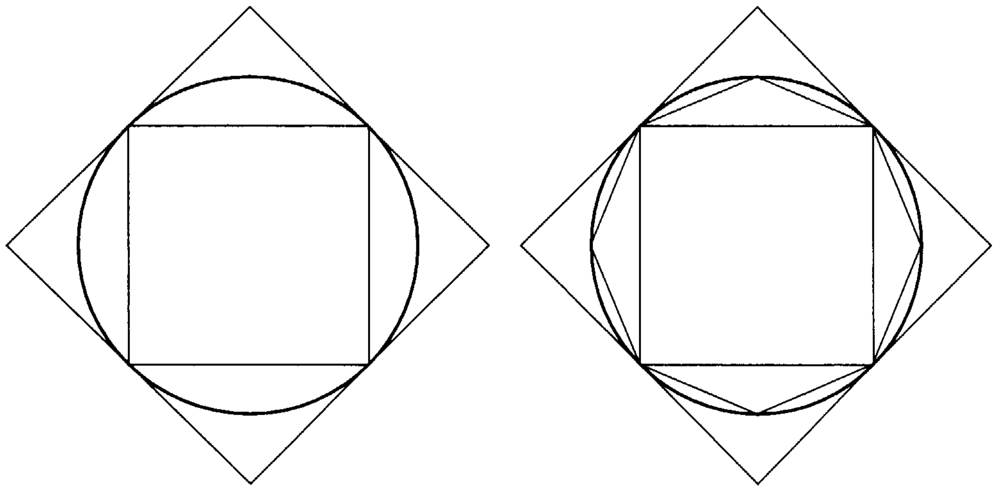
Module C7

Total change – an introduction to integral calculus

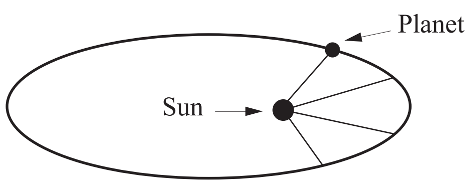
Table of Contents

Introduction

In this final section we will investigate another important aspect of calculus called integral calculus. One of the most important applications of integral calculus is to find areas and volumes. These problems have been considered by practitioners since the earliest times with the Greek mathematician Eudoxus (about 400 BC) credited with developing one of the first steps in calculus. He found an approximate area of the circle by sandwiching it between polygons with more and more sides.



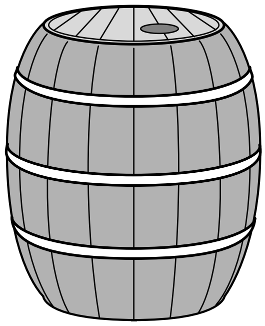
This was developed further by Archimedes to include many types of shapes, but it wasn't until the 17th Century when Johannes Kepler, during the course of his astronomical investigations, wanted a method for finding areas of sectors that ideas moved ahead. Kepler was interested in two things. Firstly, as an astronomer, he wanted to find the area of an ellipse formed by the path of planets around the sun. He approximated this by using areas of triangles with their vertex at the sun.



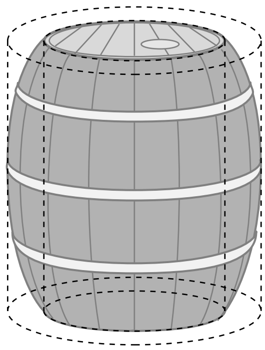
Secondly, as a connoisseur in wine, so the story goes, he wanted to find the exact method to calculate the volume of wine in kegs. Kepler thought of the wine barrel as being made up of infinitely many infinitely thin disks, the sum of whose areas became the volume.

From a more every day perspective, suppose you were a wine merchant in the 17th Century. In those days the kegs were all hand made, and each one would hold a different amount of wine. You would need to know approximately how much wine you were selling to the innkeepers. Just as important the innkeepers would want to keep a check on you to ensure you were not overcharging on the amount of wine bought.

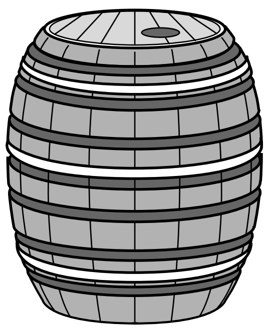
In the diagram below, think about the ways you could estimate the volume of wine in the keg, without pouring out the wine litre by litre, or by submerging the keg in a larger container of water.



