the 2nd and 2nd last terms etc.

But, $a_1 + a_n = a_1 + \underbrace{a_1 + (n-1)d}_{a_n} = 2a_1 + (n-1)d$

And $a_2 + a_{n-1} = \underbrace{a_1 + d}_{a_2} + \underbrace{a_1 + (n-2)d}_{a_{n-1}} = 2a_1 + (n-1)d$

Convince yourself this is correct by expanding the brackets on both sides

It appears that each pairing of the terms gives a sum of $2a_1 + (n-1)d$. So how many pairs of such terms are there?

If there were *n* terms to start with and we have broken them up into groups of two there must now be $\frac{n}{2}$ pairs. So,

No need to learn this proof

$$S_n = a_1 + a_2 + a_3 + a_4 + \ldots\ldots\ldots\ldots\ldots\ldots + a_{n-1} + a_n$$
$$= (a_1 + a_n) + (a_2 + a_{n-1}) + (a_3 + a_{n-2}) + \ldots\ldots.$$
$$= \frac{n}{2} \times \left[2a_1 + (n-1)d\right]$$

The sum of the first *n* terms of an arithmetic sequence is $S_n = \frac{n}{2} \times \left[2a_1 + (n-1)d\right]$, where a_1 is the first term and *d* the constant.

Note that the sum of the terms of an arithmetic sequence is called an **arithmetic series**, and *d* is often called an **arithmetic or common difference**.

Example

Find the 100th term and the sum of the first 100 terms of the arithmetic sequence whose first four terms are 6, 3, 0, –3.

The first term of the sequence is 6, so $a_1 = 6$. To find the constant, *d*, that is added to each term subtract the first two terms, $d = 3 - 6 = -3$. To confirm this, subtract the next two terms, $d = 0 - 3 = -3$.

To find the 100th term we can use the formula, $a_n = a_1 + (n-1)d$, where $n = 100$, $d = -3$, $a_1 = 6$
$a_{100} = a_1 + (n-1)d = 6 + (100-1) \times -3 = -291$.

To find the sum of the first 100 terms use, $S_n = \frac{n}{2} \times \left[2a_1 + (n-1)d\right]$,

$$S_{100} = \frac{100}{2} \times \left[2 \times 6 + (100-1) \times -3\right] = -14\,250.$$

So the 100th term is –291, and their sum is –14 250.

Example

An young worker commences work on an annual salary of $25 000 and will receive an annual increment of $2500. When will she earn $50 000 per year and what would have been her total earnings up to this time.

The worker's income is increasing by a constant value of $2500 per year, so the income earned each year represents an arithmetic sequence. In the sequence $a_1 = 25000, d = 2500$.

To find when the apprentice will earn $50 000 per year, find the term which has a value of $50 000, using the formula for the nth term, $a_n = a_1 + (n-1)d$ where the nth term is 50 000, $a_1 = 25\,000,\ d = 2\,500$

$$\begin{aligned} 50\,000 &= 25\,000 + (n-1) \times 2\,500 \\ 25\,000 &= (n-1) \times 2\,500 \\ n-1 &= 10 \\ n &= 11 \end{aligned}$$

After 11 years of work the worker will earn $50 000.

To find the total amount earned over this time you need to calculate the sum of 11 terms of the arithmetic sequence using the formula $S_n = \frac{n}{2} \times \left[2a_1 + (n-1)d\right]$.

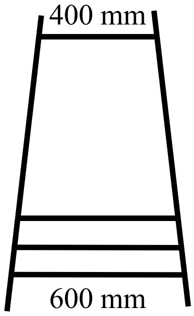
$$S_{11} = \frac{11}{2} \times \left[2 \times 25\,000 + (11-1) \times 2\,500\right] = 412\,500$$

The total income over 11 years is $412 500.

Activity 3.10

1. What will be the 52nd term of the sequence: 12, 15, 18, . . . ?
2. On the birth of her nephew, Aunt Beatrice deposited \$50.00 in a trust account. On his first birthday she deposited \$100, on his second birthday she deposited \$200 and on his third birthday \$300. If she continued in this fashion until his 21st birthday, how much in total will she have deposited in this trust account?
3. Find the sum of the first 20 terms of the following series:

   2 + 8 + 14 + . . .
4. Engineers have estimated that wheel bearings begin to reduce in size very quickly when lubrication has failed. In one trial using bearings of radius 5.000 mm, it was found that in one hour of use the radius decreased by 25μm. After the second hour of use in the trial the radius decreased by a further 40μm and after the third it decreased by 55μm.
   (a) If this pattern continued in the trial, how much would be removed from the radius of the bearing after the 15th hour?
   (b) How long would it take before the wheel bearing is reduced to one half of its original size?
5. A step ladder has a number of rungs which decrease uniformly from 600 mm on the base to 400 mm on the top (see diagram below). If a total of 3500 mm of wood is used to make the rungs, how many are in the ladder?



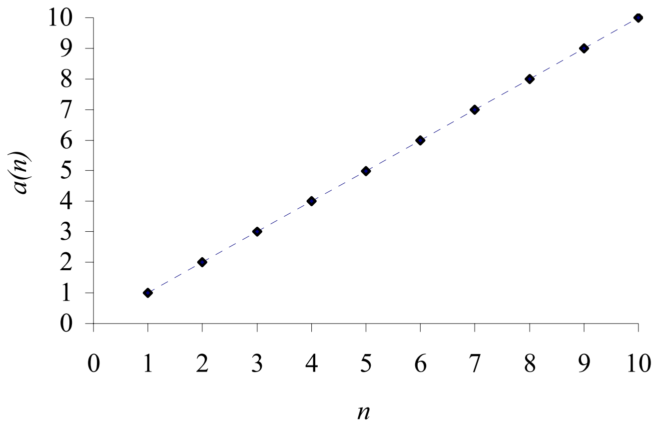
Now that you are more familiar with the nature of arithmetic sequences and series let's go back and see why we have included these in the section on functions.

Think again about the problem with the stone steps. In the design of this the number of stones in each level was part of an arithmetic sequence. The *n*th term of that sequence is given by the formula

$a(n) = a_1 + (n-1)d$, where $a_1 = 1,\ d = 1$, so
$a(n) = 1 + (n-1)\times 1 = n$

So the function defining the *n*th term is $a(n) = n$. This function is a linear function with a restricted domain of all positive integers. It is not a continuous function, but as *n* increases the value of the function approaches infinity. Notice however, that it compares well with previous linear functions in that terms of the function are calculated by adding a constant to a previous term. The function would look like this.

Figure 3.29: *n*th term of an arithmetic sequence, $a(n) = n$



(the dashed line indicates that the function is not continuous between the points indicated)

If the constant had been a negative number then the function for finding the *n*th term would still be a linear function but the gradient would be negative.

Similarly, we said that the total number of stones was determined by the function $S(n) = 1 + 2 + 3 + 4 + 5 + 6 + ... + n$, now that we know that the sum of *n* terms of an arithmetic sequence is $S_n = \frac{n}{2} \times \left[2a_1 + (n-1)d\right]$ in this case $a_1 = 1, d = 1$, so

$$S(n) = 1 + 2 + 3 + 4 + 5 + 6 + ... + n$$

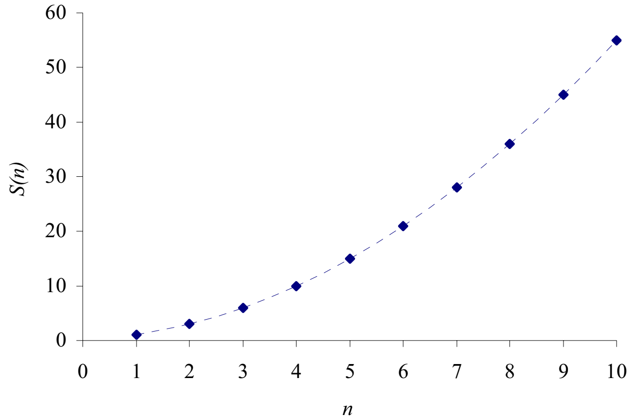
$$S(n) = \frac{n}{2}\left[2 \times 1 + (n-1) \times 1\right]$$

$$S(n) = \frac{n^2 + n}{2}$$

Notice the structure of the formula for the function. It is a quadratic function with a restricted domain of all positive integers. It will have a parabolic shape, with only part of the parabola included in the graph. It is not a continuous function, and in this case as *n* increases the value of the function approaches infinity.

A graph of the function would look like this.

Figure 3.30: Sum of *n* terms of the arithmetic sequence, $S(n)=\dfrac{n^2+n}{2}$



(the dashed line indicates that the function is not continuous between the points indicated)

If the constant had been a negative number then the function for finding the sum of *n* terms would still be a quadratic function but the coefficient of the squared term would be negative and so the parabolic shape would be inverted.

The stone steps problem involved a sequence of numbers that changed by adding a constant. Other sequences can change by multiplying by a constant. For example, we all like to save money. If we deposit $3 000 in an account that pays 3% interest per year, compounded annually, how much would we have after a period of time. In compound interest investments, the interest is calculated regularly on the **principal** (amount originally invested) plus interest. So after year one we would have the original principal plus 3% of that principal. In fact this means that we would have 103% of the principal. We can work this out mathematically as follows.

$$\begin{aligned}\text{Total earned} &= \text{principal} + 3\% \text{ of principal}\\ &= 3000+\frac{3}{100}\times 3000\\ &= 3000+0.03\times 3000\\ &= 3000(1+0.03)\\ &= 3090\end{aligned}$$

or $3090

We could perform the same calculations over five years as shown in the table below.

Year 0	3000
Year 1	$3000(1+0.03)=3000\times 1.03$
Year 2	$(1+0.03)\times 3000\times(1.03)=3000\times(1.03)^2$
Year 3	$(1+0.03)\times 3000(1+0.03)^2=3000\times(1.03)^3$
Year 4	$(1+0.03)\times 3000(1+0.03)^3=3000\times(1.03)^4$
Year 5	$(1+0.03)\times 3000(1+0.03)^4=3000\times(1.03)^5$

So we again have a sequence of numbers,

$$3000,\ 3000\times 1.03,\ 3000\times(1.03)^2,\ 3000\times(1.03)^3,\ 3000\times(1.03)^4,\ 3000\times(1.03)^5$$

We call this sequence a **geometric sequence**, because each year increases by a constant multiple.

Each term is determined by multiplying the previous term by the constant 1.03. Other examples of geometric sequence would be:

2, 6, 18, …….	Multiply each term by 3
1, –2, 4, –8, ….	Multiply each term by –2
16, 8, 4, …..	Multiply each term by $\frac{1}{2}$
$2,\ 2\sqrt{2}, 4, 4\sqrt{2}, 8,$.......	Multiply each term by $\sqrt{2}$

A geometric sequence is a list of numbers in which the terms increase or decrease by multiplying by a constant.

Again it is useful to know how to calculate any term of a sequence and the sum of the terms of a sequence.

If the first term of a geometric sequence is a_1 and the constant multiple, r, then the second term is

$$a_2 = a_1 r$$
$$a_3 = a_2 r = \underbrace{(a_1 r)}_{a_2} r = a_1 r^2$$
$$a_4 = a_3 r = \underbrace{a_1 r^2}_{a_3}\, r = a_1 r^3$$
$$a_5 = a_4 r = \underbrace{a_1 r^3}_{a_4}\, r = a_1 r^4$$

You might have noticed the pattern that has developed.

The nth (a_n) term of a geometric sequence is given by $a_n = a_1 r^{n-1}$, where a_1 is the first term and r the constant.

We can use an algebraic process to find the sum of a geometric sequence.