Using the ideas of Archimedes and Kepler, one way you could do it is by thinking of the keg as a cylinder – taking the maximum and minimum volume of cylinders and averaging them.



Another way of thinking about it is to think of 'slicing' the keg into a number of cylinders and then adding them up; the more cylinders you took the more accurate you would be.



The concepts developed by Eudoxus, Archimedes and Kepler, and later by others such as Leibniz and Newton form the basis of integral calculus. We will look at some of these basic concepts, in the calculation of areas and volumes. More specifically on successful completion of this module you should be able to:

- find areas using various geometric methods
- find areas under curves using definite integrals
- demonstrate an understanding of the relationship between differentiation and integration
- find indefinite integrals
- apply calculus to various practical situations.

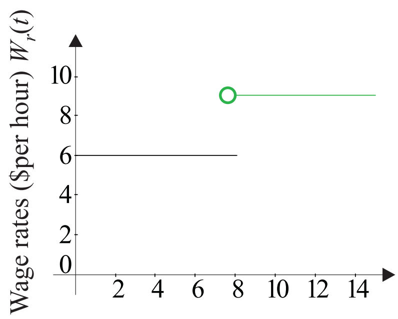
7.1 Area under the curve

In this module we will concentrate on the use of integral calculus in terms of areas under curves. As you study more maths, you will find much more widespread uses for integral calculus.

Consider the following:

A young wage earner is paid \$6 per hour for the first 8 hours and then \$9 per hour for any overtime. We can show this graphically in figure 7.1.

Figure 7.1: Wage rate over hours worked



We can also express it algebraically as

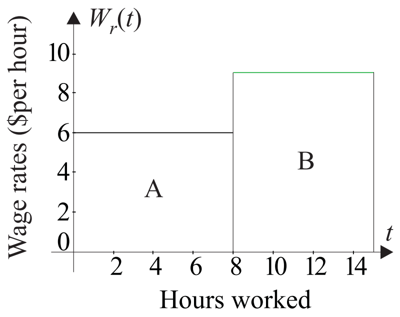
$$W_r(t) = 6; \quad t \le 8$$
$$\quad\;\; = 9; \quad t > 8$$

where t = time in hours and W_r is the wage rate in \$ per hour.

If the wage earner worked 15 hours altogether in a shift, then the total wages would be $6 \times 8 + 9 \times 7 = \$111$.

Look at this graphically. In figure 7.2, 6 × 8 is the area of rectangle A and 9 × 7 is the area of the rectangle B. So the total wage is the area under the function from $W_r(t)$ from $0 \le t \le 15$.

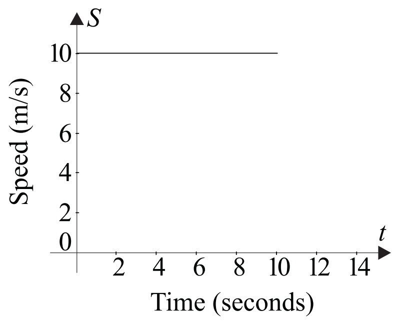
Figure 7.2: Wage rate over hours worked and total wage (A + B)



Let's look at another example.

If a car is travelling at a constant rate of 10 m/s for 10 seconds, this can be expressed graphically as in figure 7.3.

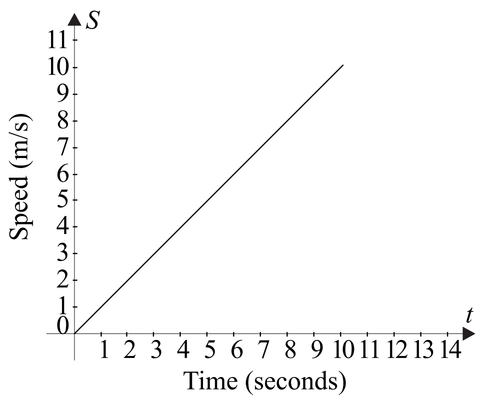
Figure 7.3: Car's speed over time



Recall that $\text{speed} = \dfrac{\text{distance}}{\text{time}}$ and therefore, the total distance travelled is 100 metres. Again this can be seen graphically as the area of the rectangle with dimensions 10 × 10.

What if the car was gradually increasing its speed at an acceleration of 1 m/s per second for 10 seconds, (i.e. constant acceleration)? The graph is depicted in figure 7.4.