Let the sum of the *n* terms above be, S_n

Do not learn this proof

$$S_n = a_1 + a_2 + a_3 + a_4 + ... + a_{n-1} + a_n$$

$$S_n = a_1 + a_1r + a_1r^2 + a_1r^3 + ...a_1r^{n-2} + a_1r^{n-1}$$ Substitute in the formula for each term

$$rS_n = r(a_1 + a_1r + a_1r^2 + a_1r^3 + ...a_1r^{n-2} + a_1r^{n-1})$$ Multiply both sides by *r*, then expand brackets

$$rS_n = a_1r + a_1r^2 + a_1r^3 + ...a_1r^{n-1} + a_1r^n$$

$$rS_n + a_1 = a_1r + a_1r^2 + a_1r^3 + ...a_1r^{n-1} + a_1r^n + a_1$$ Add a_1 to both sides

$$rS_n + a_1 = \underbrace{a_1 + a_1r + a_1r^2 + a_1r^3 + ...a_1r^{n-1}}_{S_n} + a_1r^n$$ Rearrange to put a_1 at front of RHS

$$rS_n + a_1 = S_n + a_1r^n$$ Replace grouped terms by S_n

$$rS_n - S_n = a_1r^n - a_1$$

$$S_n(r-1) = a_1(r^n - 1)$$

$$S_n = \frac{a_1(r^n-1)}{r-1} \quad \text{or} \quad S_n = \frac{a_1(1-r^n)}{1-r}$$ Make S_n the subject of the formula
Multiply numerator and denominator by –1

The sum of the first *n* terms of a geometric sequence is $S_n = \frac{a_1(1-r^n)}{1-r}$, where a_1 is the first term and *r* the constant multiple.

Note that the sum of the terms of a geometric sequence is called a **geometric series** and *r* is often called the **common ratio**.

Example

Find the 100th term and the sum of the first 100 terms of the geometric sequence whose first four terms are 1, –2, 4, –8.

The first term of the sequence is 1, so $a_1 = 1$. To find the constant, *r*, divide the second term by the first term, $r = \frac{-2}{1} = -2$. To confirm this divide the next two terms in the same way, $r = \frac{4}{-2} = -2$.

To find the 100th term we can use, $a_n = a_1r^{n-1}$, where $n = 100,\ r = -2,\ a_1 = 1$

$a_{100} = 1 \times (-2)^{100-1} = (-2)^{99} \approx -6.34 \times 10^{29}$

To find the sum of the first 100 terms use, $S_n = \frac{a_1(1-r^n)}{1-r}$,

$$S_{100} = \frac{1 \times (1-(-2)^{100})}{1-(-2)} \approx -4.226 \times 10^{29}$$

So the 100th term is approximately -6.34×10^{29}, and their sum is -4.226×10^{29}.

Example

A wheat farming couple harvests 1000 tonnes of wheat in their first year on a property. Each year their harvest increases by 10% over the previous year. How much will they harvest in their eighth year? How much wheat will they have sold altogether in their first eight years on the property?

As the increase is 10% over the previous year the weight of wheat harvested each year forms a geometric series with a common multiple of 1.10. So to find the amount of wheat harvested in the eighth year we need to find the eighth term of the geometric sequence with the first term, $a_1 = 1000$, and a common multiple, $r = 1.10$.

$a_8 = 1000 \times 1.10^{8-1} = 1948.7171$

The farmers harvest approximately 1949 tonnes in their eighth year.

To calculate the total amount of wheat harvested over eight years we need to find the sum of the arithmetic sequence using the formula, $S_n = \dfrac{a_1(1-r^n)}{1-r}$

$S_8 = \dfrac{1000 \times (1-1.10^8)}{1-1.10} = 11435.8881$

The farmers would harvest approximately 11 436 tonnes over the eight years.

Activity 3.11

1. (a) Explain why the following sequence is a geometric sequence.
   125, 25, 5, 1, . . .

   (b) Find the 10th term of the sequence above.

2. Find the sum of the first 15 terms of the following sequence:
   8, 12, 18, 27, . . .

3. Imagine that you have been employed by a rich eccentric to undertake some dangerous work for a period of 30 days. Your boss offers you two alternative payment options. The first is to receive \$1 000 000 in cash at the completion of the work, and the second option is a bit more bizarre. In the second option the boss will pay you one cent if you last the first day, two cents if you last the second day, four cents if you last the third day, eight cents if you last the fourth day and so on. Assuming that you make it to the end of the 30 days, which option should you take?

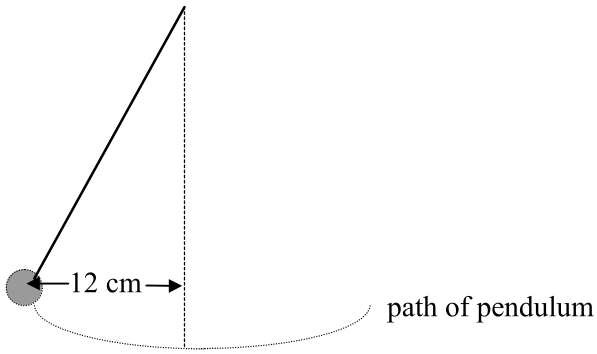
4. Scientists observing growth of a bacterial colony, were able to estimate the number of bacterium present on three successive days. These are listed below:

Time (in days)	Number of bacteria ($\times 1000$)
0	20
1	22
2	24.2

(a) Find the number of bacteria present after 6 days.

(b) Assuming that the bacteria continued to multiply at the same rate, how long would it take before their population had doubled?

5. A pendulum is given an initial swing, so that to begin with it reaches a distance of 12cm from its central position (see diagram below):



Due to frictional forces, the pendulum will not swing out as far on its second swing. Assume that the following measurements were made:

Swing number	Distance from central position (cm)
1	12
2	11.4
3	10.83

(a) If this pattern were to continue, how far should the pendulum swing on its 20th swing.

(b) What about after its 200th swing.

(c) Will the pendulum ever stop?

Let's now see how geometric sequences and series are related to functions.

If we return to the savings question we had the geometric sequence

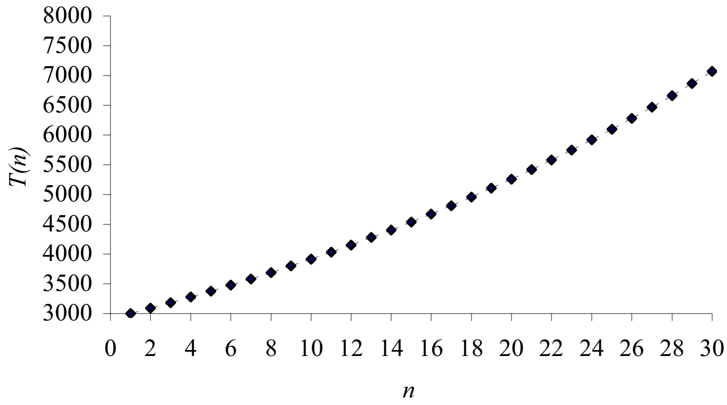
$$3000, 3000\times 1.03, 3000\times(1.03)^2, 3000\times(1.03)^3, 3000\times(1.03)^4, 3000\times(1.03)^5, ...$$

The *n*th term can be represented by the function,

$$\begin{aligned} T(n) &= 3000\times 1.03^{n-1} \\ &= 3000\times 1.03^n \times 1.03^{-1}\,. \\ &\approx 2912.62\times 1.03^n \end{aligned}$$

This function is an exponential growth function with a restricted domain of all positive integers. It is not a continuous function and as n increases the value of the nth term approaches infinity. Notice that it compares well with previous exponential functions in that terms of the function are calculated by multiplying the previous term by a common multiple. The graph of the function looks like this.

Figure 3.31: *n*th term of the geometric sequence, $T(n) \approx 2912.62 \times 1.03^n$



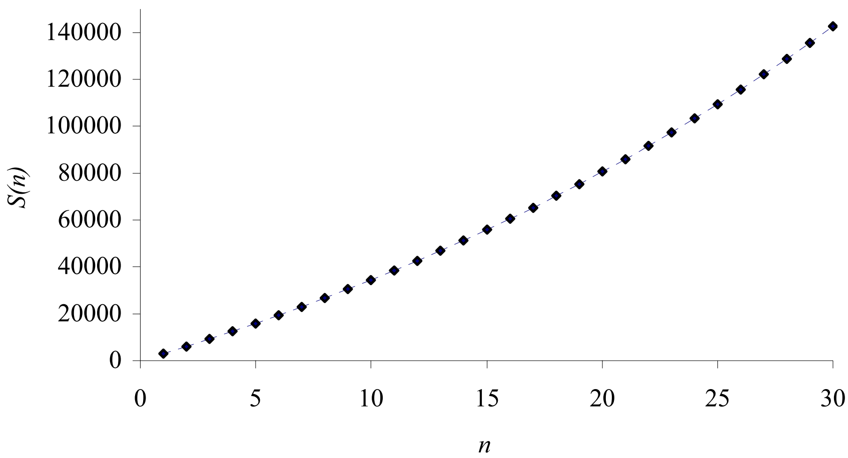
The sum of the same sequence of numbers can be generated from the function,

$$\begin{aligned} S_n &= \frac{3000 \times (1.03^n - 1)}{1.03 - 1} \\ &= 100\,000 \times (1.03^n - 1) \\ &= 100\,000 \times 1.03^n - 100\,000 \end{aligned}$$

Rearrange the formula to put it in a more familiar form $y = A \times 1.03^n + B$.

If we graph this function realizing that it has a restricted domain of all positive integers it will look like this.

Figure 3.32: Sum of *n* Terms of the geometric sequence, $S_n = 100\,000 \times 1.03^n - 100\,000$



This function is also an exponential growth function. It is not continuous because of its restricted domain but as the value of *n* increases then the value of the function approaches infinity.

What would happen to the size of the *n*th term and the sum of the terms of a geometric sequence if the constant multiple was less than one. Let's think about the following situation.

A bushwalker is lost. He knows that if he walks along the creek, the nearest farm house is 25 km away. On the first day he walks 12 km, on the second 6 km, on the third 3 km. Will he ever get to the farm house?

In this case a sequence of numbers is generated relating to each day of travel.

$12, 6, 3, \frac{3}{2}, \frac{3}{4}, \ldots..$

This sequence is a geometric sequence in which the constant multiple is $\frac{1}{2}$ and the first term is 12. Using our knowledge of geometric sequences we can find the functions that predict the *n*th term and the sum of *n* terms.

The *n*th term, $T(n)$

$$\begin{aligned} T(n) &= 12\times(\frac{1}{2})^{n-1} \\ &= \frac{12\times 1^{n-1}}{2^{n-1}} \\ &= \frac{12\times 1}{2^n\times 2^{-1}} \\ &= 24\times 2^{-n} \end{aligned}$$

This is the more familiar form of $y = A\times 2^{-n}$

Sketch the function for yourself on Graphmatica to get a feel for its shape. Note that it is not possible to make it look discontinuous on this package yet. It is an exponential decay function with a restricted domain of all positive integers. It will not be continuous and will have an asymptote at $T(n) = 0$, which means that as *n* gets very large the value of the *n*th term approaches zero or $\lim_{n\to\infty} T(n) = 0$.

Let's now think about the sum of the terms of a geometric series in which the constant multiple is less than 1. If the value of the *n*th term is approaching zero then we might expect that the sum of *n* terms as *n* gets very large approaches a constant value or a limit. Let's see. The formula for the sum of *n* terms for the bushwalking situation is:

$$S(n) = \frac{12\times\left[1-(\frac{1}{2})^n\right]}{1-\frac{1}{2}}$$

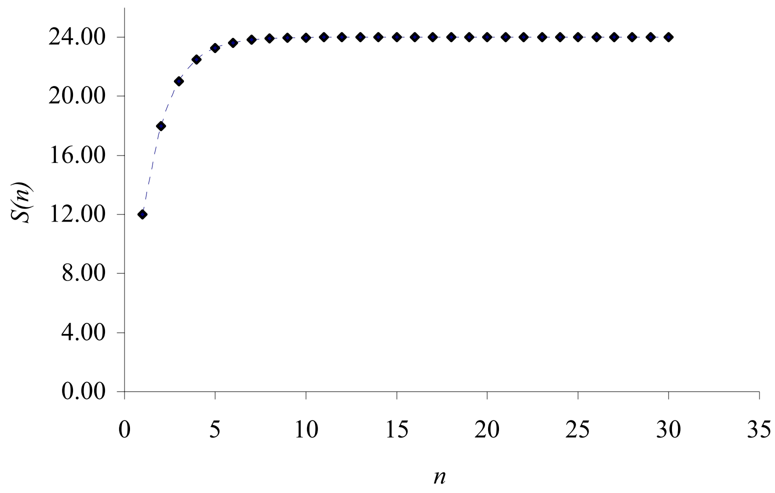
$$= 24\left[1-(\frac{1}{2})^n\right]$$ Evaluate denominator to get $\frac{1}{2}$, then simplify numerator to 24

$$= 24 - 24(\frac{1}{2})^n$$ Remove square brackets

$$= 24 - 24\times 2^{-n}$$ Notice the more familiar form $y = A - B\times 2^{-n}$

Notice that it has an exponential-like form. If we sketch it, it will look like this.