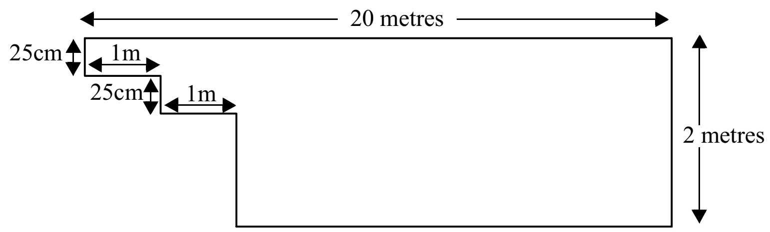
Figure 7.4: Speed of accelerating car over time



How much distance has been covered now? Can you guess that it will be half the distance of the previous example? Again it can be described as the area under the function. This time it is the area of a triangle, so the area is $\frac{10 \times 10}{2} = 50 \text{ metres}$.

Activity 7.1

1. An inground swimming pool is designed with 2 wide steps leading down from one end, the cross section as shown: The pool is 20 m long at water level and 2 m in depth. Each step is 25 cm deep and 1m wide.



   (a) Draw a graph showing water **depths** over the 20 m length.

   (b) Find the area of this cross section and hence find the capacity of a 7 m wide pool.

2. A household electricity account for a ninety day period during winter is as follows:

   Domestic use: 3 000 kWh @ 11 cents per kWh (incl. GST)

   Economy (controlled supply): 600 kWh @ 7 cents per kWh (incl. GST).

   Show this information graphically and hence find the total electricity account.

3. A real estate agent's commission is calculated on the selling price of the property he sells such that he receives 5.5% on the first $18 000 and 2.75% on anything in excess of that amount. Use a graph to calculate the total commission on a house sold for $165 000.

4. From a stationary position a motor cyclist accelerates at a constant rate for the first 30 seconds until she reaches a speed of 25 m/s (90 km/h). She then continues at this speed for another 30 seconds before decelerating (constantly) to a stop 20 seconds later. Use a graph of speed against time to find the distance covered.

In the previous examples we assumed the speed was constantly increasing and in the activity the cross sectional area was regular so we could use standard formulas and our knowledge of geometry to calculate the areas. Suppose the speed varied over time or we wanted to find the cross sectional area of a dam. How could we do this?

We could attempt it like Archimedes or Kepler, by making approximations. Let's look at two different ways of doing this with a number of different examples. You may think of other ways.

1. Averaging the upper and lower sums of rectangles.

2. Take a series of trapeziums.

Example

A car travels at varying speeds over a 10 second period as depicted in figure 7.5. When we first look at the car, it is travelling at 3 m/s and speeds up; then it keeps an almost constant speed before speeding up again. Over the 10 second period, what is the total distance travelled?

Figure 7.5: Car's velocity over time

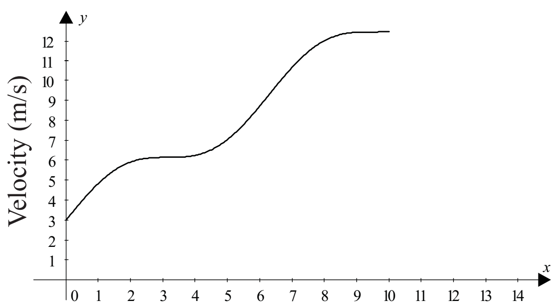
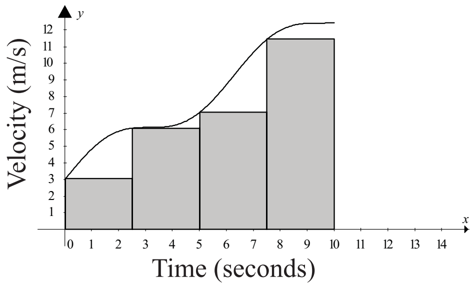


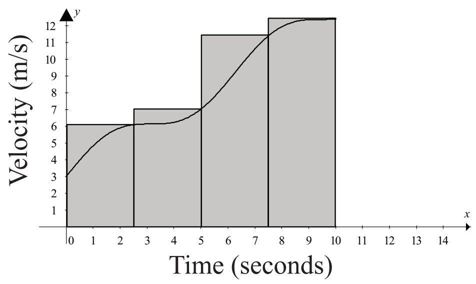
Figure 7.6: Finding the area using rectangles mainly below the curve



Take a series of rectangles of equal width mainly below the curve. Find the area of these rectangles (figure 7.6).

The small area
≈ 2.5 × 3 + 2.5 × 6 + 2.5 × 7 + 2.5 × 11.4
≈ 68.5

Figure 7.7: Finding the area using rectangles mainly above the curve



Take a series of rectangles above the curve and find this area (figure 7.7)

The large area
≈ 2.5 × 6 + 2.5 × 7 + 2.5 × 11.4 + 2.5 × 12.2
≈ 91.5

Since one will be slightly smaller than the real area and the other slightly larger the average of these two will give a better estimate of the area.

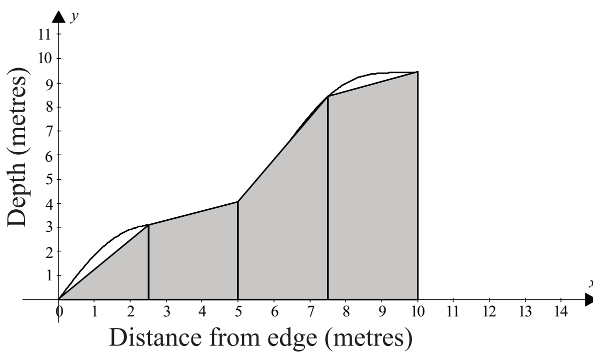
The average of these two areas is $\frac{68.5+91.5}{2}=\frac{160}{2}=80$

The total distance travelled will be approximately 80 metres.

Example

A small dam has a vertical retaining wall at one end. A cross section of the dam is shown in figure 7.8. Find the area of the cross section.

Figure 7.8: Finding the area using trapeziums



Divide the area into trapeziums with **equal bases**, then add the areas of these trapeziums (The area of a trapezium is the average of the two sides times the base).

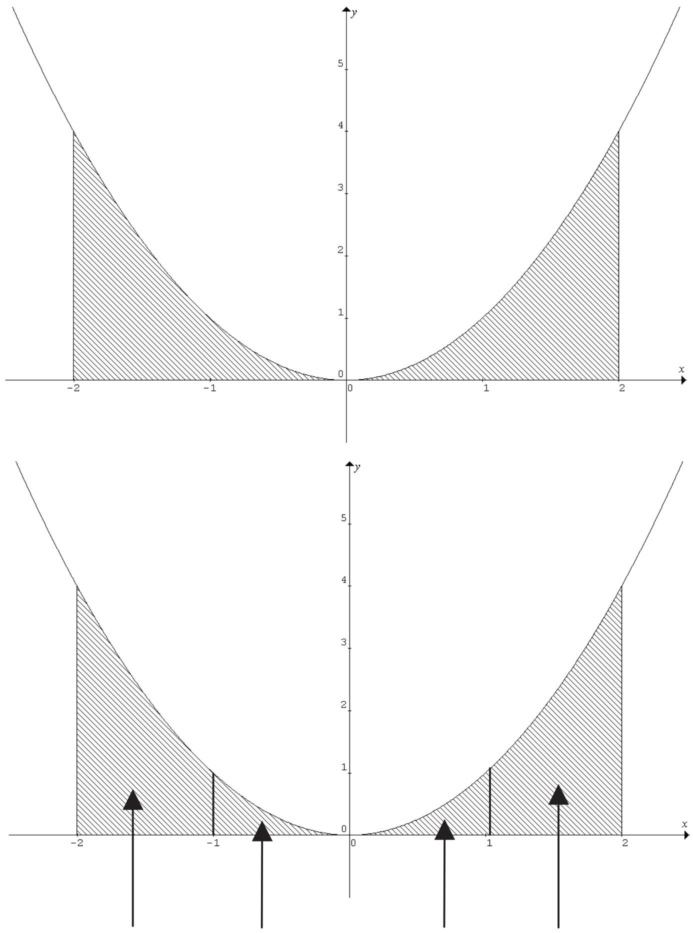
$$\text{Area} \approx \frac{3+0}{2}\times 2.5+\frac{3+4}{2}\times 2.5+\frac{4+8.2}{2}\times 2.5+\frac{8.2+9.3}{2}\times 2.5$$
$$\approx 2.5\times(1.5+3.5+6.1+8.75)$$
$$\approx 49.625 \text{ sq. metres}$$

It doesn't matter where the function lies on the Cartesian plane. Consider the following examples.

Example

Find the area of the shaded region below i.e., using 4 trapeziums.