Figure 3.33: Sum of a geometric series first term 12 and $r=\frac{1}{2}$



It is not a continuous function because of the restricted domain of all positive integers. It also has a horizontal asymptote but not along the horizontal axis. This time the asymptote is at $S(n)=24$. This means that as *n* gets very large and approaches infinity, the sum of *n* terms of the series approaches a constant value of 24 or in symbolic notation, $\lim_{n\to\infty} S(n)=24$. This means that in theoretical terms the bushwalker would never reach the house 25 km down the creek. (Hopefully they will be able to hear him from where he is!)

This result can be applied to all geometric functions with a constant multiple between –1 and 1. We say

The sum of *n* terms as *n* approaches infinity of a geometric series with *r* between –1 and 1 is $\boldsymbol{S_\infty=\frac{a_1}{1-r}}$.

Let's think about it algebraically.

$$S_n=\frac{a_1(1-r^n)}{1-r}$$

$$S_n=\frac{a_1-a_1r^n}{1-r}$$

$$S_n=\frac{a_1}{1-r}-\frac{a_1r^n}{1-r}$$

Now if *r* is a fraction between –1 and 1, when *n* approaches infinity then r^n approaches zero e.g. $(\frac{1}{2})^{1000}=\frac{1}{2^{1000}}\to 0$. This means that as *n* get very large the $S_n=\frac{a}{1-r}$.

Example

Repeating decimals are rational numbers and so can be rewritten in fractional form. Calculators do this. One way that they could do it is knowing that a repeating decimal can be written as a geometric series. Using your knowledge of geometric series rewrite the repeating decimal 0.12121212…. as a geometric series, find its sum to infinity and thus write the decimal as a fraction. Sketch the graph of the function used to calculate the sum of *n* terms of the series to confirm your answer.

The decimal can be rewritten as

$$0.1212121212.....=0.12+0.0012+0.000012+0.00000012+....$$

This is a geometric series with the first term of 0.12 and the common ratio of 0.01.

The sum to infinity of the series is,

$$\begin{aligned}S_\infty &= \frac{a_1}{1-r}=\frac{0.12}{1-0.01}\\ &= \frac{(\frac{12}{100})}{(\frac{99}{100})}\\ &= \frac{12}{100}\div\frac{99}{100}\\ &= \frac{12}{100}\times\frac{100}{99}\\ &= \frac{4}{33}\end{aligned}$$

To confirm this answer we need to sketch the graph of the function used to predict the sum of *n* terms of the geometric sequence. It is often easier to do this if it is in a more conventional exponential form.

$$S(n)=\frac{a_1(1-r^n)}{1-r}$$

$$S(n)=\frac{0.12\times(1-0.01^n)}{1-0.01}$$

$$S(n)=\frac{\frac{12}{100}\times(1-0.01^n)}{\frac{99}{100}}$$

$$S(n)=\frac{4}{33}\times(1-0.01^n)$$

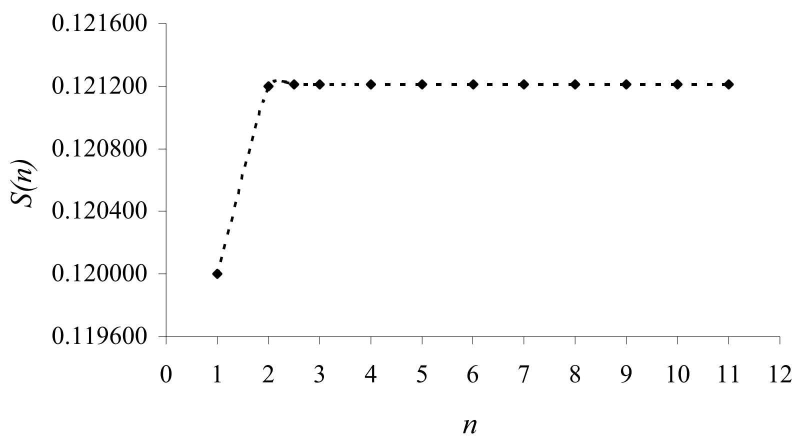
$$S(n)=\frac{4}{33}-\frac{4}{33}\times0.01^n$$

$$S(n)=\frac{4}{33}-\frac{4}{33}\times100^{-n}$$

This is now in the more familiar form $y=A-B\times100^{-n}$.

You might notice that when the function is in this form it is easier to see that if n gets very large then the term $\frac{4}{33}\times 100^{-n}$ will approach zero, so the whole function will approach the value of $\frac{4}{33}$. The graph below confirms this.

Figure 3.34: Sum of a geometric series, first term 0.12 and $r=\frac{1}{100}$



Example

A child accidentally ingests 20 mg of a medication. Doctors say that 50% of the drug is metabolized each day so that after the first day the child has 10 mg after the second day 5 mg. Generate a geometric series to represent the situation. If the medication has no effect after it reaches 0.039 mg when will the child be virtually free of the medication. Will the child ever be absolutely free of the medication?

The geometric series representing this situation is $20,10,5,2.5,1.25,......$, it has a first term of 20 and a common ratio of 0.5. The nth term of the sequence can be evaluated by the function $T(n)=20\times 0.5^{n-1}$. If $T(n)=0.039$,

$$\begin{aligned}
0.039 &= 20\times 0.5^{n-1}\\
0.00195 &= 0.5^{n-1}\\
\ln(0.00195) &= \ln(0.5^{n-1})\\
\ln(0.00195) &= (n-1)\times\ln(0.5)\\
n-1 &= \frac{\ln(0.00195)}{\ln 0.5}\\
n &= \frac{\ln(0.00195)}{\ln 0.5}+1\\
n &\approx 10.002
\end{aligned}$$

The medication will reach an ineffective level after 10 days.

To find out if the medication will ever entirely leave the child's body think about what happens to the *n*th term as *n* approaches infinity. First rearrange the *n*th term into a more recognizable exponential function.

$$T(n) = 20 \times 0.5^{n-1}$$

$$T(n) = 20 \times (\frac{1}{2})^{n-1}$$

$$T(n) = \frac{20 \times 1}{2^{n-1}}$$

$$T(n) = \frac{20 \times 1}{2^n 2^{-1}}$$

$$T(n) = 40 \times 2^{-n}$$

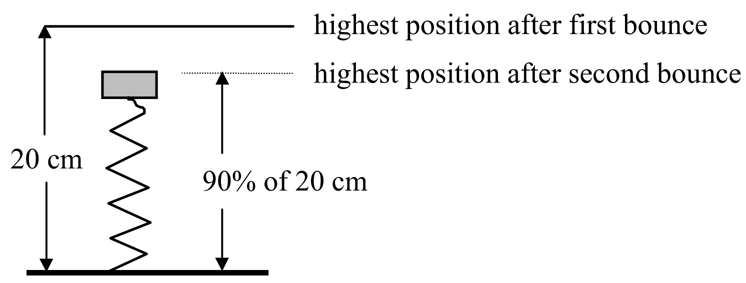
The *n*th term of the geometric sequences is represented by an exponential decay function, so as *n* gets very large the value of the *n*th term approaches zero, but does not reach it absolutely. The child will never be entirely free of the medication.

Activity 3.12

1. Find the sum to infinity of the following sequence:
   125, 25, 5, 1, . . .
2. Express the repeating decimal 0.999..... as a geometric sequence and consequently show that its infinite sum is 1.
3. Find the sum of the series: $15 + 10 + 6\frac{2}{3} + ...$
4. A person wishes to drain a water tank. The rate at which the water empties from the tank depends on how much water is still in the tank. The person notes that after 30 minutes the tank is half full, after another 30 minutes it is one quarter full, and after a further 30 minutes the tank is one eighth full. If the tank continues to leak in this pattern,

   (a) how long will it take before it contains one fiftieth of its original volume;

   (b) will it ever be entirely empty?
5. The back spring on a car has the shock absorber removed, consequently when a pressure is exerted on the spring, it will continue to bounce for some time. Engineers, developing these springs, have found that in such circumstances the spring will return to a position which is 90% of its position on the previous bounce (see diagram below).



In one experiment, a force was applied to the car, so that it initially rose to 20 cm above its rest position.

(a) Complete the following table showing the height above the rest position compared with the number of bounces.

Number of bounces	1	2	3	4	5	6	7
Height of bounce from rest position (cm)	20	18					

(b) If it continues to bounce, how many oscillations will it take before it has reached a distance 8 cm above its rest position?

6. A small child is initially given a push whilst on a swing. On the first complete swing, she travels a distance of 2.4 m and then on each subsequent swing she travels only 80% of the distance previously travelled.

(a) Complete the following table:

swing number	1	2	3
Distance of the swing (m)	2.4	1.92	

(b) How far will she have travelled in total after her 4th swing?

(c) Develop an equation of a function used to predict the sum of n terms of this sequence.

(d) Draw a graph showing the total distance travelled against the number of swings.

(e) Will the child ever travel more than a total distance of 12 m?

Something to talk about…

Are all exponential functions geometric sequences or series? Think of some examples for yourself and switch between the two forms. Share your opinions with your colleagues or the discussion group.

That's the end of this module. You will have experienced a lot of new algebraic techniques and a toolbox of different functions to arm you for your future studies in this unit and later.

But before you are really finished you should do a number of things:

1. Have a close look at your action plan for study. Are you on schedule? Or do you need to restructure you action plan or contact your tutor to discuss any delays or concerns?

2. Make a summary of the important points in this module noting your strengths and weaknesses. Add any new words to your personal glossary. This will help with future revision.

3. Practice some real world problems by having a go at 'A taste of things to come'.

4. Check your skill level by attempting the post-test.

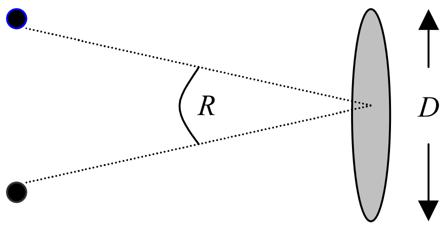
3.4 A taste of things to come

1. Surveyors use telescopes of one form or another. The purpose of a telescope in this area is to create for an observer a picture of the cross-wire's position on the target with the greatest clarity and precision. Optical factors to be considered included resolving power, magnification, definition, eye distance, size of pupil and field of view. Two of these are discussed below.

   The ability of a lens to show detail is called resolving power. It is measured as the smallest angular distance, expressed in seconds of arc, between two points just far enough apart to be distinguished as separate objects rather than a blurred one. The maximum resolving power that theoretically can be attained with a perfectly made telescope depends entirely on the diameter of that part of the objective lens actually used (the effective aperture). The resolving power of an objective lens is independent of magnification. It can be computed by the empirical formula:

$$R = \frac{140}{D},$$

   where R is the angle that can be resolved, in seconds, and D the diameter of the lens aperture in millimetres. For example if the objective lens of a certain telescope has an aperture of 30 mm in diameter, its resolving power is about 4.7 seconds. The accepted standard for resolving power of a human eye is 60 seconds. See diagram below:



   (a) The specification of telescope manufacturers could include a graph describing the relationship between resolving power and diameter of the lens. Complete a graph of this function.

   (b) What do you think the domain and range of the function should be?

   (c) What would be the resolving power of a 40 mm camera lens?

   (d) Using a 40 mm lens how many times would the image have to be magnified so that it was resolvable by the human eye?

2. In the last few sections of this module, you studied arithmetic and geometric sequences as examples of discrete functions. That is, functions which are only defined for discrete values of the domain. Another example of such a function is that of the energy levels of separate atoms. Niels Bohr in 1913 demonstrated that the Hydrogen atom when excited only emitted energy at certain discrete levels. The energy was found to be given by the function:

$$E_n = \frac{-13.6}{n^2},$$

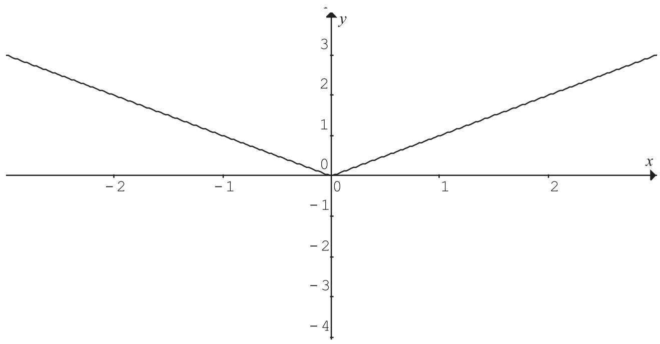
where n is the energy level, $n = 1, 2, 3, ...$ and E_n the energy (in electron volts) at the nth energy level.

(a) Calculate the energy emitted at the energy levels $n = 1, 2, 3$ and 4.

(b) The first two spectral lines in the Balmer series, correspond to the light given off when electrons move from the 3rd and 4th energy levels down to the second energy level. How much energy will be given off in each of these two cases?

3.5 Post-test

1. Determine whether the following relation is a function, and if so describe the type of function.



2. If $B(t) = 13 - t$ and $I(t) = 2t^2 - 3$ find the following:

   (a) $B(t) \times I(t)$

   (b) $I(B(t))$

3. Find the solutions of the following equation:

   $f(x) = x^3 - 3x^2 - 13x + 15 = 0$

4. Sketch the graph:

   $y = x^3 - 3x^2 - 13x + 15$

5. Examine the behaviour of the function $y = \dfrac{3}{x-2} + 1$ by answering the following questions.

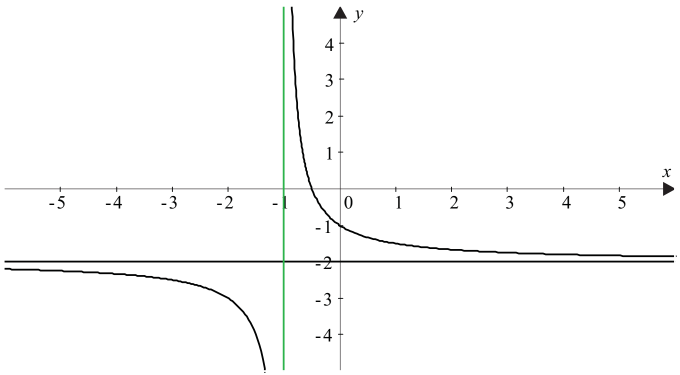
   (a) Confirm that it is a rational function

   (b) Find the vertical and horizontal intercepts.

   (c) Determine any asymptotes

   (d) Hand draw a sketch of the graph.

6. Using the knowledge developed in Q5, suggest a possible equation for the following function:



7. The amount of charge on a capacitor is known to decrease according to the following relation: $Q = Q_0 e^{-kt}$, where Q is the amount of charge remaining after t seconds and Q_0 the initial charge. In a certain experiment, a capacitor with an initial charge of 2.17×10^{-4} C was allowed to discharge for 3 seconds, when its charge was then found to be 1.5×10^{-4} C.

   (a) Determine the exact equation of the function.

   (b) How much charge is remaining after 6 seconds?

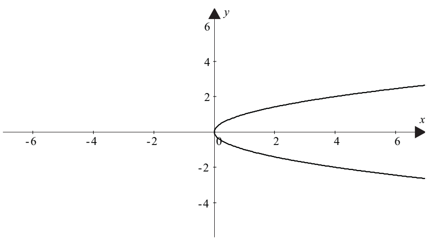
8. What will be the sum of the first 20 terms of the following sequence:
   12, 16, 20, . . .

9. A ball is dropped from a height of 12.8 m. If it bounces to $\frac{5}{8}$ of its previous height on each bounce, through what total distance will it travel when it has hit the ground for the sixth time?

3.8 Solutions

Solutions to activities

Activity 3.1

Relation	If function, what type?	Reason
$h = t^2 - 3$	A many-to-one function	For each value of t there is only one value of h, therefore it is a function. However it is possible that two values of t give the same value of h, e.g. $t = -1$ and $t = 1$. Therefore it is a many-to-one function.
	Not a function.	For each value of x there are nearly always two values of y. If we were to draw a vertical line on the graph it would cross the relation twice for $x > 0$.
(1,–2), (2,3), (3,0), (1,–4), (–2,3)	Not a function.	When the independent variable takes the value of 1 we see that the dependent variable can take two possible values, i.e. –2 and –4.
Telstra currently charges a network access fee of \$11.65 each month and then \$0.25 for each local call. What is the relation between the total monthly charge and the number of local calls made each month?	One-to-one function	For a given number of local calls there will be only one possible charge (note we are not considering STD and other calls). The opposite also applies, in that for any given charge it is possible to evaluate the exact number of local calls made.
(2,3), (5,1), (–1,3), (7,0), (–3,1)	Many-to-one function	For each different value that the independent variable assumes there is only one possible value for the dependent variable. Therefore the relation is a function. However two values of the independent variable give a result of 3, so the function is a many-to-one function.

Activity 3.2

1. (a) 'The value of the function *f* at 12 is 3'.

   (b) 'The value of the function *h* at –2 is 12.75'.

2. (a)
$$\begin{aligned} p(t) &= 3t^2 - 2 \\ p(0.5) &= 3\times 0.5^2 - 2 \\ &= 3\times 0.25 - 2 \\ &= -1.25 \end{aligned}$$

   (b)
$$\begin{aligned} p(t) &= 3t^2 - 2 \\ p(m+2) &= 3(m+2)^2 - 2 \\ &= 3(m^2+4m+4) - 2 \\ &= 3m^2+12m+12-2 \\ &= 3m^2+12m+10 \end{aligned}$$

3. (a) This expression will determine the strength of the magnetic field when there are twice as many turns of wire.

   (b) This expression gives the strength of the magnetic field when we leave the original coil alone but add from an external source (for example a standard bar magnet) an extra two units of magnetic strength.

4. (a)
$$\begin{aligned} h(m) - p(m) &= (4m^2-9) - (2m-3) \\ &= 4m^2 - 9 - 2m + 3 \\ &= 4m^2 - 2m - 6 \end{aligned}$$

   (b)
$$\begin{aligned} p(m)\times h(m) &= (2m-3)(4m^2-9) \\ &= 8m^3 - 12m^2 - 18m + 27 \end{aligned}$$

   (c)
$$\begin{aligned} \frac{p(m)}{h(m)} &= \frac{2m-3}{4m^2-9} \\ &= \frac{\cancel{(2m-3)}}{\cancel{(2m-3)}(2m+3)} \\ &= \frac{1}{2m+3} \end{aligned}$$

   (d)
$$\begin{aligned} p(h(m)) &= p(4m^2-9) \\ &= 2(4m^2-9) - 3 \\ &= 8m^2 - 18 - 3 \\ &= 8m^2 - 21 \end{aligned}$$

5. $$\begin{aligned} f(g(x)) &= 2[g(x)]^2 \\ &= 2[x^2+2]^2 \\ &\text{or } 2[x^4+4x^2+4] \end{aligned}$$

$$\begin{aligned} g(f(x)) &= [f(x)]^2 + 2 \\ &= [2x^2]^2 + 2 \\ &= 4x^4 + 2 \end{aligned}$$

Activity 3.3

1. $h(0)=2$ and $h(p)=0$ when $p=-2$ and 2.

2. $h(0)=-4$ and $h(t)=0$ when $t=-4,-1$ and 2.

3. $P(0)=-2$ and $P(a)=0$ when $a=-3,-1,1$ and 3.

Activity 3.4

Which of the following expressions represent polynomials and what is their degree?

Expression	Is it a polynomial?	Degree
1. $h=12t-2t^2+\dfrac{3}{t}$	No, as the last term can be written as $3t^{-1}$ and all powers in a polynomial must be positive or zero.	Not applicable
2. $r(x)=12x+3x^5-2x^2+4$	Yes, however, it would normally be written with the highest power first, i.e. $r(x)=3x^5-2x^2+12x+4$	5
3. $P(r)=5r^4-2r^3+12$	Yes.	4
4. $y=12x-5x^4+2x^{0.5}$	No, as the power of x is 0.5, which is not a whole number.	Not applicable
5. $F(w)=5w+3-\dfrac{2}{w}\times w^2$	Yes. The expression can be simplified to one with no negative powers, i.e. $F(w)=5w+3-2w$ $=3w+3$ $=3w^1+3$	1

Activity 3.5

1. Firstly we need to consider how the function $h(x) = x^3 - 3x^2 - x + 3$ behaves when $x \to \infty$.

   Consider a large value for x, such as $x = 10000$,
   $h(10\,000) \approx 999\,699\,990\,000$ which is very large, therefore as $x \to \infty, h(x) \to \infty$.

   Then we need to consider how the function behaves when $x \to -\infty$

   Consider a small value for x, such as $x = -10000$,
   $h(-10\,000) = -1\,000\,299\,989\,997$, which is very small, therefore as $x \to -\infty, h(x) \to -\infty$.

2. Consider a large value for x, such as $x = 10000$,
   $y = -(10\,000)^2 + 6 \times 10\,000 - 12$
   $= -99\,940\,012$ which is very small
   Therefore as $x \to \infty, y \to -\infty$.

   Consider a small value for x, such as $x = -10000$,
   $y = -(-10\,000)^2 + 6 \times -10\,000 - 12$
   $= -100\,060\,010$ which is very small
   Therefore as $x \to -\infty, y \to -\infty$.

3. Consider a large value for x, such as $x = 10000$,
   $P(10\,000) = -9.996 \times 10^{15}$ (in case you have forgotten, this is the same as –9996 with 12 zeros after it) and it is a very small number.
   Therefore as $x \to \infty, P(x) \to -\infty$.

   Consider a small value for x, such as $x = -10000$,
   $P(-10000) = -1.0004 \times 10^{16}$, which is also very small.
   Therefore as $x \to -\infty, P(x) \to -\infty$.

4. Consider a large value for x, such as $x = 10000$,
   $f(10000) \approx -9.998 \times 10^{11}$ and this is a very small number.
   Therefore as $x \to \infty, f(x) \to -\infty$.

   Consider a small value for x, such as $x = -10000$,
   $f(-10000) \approx 1.0002 \times 10^{12}$, which is a very large number.
   Therefore as $x \to -\infty, f(x) \to \infty$.

5. Consider a large value for x, such as $x = 10000$,
   $y \approx 3.0002 \times 10^{16}$ and this is a very large number.
   Therefore as $x \to \infty, y \to \infty$.

   Consider a small value for x, such as $x = -10000$,
   $y \approx 2.9998 \times 10^{16}$, which is a very large number.
   Therefore as $x \to -\infty, y \to \infty$.

Note: Graphmatica could be used to graph each function and to verify that each function behaves as we have predicted.

Activity 3.6

1. (a) This is a simple quadratic expression, hence:

$$x^2 + 7x + 12 = (x + 3)(x + 4)$$

(b) Instead of using the guess and check method in this question, you should observe that the common factor p is present in each term (we call p a common factor), therefore we now have a quadratic in the brackets to factorize:

$$\begin{aligned} p^3 + 7p^2 + 12p &= p(p^2 + 7p + 12) \\ &= p(p + 3)(p + 4) \end{aligned}$$

(c) Before we find the factors, we need to find the solutions to the equation $x^3 + x^2 - 25x - 25 = 0$ and we would expect there to be at most 3 of these.

Possibilities would be: $\pm 1, \pm 5, \pm 25$, so use trial and error:

Possible solutions	**Value of function**
1	–48
–1	0
5	0
–5	0
25	not worth doing as we have the 3 solutions

Therefore we see the solutions are: $x = -1, x = 5$ and $x = -5$.

The factors of the expression are: $(x + 1)$, $(x - 5)$, $(x + 5)$, so that $x^3 + x^2 - 25x - 25 = (x + 1)(x - 5)(x + 5)$

(d) We need to find the solutions to the equation: $a^3 + 2a^2 - a - 2 = 0$ and we would expect that there be at most 3. Possibilities are: $\pm 1, \pm 2$.

Possible solutions	**Value of function**
1	0
–1	0
2	12
–2	0

Therefore we see the solutions are: $a = 1, a = -1$ and $a = -2$.

The factors of the expression are: $(a - 1)$, $(a + 1)$, $(a + 2)$

(e) The common factor m is present in both terms, therefore we can factorize it out:

$$\begin{aligned} m^3 - 4m &= m(m^2 - 4) \\ &= m(m + 2)(m - 2) \end{aligned}$$

2. (a) This is a quadratic so it is simpler to firstly factorize and then solve:

$$x^2 + 5x + 6 = 0$$
$$(x + 2)(x + 3) = 0$$

Therefore the solutions are: $x = -3, x = -2$.