Figure 7.9: Area under a curve in quadrants 1 and 2

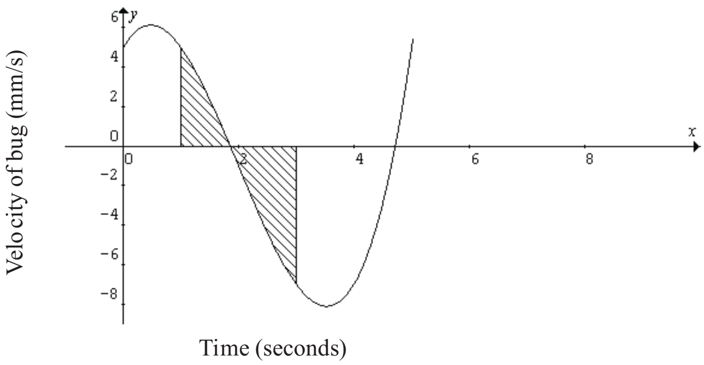


$$\text{Area} \approx \left(\frac{4+1}{2}\times 1+\frac{1+0}{2}\times 1\right)+\left(\frac{1+0}{2}\times 1+\frac{4+1}{2}\times 1\right)$$
$$\approx 3+3$$
$$\approx 6$$

Example

A bug is moving in a tube as depicted in figure 7.10. At the beginning, it is moving at 5 mm/s towards one end and speeding up. It then starts to slow down and stops after about 1.85 seconds. It then starts to move away from that end of the tube. What is the approximate distance it has moved between the first and third second?

Figure 7.10: The velocity of a bug over time



We need to find the area from $x = 1$ to $x = 3$.

To do this we can find the area of 2 triangles.

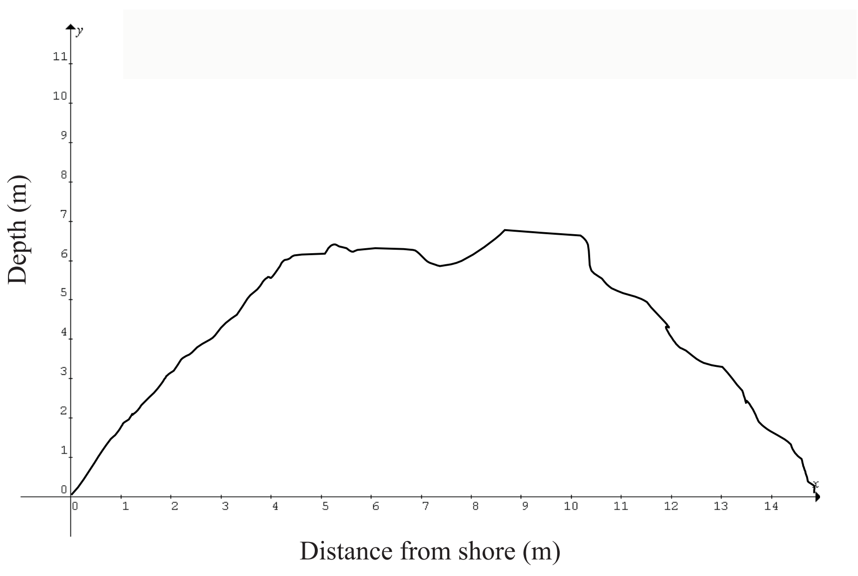
$$\text{Area} \approx \frac{0.85 \times 5}{2} + \frac{1.15 \times 7}{2}$$
$$\approx 6.15$$

So the bug has moved about 6.15 mm in the 2 seconds.

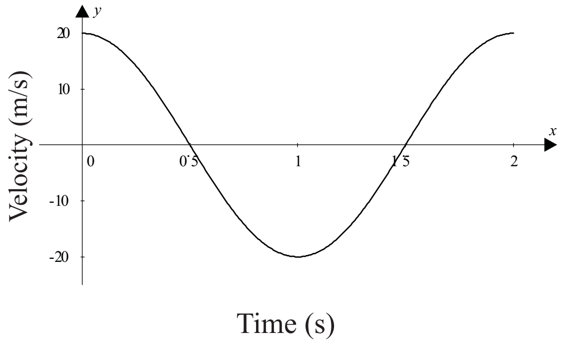
In this case the area under the curve does not represent a physical area. We are multiplying velocity in millimetres per second by time in seconds, so out answer will be in millimetres ($\frac{mm}{\not s} \times \frac{\not s}{1}$). When finding the area under the curve in practical problems like this, it is always wise to carefully check the units in your answer.

Activity 7.2

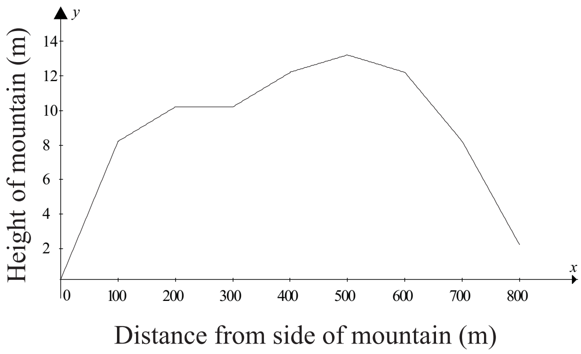
1. Find the approximate cross-sectional area of a dam using 6 trapeziums.



2. The velocity of a pendulum is recorded over one complete oscillation shown below. Timing starts as the pendulum passes the vertical position (maximum velocity) and swings toward the right. Half a second later it reaches the right side extension, stops momentarily, and then starts to swing back, passing the vertical position one second from the start. Use 3 (or 4) triangular areas to find the approximate distance travelled by the pendulum in 2 seconds.



3. The side of a mountain is to be cut away to improve an existing highway that currently winds over the elevation. The cross section is as shown.



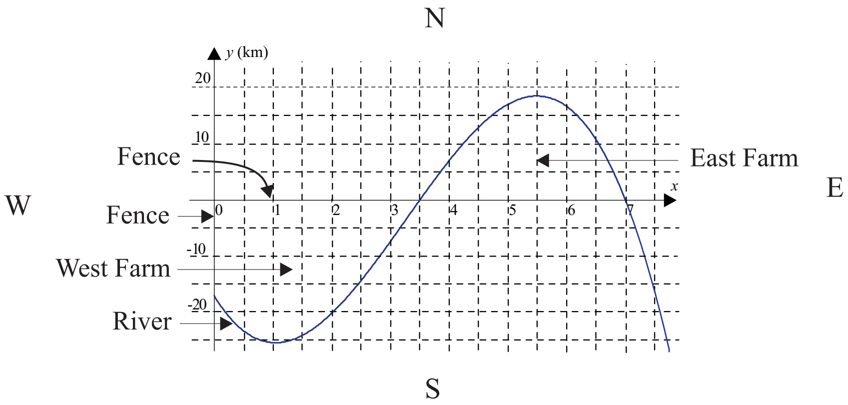
Use 8 trapeziums to find the approximate area of the 800 metre long cross section.

4. The following table gives widths (y) of a lake taken at regular intervals along one side (x) in metres.

x	0	5	10	15	20	25	30
y	14.2	14.8	12	13.6	13.6	13.6	12

Use the trapezoidal method to find the approximate area of the surface water.

5. Two farms are bounded by a straight road running east-west and a winding river. In addition one of the farms has a fence along its western border, as in the diagram below.



By calculating the areas of trapeziums at 0.5 km intervals, determine which farm is larger and by approximately how many hectares. (100 ha = 1 km^2)

7.2 The definite integral

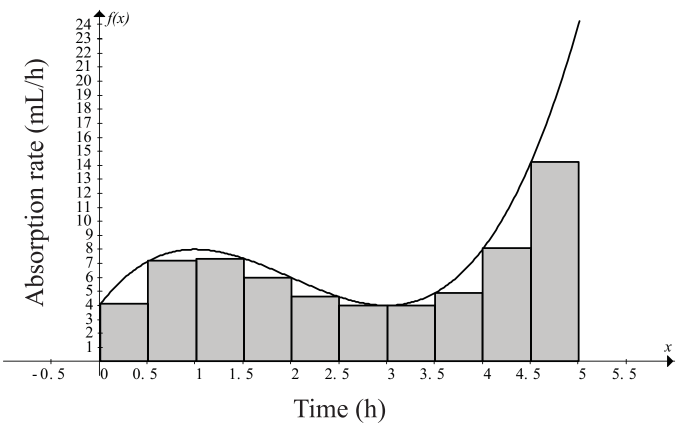
So far we have looked at finding the area under a curve by approximate methods, but this can take a lot of tedious calculations if the curve is not a regular geometric shape.

Let's now look at an easier way of calculating the area under the curve, when the curve can be expressed algebraically. But first let's treat areas in the same manner as the derivative in the previous module – by using the idea of a limit.

Example

The change in the rate at which alcohol is absorbed into the bloodstream (in millilitres per hour) of a drinker over a 5 hour period was monitored and is illustrated in figure 7.11. Find the total amount of alcohol in the bloodstream in the 5 hours period.