(b) Possible solutions for this equation are: $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30$, which is a lot to try, so we will start with the small ones and keep going until we get three solutions.

Possible solutions	**Value of function**
1	–24
–1	–24
2	0
–2	–12
3	48
–3	0
5	240
–5	0

Therefore we see that the solutions are: $p = 2, -3, -5$ or writing these in order: $p = -5, -3, 2$.

(c) As the common factor $3m$ is present in all three terms it may be simpler to factorize the expression instead of using the longer 'guess and check' method used above.

$$\begin{aligned} 3m^3 + 15m^2 + 18m &= 3m(m^2 + 5m + 6) \\ &= 3m(m + 2)(m + 3) \end{aligned}$$

Therefore the solutions are; $m = 0, -2, -3$ or writing these in order: $m = -3, -2, 0$.

(d) Possible solutions for this equation are: $x = \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$.

Possible solutions	**Value of function**
1	8
–1	24
2	0
–2	20
3	0
–3	0

Therefore we see the solutions are: $x = -3, 2, 3$.

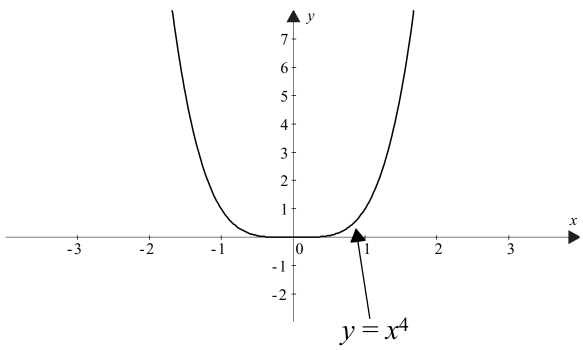
(e) Possible solutions for this equation are: $a = \pm 1, \pm 7$.

Possible solutions	**Value of function**
1	0
–1	0
7	672
–7	0

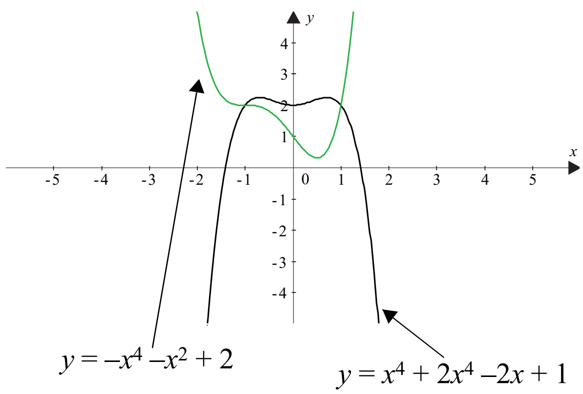
Therefore solutions for this equation are: $a = -7, -1, 1$.

Activity 3.7

1. (a) The graph of $y = x^4$ is shown below:



(b) and (c) The graphs of $y = x^4 + 2x^3 - 2x + 1$ and $y = -x^4 + x^2 + 2$ are shown below:

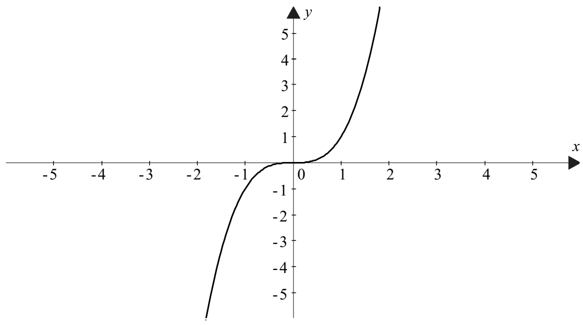


The vertical intercept for each function is the same as the constant term in its original equation. Consequently: (a) $y = 0$ (b) $y = 1$ (c) $y = 2$.

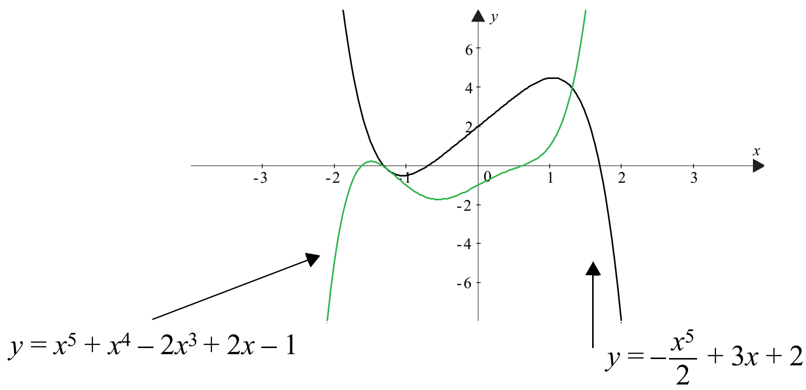
In questions (a) and (b) the functions both approach infinity as $x \to \infty$ and $x \to -\infty$. One way of getting a good feel for what happens to the function at its extremities is to 'zoom out' and look at it from a distance. Using Graphmatica this involves changing the domain to say $-50 < x < 50$ and the range to $-50 < y < 50$. From this distance you see the two functions look fairly similar. The third function approaches negative infinity as both $x \to \infty$ and $x \to -\infty$.

Hopefully you will observe that the negative sign before the x^4 term in question (c), has the effect of reflecting a normal quartic (one with a degree of 4) over the x-axis.

2. (a)



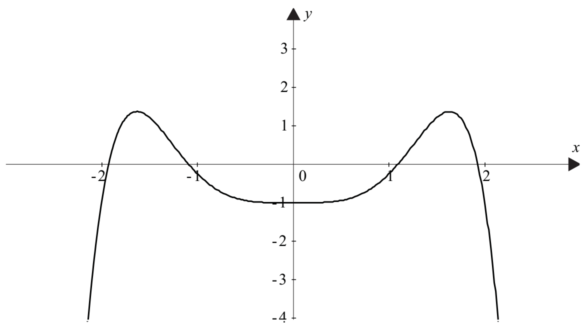
(b) and (c)



The vertical intercepts of the functions are the constant terms in the original equations, consequently: (a) $y = 0$ (b) $y = -1$ (c) $y = 2$.

The first two functions both behave in a similar manner at their extremities, that is when $x \to \infty$, $y \to \infty$ and when $x \to -\infty$, $y \to -\infty$. You will see that if you use Graphmatica to ‘zoom out’, both functions will look similar. In fact they will have a similar shape to the cubic functions discussed in your notes. All polynomial functions with an odd degree, for example 3,5 etc. will have this basic shape when they are viewed from a distance. The higher their degree, the steeper they will appear.

3. Since the polynomial function has an even degree, from a distance it will appear similar to a parabola, except with much steeper sides. The negative sign before the x^6 term will mean that it extends downwards. Consequently the function will approach negative infinity as $x \to \infty$ and as $x \to -\infty$. The constant term of -1 will mean that it intercepts the vertical axis at $y = -1$. Lets see what it actually looks like:



$$y = -\frac{x^6}{4} + x^4 - 1$$

4. (a) We do not need to expand the function to know that if it were expanded, the highest power of x would be 4. Consequently the degree of the function will be 4.

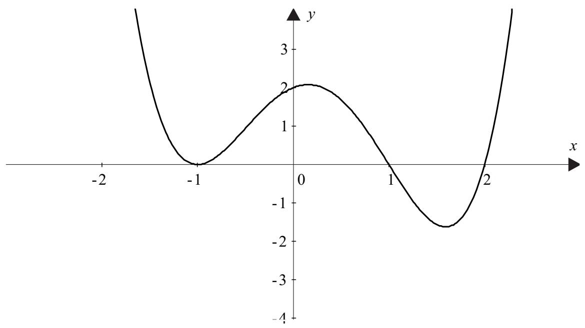
   (b) If we multiply the constants in each factor of the function we will obtain the constant term, i.e. $-2 \times 1^2 \times -1 = 2$. Consequently the function will cut the vertical axis at $y = 2$.

   (c) To find the horizontal intercepts, substitute $y = 0$ into the equation and solve

   $$0 = (x-2)(x+1)^2(x-1)$$

   therefore the horizontal intercepts will be: $x = 2$, $x = -1$ and $x = 1$ (recall that the function will just touch the x-axis at $x = -1$).

5. Keeping in mind the behaviour of the function at its extremities, plot the intercepts and then join these with a continuous line. The result from Graphmatica is:



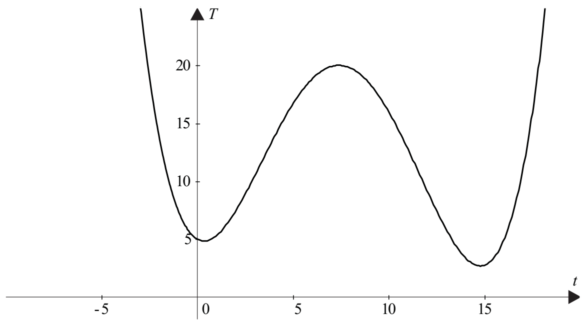
$$y = (x-2)(x+1)^2(x-1)$$

6. (a) We need to substitute $t = 2$ into the equation,

$$\begin{aligned} T &= 0.006t^4 - 0.18t^3 + 1.4t^2 - 0.9t + 5 \\ &= 0.006 \times 2^4 - 0.18 \times 2^3 + 1.4 \times 2^2 - 0.9 \times 2 + 5 \\ &\approx 7.5\,°\text{C} \end{aligned}$$

(b) We would expect there to be at most 3 turning points, as the function has a degree of 4. We would expect that it would cut the vertical axis at 5.

(c) The graph is shown below. The domain has initially been set to $-12 < t < 24$ and the range $0 < T < 30$.



We see that this is not suitable as it is unlikely that the temperatures would soar before 4 am and after 4 pm. Therefore the function is only a suitable model for temperature fluctuation for the period when the temperatures were taken, i.e. between $t = 0$ and $t = 14$. Consequently the domain should be restricted further to $0 < t < 14$.

(d) The temperature is rising between 0.3 and 7.3 hrs after 4 am. (It may rise again, but that is outside the domain and the model may not work then.)

(e) The temperature is falling between 7.3 hours and 14 hours.

Activity 3.8

1. (a) Since t is the time since 1894, we substitute $t = 0$ into the equation,

$$\begin{aligned} P &= 3.2e^{0.0169t} \\ &= 3.2e^{0} \\ &= 3.2 \end{aligned}$$

Consequently the population in 1894 was approximately 3.2 million.

(b) Since t is the time since 1894, we substitute $t = 2010 - 1894 = 116$ into the equation,

$$\begin{aligned} P &= 3.2e^{0.0169t} \\ &= 3.2e^{0.0169\times 116} \\ &\approx 22.7 \end{aligned}$$

Therefore we would expect that the population would be about 22.7 million in 2010.

(c) We need to substitute $P = 10$ into the equation and solve,

$$\begin{aligned} 10 &= 3.2e^{0.0169t} \\ \frac{10}{3.2} &= e^{0.0169t} \\ \ln(3.125) &= 0.0169t \\ t &\approx 67.4 \end{aligned}$$

Therefore the population of Australia should have reached 10 million in approximately 1894+67, that is 1961.

(d) To determine how quickly the population was changing between 1954 and 1955, we could calculate average rate of change from 1954 to 1955.

In 1954, $t = 1954 - 1894 = 60$

$$\begin{aligned} P &= 3.2e^{0.0169t} \\ &= 3.2e^{0.0169\times 60} \\ &\approx 8.82 \end{aligned}$$

In 1955, $t = 61$ and therefore

$$\begin{aligned} P &= 3.2e^{0.0169t} \\ &= 3.2e^{0.0169\times 61} \\ &\approx 8.97 \end{aligned}$$

Consequently the average rate of change in population is:

$$\begin{aligned} \frac{\Delta P}{\Delta t} &\approx \frac{8.97 - 8.82}{1} \\ &\approx 0.15 \end{aligned}$$

That is the population was changing at 0.15 million per year.

(e) The y-intercept indicated the population at the start of the period being studied i.e. 1894.

2. (a) We can't find the actual growth, since we don't know N_0, but we can find the growth in terms of N_0.