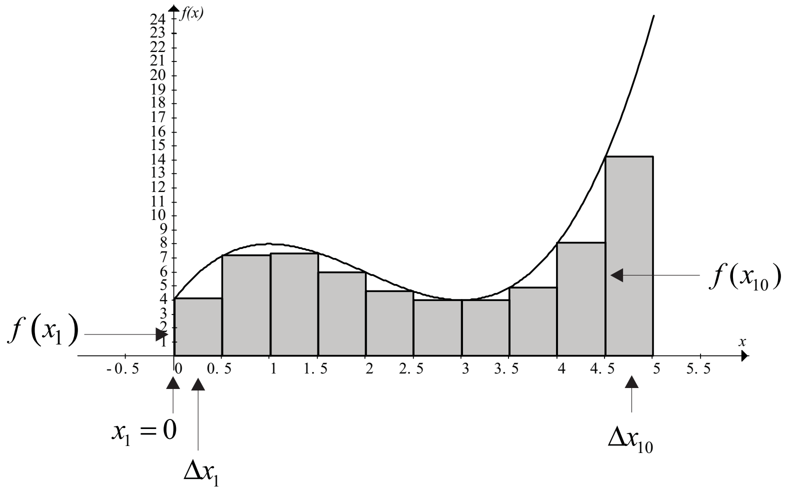
Figure 7.11: Absorption rate of alcohol over time



If we divide the total area in figure 7.11 into 10 rectangles each 0.5 wide, the area of the 10 rectangles is about 32 millilitres (about three beer's worth!).

Let's generalise the area of the rectangles in the example above. Let the base of each rectangle be Δx_i and the height of the rectangle $f(x_i)$ (similar to the notation we used in the last module).

Figure 7.12: Finding the area under the curve using a series of rectangles



So the first rectangle in figure 7.12 will be $f(x_1) \times \Delta x_1$. All the rectangles will have the same width of Δx. The area of the last rectangle is $f(x_{10}) \times \Delta x_{10}$, where $f(x_{10}) = f(4.5) \approx 14$. If we continue like this, then the area will be the sum of these rectangles. The area itself is a function, since for each value of *x* (i.e., the upper boundary of the area) there is one value for the area. So we call this new function $F(x)$.

$$F(x) = f(x_1) \times \Delta x_1 + f(x_2) \times \Delta x_2 + f(x_3) \times \Delta x_3 + f(x_4) \times \Delta x_4 ......f(x_{10}) \times \Delta x_{10}$$

In sigma notation this is

$$= \sum_{i=1}^{10} f(x_i) \times \Delta x_i$$

(If you are unsure of sigma notation, look at the module 2 again or contact your tutor.)

Now if we take these rectangles and make them thinner and thinner, we would get closer and closer to the actual area. In the alcohol example if we took 20 rectangles the area would be about 35.3 mL; if we took 40, the area would be about 37 mL.

Generally speaking if we took *n* rectangles, the area under the curve from $x = 0$ to $x = a$ (the right hand boundary) would become:

$$F(x) = f(x_1) \times \Delta x_1 + f(x_2) \times \Delta x_2 + f(x_3) \times \Delta x_3 + f(x_4) \times \Delta x_4 .....$$

$$F(x) = \sum_{i=1}^{n} f(x_i) \times \Delta x_i$$

The limit of these sums is called the **definite integral**.

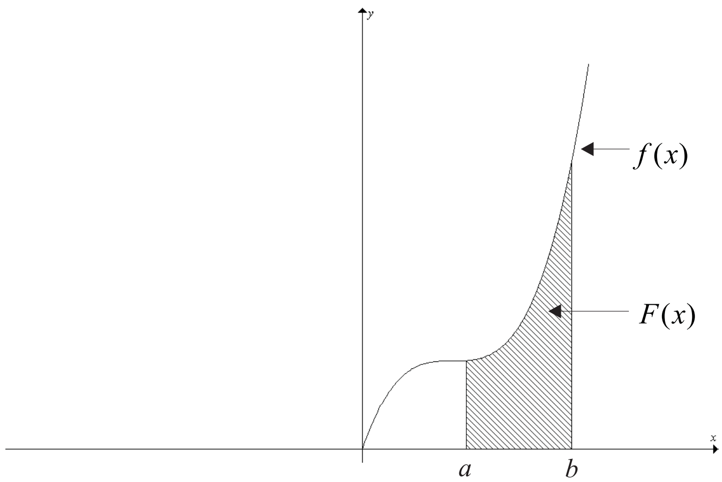
$$\lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\Delta x\right) = \int_0^a f(x)\ dx$$