In 1999, $t = 0$, so $N = N_0$

In 2004, $t = 5$, so $N = N_0(1.2)^5$

$$\begin{aligned}\text{Average rate of growth} &= \frac{\text{change in height}}{\text{change in horizontal distance}}\\ &= \frac{N_0(1.2)^5 - N_0}{5-0}\\ &= \frac{N_0((1.2)^5 - 1)}{5}\\ &= \frac{N_0(1.48832)}{5}\\ &= N_0 \times 0.297664\end{aligned}$$

(b) We can do the same for the years 2004 and 2009

In 2004, $t = 5$, so $N = N_0(1.2)^5$

In 2009, $t = 10$, so $N = N_0(1.2)^{10}$

$$\begin{aligned}\text{Average rate of growth} &= \frac{\text{change in height}}{\text{change in horizontal distance}}\\ &= \frac{N_0(1.2)^{10} - N_0(1.2)^5}{10-5}\\ &= \frac{N_0((1.2)^{10} - (1.2)^5)}{5}\\ &\approx \frac{N_0(3.7034)}{5}\\ &\approx N_0 \times 0.7407\end{aligned}$$

(c) The average rate of growth for the five year period from 2004 to 2009 is about 2.5 times the growth rate from 1999 to 2004. $\left(\text{since } \dfrac{N_0 \times 0.7407}{N_0 \times 0.297664} \approx 2.5\right)$

3. (a) We need to substitute $A = 0.5$, $A_0 = 1$ and $t = 1620$ into the equation,

$$\begin{aligned}A &= A_0 e^{-kt}\\ 0.5 &= 1e^{-k1620}\\ \ln(0.5) &= -k \times 1620\\ k &= \frac{\ln(0.5)}{-1620}\\ &\approx 0.00043\end{aligned}$$

(b) We need to substitute $A_0 = 10$, $A = 1$, $k = 0.00043$ into the equation and solve for t

$$
\begin{aligned}
A &= A_0 e^{-kt} \\
1 &= 10e^{-0.00043t} \\
0.1 &= e^{-0.00043t} \\
\ln 0.1 &= -0.00043t \\
t &\approx 5355
\end{aligned}
$$

It would take approximately 5355 years for 10g of Radium to decay to 1g.

(c) We need to calculate the amount that remains after 5 years, then the amount after 6 years and examine how much has decayed during the 5th year.

After 5 years the amount remaining would be:

$$
\begin{aligned}
A &= 10e^{-0.00043t} \\
A &= 10e^{-0.00043\times 5} \\
&\approx 9.979 \text{ g}
\end{aligned}
$$

After 6 years, the amount remaining would be:

$$
\begin{aligned}
A &= 10e^{-0.00043t} \\
A &= 10e^{-0.00043\times 6} \\
&\approx 9.974 \text{ g}
\end{aligned}
$$

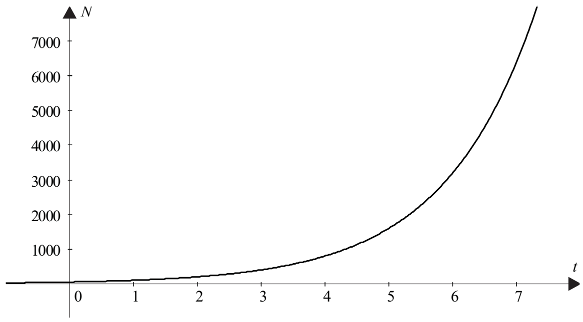
Therefore the rate of decay during the 5th year would be approximately:

$$
\begin{aligned}
\frac{\Delta A}{\Delta t} &= \frac{9.979 - 9.974}{1} \\
&\approx 0.005 \text{ g/year}
\end{aligned}
$$

4. (a) The table is shown below:

Time (days)	Number
0	50
1	100
2	200
3	400
4	800
5	1600
6	3200
7	6400

(b) The graph is shown below, drawn using Graphmatica.



(c) A function which would describe the situation would be:

$N = 50 \times 2^t$

Use the values in the table to check that it works.

5. (a) For this particular experiment the formula we will use is:

$$G = 20\log\left(\frac{v_o}{50}\right)$$

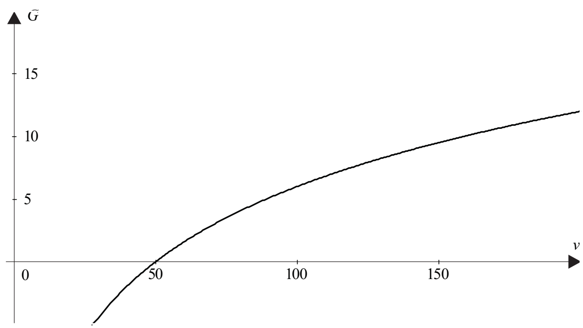
Therefore when the output voltage is 160 mV, we need to substitute $v_0 = 160$ into the above equation:

$$\begin{aligned} G &= 20\log\left(\frac{160}{50}\right) \\ &\approx 20 \times 0.505 \\ &\approx 10.1\,\text{dB} \end{aligned}$$

(b) Assuming that any one of the voltages from 50 mV to 200 mV was obtainable in this experiment, we can graph the equation:

$$G = 20\log\left(\frac{v_o}{50}\right)$$

using Graphmatica, and setting the domain to: $0 < v_0 < 200$:



Notice that when the output voltage is less than 50 mV, the gain is negative, that is the amplifier is not functioning correctly.

(c) To examine the rate at which the midband gain is changing at 100 mV, we need to look at the change in gain between say 100 mV and 105 mV.

when $v_0 = 100,\ G = 20\log\left(\frac{100}{50}\right) \approx 6.02$

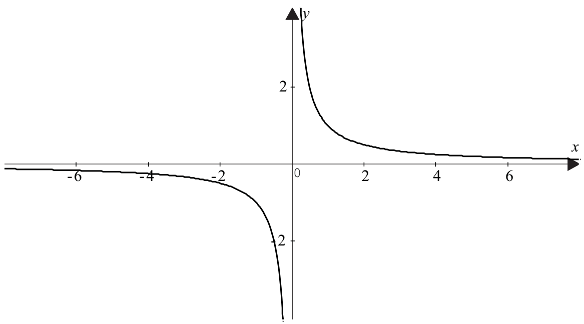
when $v_0 = 105,\ G = 20\log\left(\frac{105}{50}\right) \approx 6.44$

therefore the average rate of change of gain at this point will be approximately:

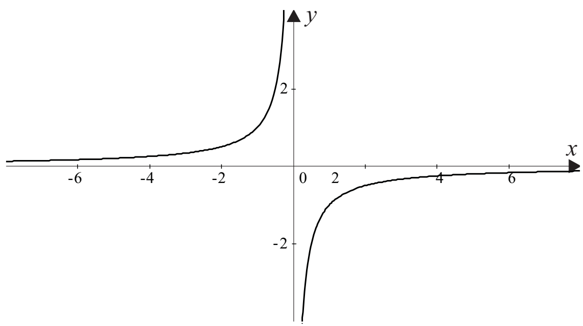
$$\frac{\Delta G}{\Delta v_0} = \frac{6.44 - 6.02}{105 - 100} = 0.084 \text{ dB/mV}$$

Activity 3.9

1. The graph of $y = \frac{1}{x}$ is shown below:

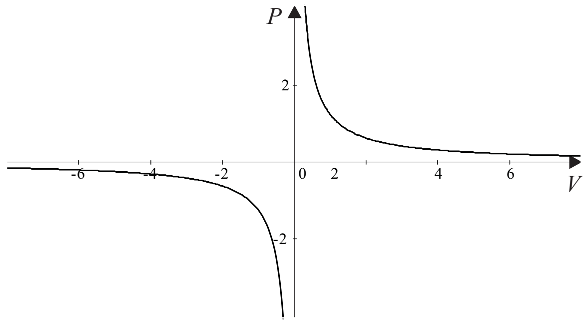


and the graph of $y = -\frac{1}{x}$ below:



You will notice that both functions have asymptotes at $x = 0$ and $y = 0$. The second function, however, occupies the second and fourth quadrants, instead of the first and third. It is a reflection of the first function about the line $x = 0$.

2. (a) The function $P = \dfrac{1.25}{V}$ is shown below:



(b) In order to represent the situation described, the domain of $P = \dfrac{1.25}{V}$ would need to be restricted to: $V > 0$ since we cannot have a negative or zero volume.

(c) From the above graph it can be seen that when $V \to \infty,\ P \to 0$. Consequently the pressure would become very small when the volume is increased.

3. (a) Since the function can be written in the form $y = \dfrac{p(x)}{q(x)}$ where both numerator and denominator are polynomial functions, the function is a rational function.

(b) To find the vertical intercept, substitute $x = 0$ into the equation, which yields $y = 3$. To find the horizontal intercept, substitute $y = 0$ into the equation and solve.

$$y = \frac{3}{x+1}$$
$$0 = \frac{3}{x+1}$$

There is no solution to this equation, as multiplying both sides by $(x + 1)$ yields the following:

$0 = 3$

Consequently there is no horizontal intercept.

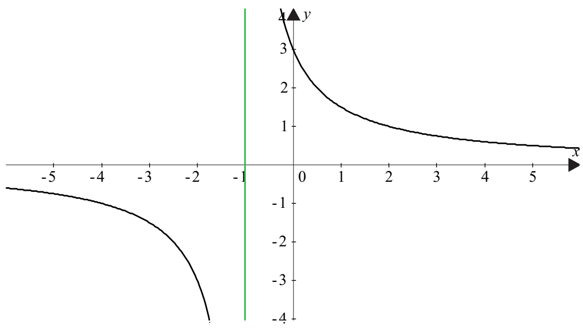
(c) Vertical asymptotes occur whenever the function is not defined, this occurs when values of the domain result in division by zero. If we examine the denominator of the function, we see that is will result in zero when $x = -1$, consequently division by zero will occur when $x = -1$, and so a vertical asymptote exists on this line.

Horizontal asymptotes can be calculated by finding what happens to the function at its extremities, that is when $x \to \pm\infty$. Using a calculator we find that when x becomes very large, the function becomes very small, that is when $x \to \infty,\ y \to 0^+$. When x becomes very small, for example –1000 000, the value of the function becomes a small negative number, that is, when $x \to -\infty,\ y \to 0^-$. Consequently the line is a horizontal asymptote.

(d) To draw a neat sketch, commence by plotting the vertical intercept (0,3). Then draw using an unbroken line the two asymptotes $x = -1$ and $y = 0$. Finally calculate and plot some points, using a table of values, for example:

x	–3	–2	–1.5	–0.5	0	1	2
y	–1.5	–3	–6	6	3	1.5	1

The graph is shown below, using Graphmatica.



4. (a) If we firstly rearrange the function, we obtain:

$$y = \frac{-2}{x-3} + 2$$
$$y = \frac{-2}{x-3} + \frac{2(x-3)}{x-3}$$
$$y = \frac{-2+2x-6}{x-3}$$
$$y = \frac{2x-8}{x-3}$$

Both the numerator and denominator of the function are themselves polynomial functions, consequently it is a rational function.

(b) To determine the vertical intercept, substitute $x = 0$ into the equation, therefore $y = \frac{8}{3}$.

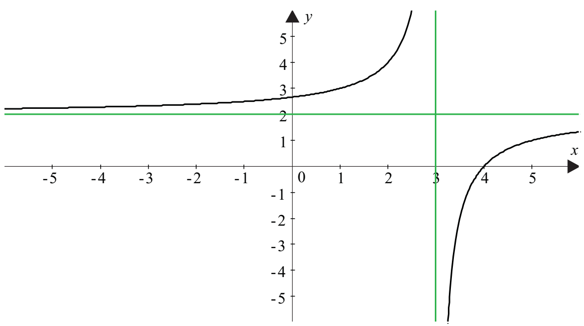
To determine the horizontal intercept, substitute $y = 0$ into the equation and solve.

$$\begin{aligned} y &= \frac{-2}{x-3} + 2 \\ 0 &= \frac{-2}{x-3} + 2 \\ -2 &= \frac{-2}{x-3} \\ -2(x-3) &= -2 \\ -2x + 6 &= -2 \\ -2x &= -8 \\ x &= 4 \end{aligned}$$

(c) To find the vertical asymptote, find for which values in the domain the function is undefined. From an inspection of the function, we see that when $x = 3$ division by zero occurs, so consequently an asymptote occurs on this line.

Horizontal asymptotes can be calculated by finding what happens to the function at its extremities, that is when $x \to \pm\infty$. Using a calculator we find that when $x \to \infty$, $y \to 2^-$ (that is the function gets close to 2, from the negative direction). Similarly when $x \to -\infty$, $y \to 2^+$. Therefore a horizontal asymptote occurs on the line $y = 2$.

(d) The function is shown below using Graphmatica.



5. (a) Average rate of change $= \dfrac{\Delta I}{\Delta R} = \dfrac{I_2 - I_{1.9}}{2 - 1.9} \approx \dfrac{6 - 6.32}{2 - 1.9}$

$$\approx \frac{-0.32}{0.1}$$

$$\approx -3.2 \text{ amps/ohm}$$

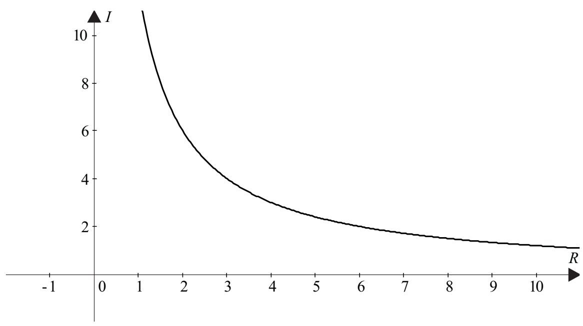
(b) Average rate of change $= \dfrac{\Delta I}{\Delta R} = \dfrac{I_{15} - I_{14.9}}{15 - 14.9} \approx \dfrac{0.8 - 0.81}{15 - 14.9}$

$$\approx \frac{-0.01}{0.1}$$
$$\approx -0.1 \text{ amps/ohm}$$

Consequently the current is decreasing more rapidly when we have small values of R then it does when R is larger.

(c) The function should be restricted to all positive values of R, that is $R > 0$.

(d) The function is shown below:



Activity 3.10

1. This forms an arithmetic sequence with a first term of 12 and a common difference of 3. Consequently $a_1 = 12$, $d = 3$ and we need to calculate the 52nd term, a_{52}.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ a_{52} &= 12 + (52-1)\times 3 \\ &= 12 + 51 \times 3 \\ &= 12 + 153 \\ &= 165 \end{aligned}$$

2. Apart from the initial deposit, the deposits made by Aunt Beatrice form an arithmetic series with a first term of \$100 and a common difference of \$100. We need to find the sum of the first 21 terms of this series. Consequently $a_1 = 100$, $d = 100$ and we need to calculate S_{21}.

$$\begin{aligned} S_n &= \frac{n}{2} \times \left[2a_1 + (n-1)d\right] \\ S_{21} &= \frac{21}{2} \times \left[2 \times 100 + (21-1)\times 100\right] \\ &= 10.5 \times \left[200 + 2000\right] \\ &= 10.5 \times 2200 \\ &= 23100 \end{aligned}$$

Therefore the total amount deposited by Aunt Beatrice will be \$23150 (including the \$50 deposited on the actual day of birth).

3. This an arithmetic series with a first term of 2, a common difference of 6 and we need to find S_{20}.

$$\begin{aligned}
S_n &= \frac{n}{2}\times\left[2a_1+(n-1)d\right]\\
S_{20} &= \frac{20}{2}\times\left[2\times 2+(20-1)\times 6\right]\\
&= 10\times\left[4+19\times 6\right]\\
&= 10\times 118\\
&= 1180
\end{aligned}$$

4. (a) If we consider the amount that is being worn from the bearing each hour we find that it forms an arithmetic sequence with first term 25μm (note the symbol μm is a micrometre, sometimes called a micron) and common difference 15μm. Consequently we need to find the 15th term of this sequence with $a_1 = 25$ and $d = 15$.

$$\begin{aligned}
a_n &= a_1+(n-1)d\\
a_{15} &= 25+(15-1)\times 15\\
&= 25+14\times 15\\
&= 235
\end{aligned}$$

If the pattern continued after the 15th hour of use an additional $235\ \mu$m will have been worn from the radius of the wheel bearing.

To reduce to one half its size we need to find when the sum of the above arithmetic series exceeds one half of the original 5 mm or rather $2500\ \mu$m.

Consequently: $a_1 = 25$, $d = 15$ and $S_n = 2500$ and we are trying to find n

$$\begin{aligned}
S_n &= \frac{n}{2}\times\left[2a_1+(n-1)d\right]\\
2500 &= \frac{n}{2}\times\left[2\times 25+(n-1)\times 15\right]\\
5000 &= n\times\left[50+15n-15\right]\\
5000 &= n\times\left[35+15n\right]\\
5000 &= 35n+15n^2\\
0 &= 15n^2+35n-5000
\end{aligned}$$

This is a quadratic equation whose solution is facilitated by the use of the quadratic formula:

$$\begin{aligned}
n &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
&= \frac{-35\pm\sqrt{35^2-4\times 15\times(-5000)}}{2\times 15}\\
&\approx -19.46 \text{ or } 17.13
\end{aligned}$$

Consequently after 18 hours the wheel bearing would have worn to over one half its original size (note that after 17 hours it would be still slightly over one half its original size).

5. The lengths of the rungs form an arithmetic sequence with first term 600 last term 400 and the sum of these terms 3500 mm.

   Note the formula for the sum of the 'n' terms of an arithmetic sequence is:

$$\begin{aligned} S_n &= \frac{n}{2}\times\left[2a_1+(n-1)d\right] \\ &= \frac{n}{2}\times\left[a_1+a_1+(n-1)d\right] \\ &= \frac{n}{2}\times\left[a_1+a_n\right] \qquad \text{(since } a_n = a_1+(n-1)d\text{)} \end{aligned}$$

   If we use this form of the equation then the solution is fairly straight forward $a_1 = 600$, $a_n = 400$ and $S_n = 3500$.

$$\begin{aligned} S_n &= \frac{n}{2}\times\left[a_1+a_n\right] \\ 3500 &= \frac{n}{2}\times\left[600+400\right] \\ 7000 &= n\times 1000 \\ n &= 7 \end{aligned}$$

   Therefore there should be 7 rungs in the ladder.

Activity 3.11

1. (a) The ratio of consecutive terms in the sequence is always the same, namely:

$$\frac{25}{125} = \frac{5}{25} = \frac{1}{5} = 0.2 = r$$

   (b) The sequence is a geometric sequence with first term $a_1 = 125$ and common ratio $r = 0.2$. We need to find the 10th term, a_{10}.

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ a_{10} &= 125 \times 0.2^9 \\ &= 6.4 \times 10^{-5} \end{aligned}$$

2. This is a geometric sequence with first term $a_1 = 8$, common ratio $r = 1.5$ and we need to find the sum of the first 15 terms, S_{15}.

$$\begin{aligned} S_n &= \frac{a_1(1-r^n)}{1-r} \\ S_{15} &= \frac{8(1-(1.5)^{15})}{1-1.5} \\ &\approx \frac{8(1-437.9)}{-0.5} \\ &\approx 6990 \end{aligned}$$

3. The first option certainly sounds OK, but the second one should be explored. If we list the amounts paid after successive days employment, we see that they form a geometric sequence.

   1c, 2c, 4c, 8c, 16c, . . .

   with first term $a_1 = 1$ and common ratio $r = 2$. Consequently we need to calculate the 30th term of this sequence.

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ a_{30} &= 1 \times 2^{29} \\ &\approx 5.37 \times 10^8 \text{ cents} \\ &\approx 5.37 \times 10^6 \text{ dollars} \end{aligned}$$

   That is in the order of 5 million dollars, so the second option is the one that you would take.

4. (a) After 6 days is actually the 7th of the geometric sequence.

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ a_2 &= 20(1.1)^{7-1} \\ a_2 &\approx 35.43 \end{aligned}$$

   Therefore there would be about 35000 bacteria present after 6 days.

   (b) The first three numbers counted are the terms of a geometric sequence, with first term $a_1 = 20$ and common ratio $r = 1.1$. We need to find for what value of n the term will exceed 40. So commence by letting $a_n = 40$.

$$\begin{aligned}
a_n &= a_1 r^{n-1} \\
40 &= 20 \times (1.1)^{n-1} \\
2 &= (1.1)^{n-1} \\
\log 2 &= \log(1.1)^{n-1} \\
\log 2 &= (n-1)\log 1.1 \\
\frac{\log 2}{\log 1.1} &= n-1 \\
n &= \frac{\log 2}{\log 1.1} + 1 \\
n &\approx 8.27
\end{aligned}$$

Therefore the 9th term of the sequence would be the first to exceed a value of 40, this represents a period of 8 days after the first observation.

5. (a) The measurements taken are the first three terms of a geometric sequence with first term $a_1 = 12$ and common ratio $r = \frac{11.4}{12} = 0.95$. We need to find the 20th term.

$$\begin{aligned}
a_n &= a_1 r^{n-1} \\
a_{20} &= 12 \times 0.95^{19} \\
&\approx 4.5
\end{aligned}$$

Consequently after the 20th swing the pendulum should reach 4.5 cm from its central position (incidentally the central position is also called its rest position).

(b) To find the size of the 200th swing, we need to find the 200th term of the above sequence.

$$\begin{aligned}
a_n &= a_1 r^{n-1} \\
a_{200} &= 12 \times 0.95^{199} \\
&\approx 4.4 \times 10^{-4} \text{ cm}
\end{aligned}$$

Therefore the size of the 200th swing would not be observable to the naked eye.

(c) In theory the pendulum would continue to oscillate forever, however, in practice frictional forces vary according to the motion they are opposing, so these forces would of course completely stop the pendulum.

Activity 3.12

1. This is a geometric sequence with first term $a_1 = 125$ and common ratio $r = 0.2$.

$$\begin{aligned} S_\infty &= \frac{a_1}{1-r} \\ &= \frac{125}{1-0.2} \\ &= \frac{125}{0.8} \\ &= 156.25 \end{aligned}$$

2. The repeating decimal $0.\overline{9} = 0.9999999... = 0.9 + 0.09 + 0.009 + ...$ This is in turn a geometric sequence with first term $a_1 = 0.9$ and common ratio $r = 0.1$.

   The infinite sum is therefore:

$$\begin{aligned} S_\infty &= \frac{a_1}{1-r} \\ &= \frac{0.9}{1-0.1} \\ &= \frac{0.9}{0.9} \\ &= 1 \end{aligned}$$

   (This is a nice little proof to convince people that $0.\overline{9} = 1$).

3. This series is formed from a geometric sequence with first term $a_1 = 15$ and common ratio $r = \frac{2}{3}$. The infinite sum is therefore:

$$\begin{aligned} S_\infty &= \frac{a_1}{1-r} \\ &= \frac{15}{1-\frac{2}{3}} \\ &= \frac{15}{\frac{1}{3}} \\ &= 45 \end{aligned}$$

4. For simplicity, we could assign an initial volume to the tank, say 10000 litres. The volumes remaining after 30 minute periods are shown below:

time (30 minutes)	0	1	2	3
Volume (litres)	10000	5000	2500	1250

   We see that these volumes form a geometric sequence with first term $a_1 = 10000$ and common ratio $r = 0.5$, consequently the nth term of this sequence will be the volume remaining after $(n - 1)$ 30 minute periods.

$$\begin{aligned}T_n &= a_1 r^{n-1}\\ &= 10000\times\left(\frac{1}{2}\right)^{n-1}\\ &= 10000\times(2^{-1})^{n-1}\\ &= 10000\times 2^{-n+1}\\ &= 10000\times 2\times 2^{-n}\\ &= 20000\times 2^{-n}\end{aligned}$$

To find when the tank is one fiftieth full we need to find when the term reaches 200 (i.e. $10000\div 50$). Therefore:

$$\begin{aligned}T_n &= 20000\times 2^{-n}\\ 200 &= 20000\times 2^{-n}\\ 0.01 &= 2^{-n}\\ \log 0.01 &= \log 2^{-n}\\ \log 0.01 &= -n\log 2\\ n &= \frac{-\log 0.01}{\log 2}\\ n &\approx 6.64\end{aligned}$$

Therefore the tank will be one fiftieth of its original volume after 7 terms, i.e. after six 30 minute periods or 3 hours.

Since the expression which determines the volume, that is $T_n = 20000\times 2^{-n}$ is an exponential function, we know that it will possess asymptotic properties. As $n\to\infty$, $T_n\to 0$. Consequently the tank will never completely empty.

5. (a)

Number of bounces	1	2	3	4	5	6	7
Height above rest position (cm)	20	18	16.2	14.58	13.12	11.81	10.63

(b) We see that the heights above shock absorber form a geometric sequence with first term $a_1 = 20$ and common ratio $r = 0.9$. The height of the *n*th bounce is the *n*th term of the sequence.

$$\begin{aligned}T_n &= a_1 r^{n-1}\\ &= 20\times 0.9^{n-1}\end{aligned}$$

To find when it will reach 8 cm from its rest position, substitute $T_n = 8$ into the above equation.

$$8 = 20 \times 0.9^{n-1}$$
$$0.4 = 0.9^{n-1}$$
$$\log 0.4 = \log 0.9^{n-1}$$
$$\log 0.4 = (n-1)\log 0.9$$
$$n-1 = \frac{\log 0.4}{\log 0.9}$$
$$n = \frac{\log 0.4}{\log 0.9} + 1$$
$$n \approx 9.7$$

After the 10th bounce at well below 8 cm.

6. (a) The completed table is shown below:

swing number	1	2	3
distance of swing m)	2.4	1.92	1.536

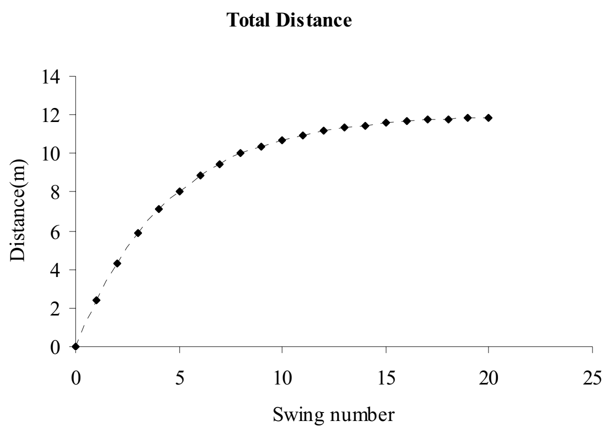
(b) The sizes of each swing form a geometric sequence with first term $a_1 = 2.4$ and common ratio $r = 0.8$. The sum of the first 4 terms can be calculated as follows:

$$S_n = \frac{a_1(1-r^n)}{1-r}$$
$$S_4 = \frac{2.4(1-0.8^4)}{1-0.8}$$
$$\approx 7.085$$

Therefore she will have travelled about 7 metres after the 4th swing.

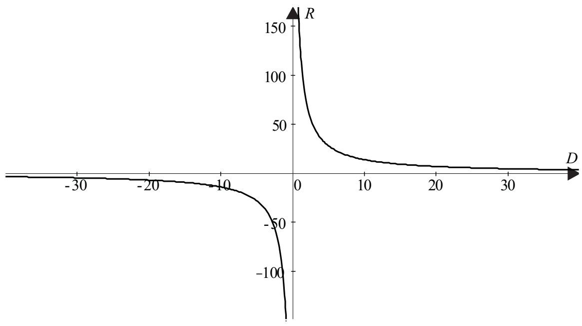
(c) $$S(n) = \frac{a_1(1-r^n)}{1-r}$$
$$= \frac{2.4(1-0.8^n)}{1-0.8}$$
$$= 12(1-0.8^n)$$
$$= 12 - 12 \times 0.8^n$$

(d)



Solutions to a taste of things to come

1. (a)



(b) The domain would be $D \geq 0$, although physical restraint would place an upper limit on this.

Range would be $R \geq 0$, with similar physical restraints placing an upper limit on the range.

(c) In this case $D = 40$, therefore:

$$\begin{aligned} R &= \frac{140}{D} \\ &= \frac{140}{40} \\ &= 3.5 \end{aligned}$$

The resolving power is 3.5 seconds or 0.0583 degrees (recall that 3600 seconds make one degree).

(d) If the human eye can only resolve when the angle is 60 seconds then the magnification of a 40 mm lens must be between 17 and 18 times, because $3.5 \times 17 = 59.5$ and $3.5 \times 18 = 63$.

2. (a) The first energy level, corresponding to $n = 1$ should be:

$$E_1 = \frac{-13.6}{1^2} = -13.6 \text{ eV}$$

Other levels are below:

$$E_2 = \frac{-13.6}{2^2} = -3.4 \text{ eV}$$

$$E_3 = \frac{-13.6}{3^2} \approx -1.5 \text{ eV}$$

$$E_4 = \frac{-13.6}{4^2} = -0.85 \text{ eV}$$

(b) The difference in energy between the fourth and the second levels is:

$$\begin{aligned}\Delta E &= E_4 - E_2\\ &= -0.85 - -3.4\\ &= 2.55\text{ eV}\end{aligned}$$

Between the third and second levels, this energy difference would be:

$$\begin{aligned}\Delta E &= E_3 - E_2\\ &= -1.5 - -3.4\\ &= 1.9\text{ eV}\end{aligned}$$

These bursts of energy are given off in packets called quantums, which appear as light of different frequencies (i.e. colours).

Solutions to post-test

1. The relation is a function as for each x-value there is only one corresponding y-value. However it is possible that two values of x give the same value of y, accordingly the function is a many-to-one function.

2. (a) $$\begin{aligned}B(t)\times I(t) &= (13-t)(2t^2-3)\\ &= 26t^2 - 39 - 2t^3 + 3t\\ &= -2t^3 + 26t^2 + 3t - 39\end{aligned}$$

   (b) $$\begin{aligned}I(B(t)) &= I(13-t)\\ &= 2(13-t)^2 - 3\\ &= 2(169 - 26t + t^2) - 3\\ &= 338 - 52t + 2t^2 - 3\\ &= 335 - 52t + 2t^2\end{aligned}$$

3. Try solutions, which are factors of 15. We would expect to get at most three solutions. Possible solutions are $\pm1, \pm3, \pm5, \pm15$. Using the guess and check method we obtain:

x	$f(x)$
1	0
–1	24
3	–24
–3	0
5	0

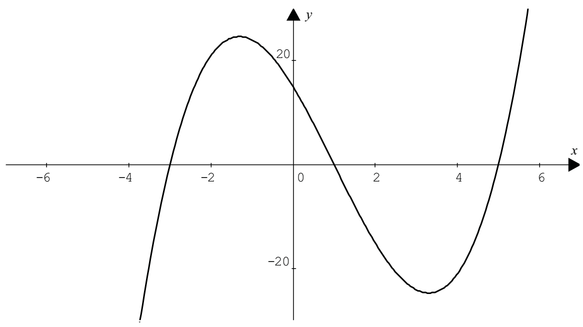
It is unnecessary to keep guessing after finding the three solutions. The solutions to the equation are $x = -3, 1, 5$.

4. We would expect that the function $y = x^3 - 3x^2 - 13x + 15$ would have at most two turning points. Also since it has a degree of three, we would expect that as $x \to \infty, y \to \infty$ and as $x \to -\infty, y \to -\infty$. To sketch the function we should also try to determine the vertical and horizontal intercepts.

   The vertical intercept will occur when $x = 0$ therefore we obtain $y = 15$.

   The horizontal intercepts will occur when $y = 0$. So we need to substitute this value into the original equation and solve. This was done in question 3, therefore the three horizontal intercepts are $x = 1$, $x = –3$, $x = 5$.

   The sketch is shown below:



5. (a)
$$y = \frac{3}{x-2} + 1$$
$$y = \frac{3}{x-2} + \frac{x-2}{x-2}$$
$$y = \frac{3+x-2}{x-2}$$
$$y = \frac{1+x}{x-2}$$

   The RHS is now in the form of one polynomial divided by another polynomial, so is a rational function.

   (b) The vertical intercept occurs when $x = 0$.

$$y = \frac{3}{0-2} + 1$$
$$y = -\frac{1}{2}$$

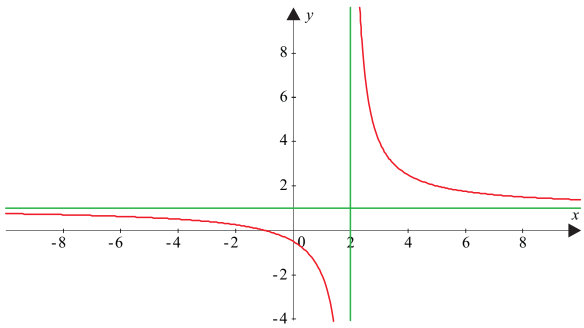
   The horizontal intercept occurs when $y = 0$.

$$
\begin{aligned}
0 &= \frac{3}{x-2}+1 \\
-1 &= \frac{3}{x-2} \\
-1(x-2) &= 3 \\
-x+2 &= 3 \\
-x &= 1 \\
x &= -1
\end{aligned}
$$

(c) To find the vertical asymptote consider where the function is undefined i.e. when $x - 2 = 0$. The vertical asymptote is the line $x = 2$.

To find the horizontal asymptote look at the function as x gets very large, as $x \to +\infty,\ y \to 1^+$ and as $x \to -\infty,\ y \to 1^-$. So $y = 1$ is the horizontal asymptote.

(d) A graph of the function $y = \dfrac{3}{x-2}+1$ is below.



6. We should be able to recognize this as a rectangular hyperbola, so therefore the corresponding function must be a rational function. A closer examination of the graph shows us that there are two asymptotes, one at $x = -1$ and the other at $y = -2$. The graph also passes through the point $(0, -1)$. Since there is an asymptote at $x = -1$ we can conclude that the denominator of the rational function may be of the form $x + 1$. Similarly the asymptote at $y = -2$ suggests that the function has a constant term of -2. One suitable equation may be:

$$y = \frac{1}{x+1} - 2$$

(Note: this also satisfies the condition that the function passes through the point $(0, -1)$)

7. We need to substitute $Q_0 = 2.17 \times 10^{-4}$, $Q = 1.5 \times 10^{-4}$ and $t = 3$ into the original equation and then solve for k.

$$
\begin{aligned}
Q &= Q_0 e^{-kt} \\
1.5\times10^{-4} &= 2.17\times10^{-4}\times e^{-k\times3} \\
0.69 &\approx e^{-k\times3} \\
\ln(0.69) &\approx \ln e^{-k\times3} \\
\ln(0.69) &= -k\times3\times\ln e \\
\ln(0.69) &= -k\times3 \\
k &\approx 0.124
\end{aligned}
$$

In order to find the amount of charge remaining after 6 seconds, we need to substitute $t = 6$ into the equation: $Q = 2.17\times10^{-4}e^{-0.124t}$, that is:

$$
\begin{aligned}
Q &= 2.17\times10^{-4}e^{-0.124\times6} \\
&\approx 2.17\times10^{-4}\times0.475 \\
&\approx 1.03\times10^{-4}\text{ C}
\end{aligned}
$$

8. This sequence is an arithmetic sequence with first term $a_1 = 12$ and common difference $d = 4$. We need to find the sum of the first twenty terms, that is S_{20}.

$$
\begin{aligned}
S_n &= \frac{n}{2}(2a_1 + (n-1)d) \\
S_{20} &= \frac{20}{2}(2\times12 + 19\times4) \\
&= 10\times100 \\
&= 1000
\end{aligned}
$$

9. The height of the first bounce will be 8m and the height of the second bounce will be 5m.

When the ball first hits the ground it will have travelled 12.8m.

When it hits on the second occasion it will have travelled $12.8 + 2\times8$.

When it hits on the third occasion it will have travelled $12.8 + 2\times8 + 2\times5$.

When it hits on the sixth occasion it will have travelled $12.8 + 2\times(8 + 5 + 3.125 + \ldots t_5)$.

The expression in the bracket represents the sum of the first five terms of a geometric sequence with first term $a_1 = 8$ and common ratio $r = \frac{5}{8} = 0.625$.

$$
\begin{aligned}
S_n &= \frac{a_1(1-r^n)}{1-r} \\
S_5 &= \frac{8(1-0.625^5)}{1-0.625} \\
&\approx 19.3
\end{aligned}
$$

Consequently when the ball hits on the sixth occasion is will have travelled a distance

$$\begin{aligned} d &= 12.8 + 2 \times (\underbrace{8 + 5 + 3.125 + \ldots t_5}_{S_n}) \\ &\approx 12.8 + 2 \times 19.3 \\ &\approx 51.4 \text{ m} \end{aligned}$$