We introduce the symbol (yes, another one!) $\int$ an old fashioned, elongated 'S' which stands for sum. So to say: 'find the sum from $i = 1$ to n', (i.e. from $x = 0$ to $x = a$), we would write $\int_0^a$. Notice we use the numbers 0 and a , since we are starting from the first value of $f(x)$, which is when $x = 0$ and go to the last value of x which is a. We want to find the area under the curve of the function $f(x)$, so we write $\int_0^a f(x)dx$. The lowest value of x is always on the bottom of the $\int$ and the highest value is on the top. The 'dx' comes from the factor Δx. Note that this is now **not** a product, but merely indicates we are integrating with respect to x. In the case above we say 'the definite integral from 0 to a of f of x, dx'.

This symbolism comes from the notation used by Leibniz who developed this calculus in the late 17th Century.

We have started our function above at zero, but it doesn't matter where the function starts, the sum of the rectangles will always give an approximation for the area under the curve.

Figure 7.13: The area under the curve of a function $f(x)$ from $x = a$ to $x = b$



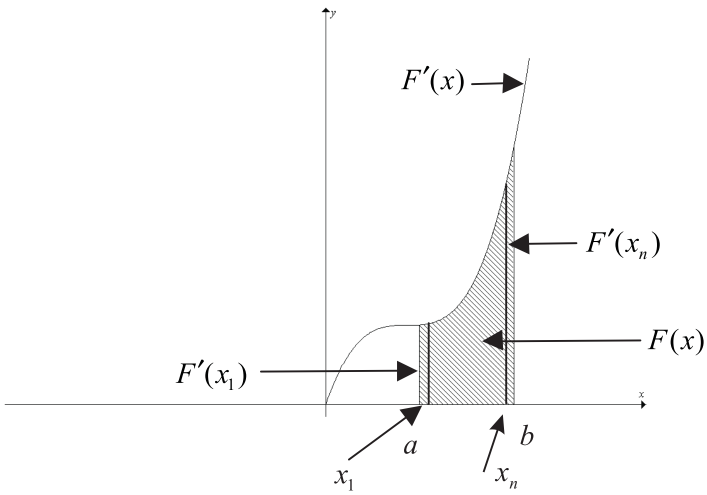
The area under the curve of a function $f(x)$ from $x = a$ to $x = b$ is:

$$F(x) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\Delta x_i\right) = \int_a^b f(x)dx$$

> Note: the definite integral is the limit of a sequence of sums. The notation used is $\int_a^b f(x)\,dx$. While the notation reminds us of how the integral is obtained, we should really treat the $f(x)\,dx$ as one whole symbol. The individual symbol *dx* by itself has no real meaning. While there is an implication that $f(x)\,dx$ is a product, this is not the case.
>
> $\int_a^b f(t)\,dt$ means we are finding the integral of a function with respect to (the independent variable being *t*) in the interval from *a* to *b*.

In the examples we have given, the definite integral gives the total change in a quantity from a rate of change of a related quantity (e.g. the total distance from the velocity; the total amount of alcohol from the rate change of alcohol in the body; total wage from the wage rate). From the last module, you know that the rate of change of a quantity $F(x)$, is given by the derivative $F'(x)$. Let's now consider depicting this function, $F'(x)$, on a graph similar to figure 7.13.

Figure 7.14: The area ($F(x)$) under the curve of a function $f(x)$ from $x = a$ to $x = b$



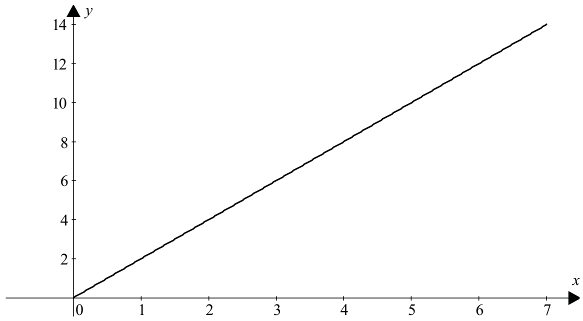
So now instead of $f(x)$ we have $F'(x)$ and the area under the curve of the function $F'(x)$, i.e.

$$\text{total change in } F \text{ between } a \text{ and } b = \sum_{i=1}^{n} F'(x_i) \times \Delta x_i = \int_a^b F'(x)dx$$

So if we knew an algebraic formula for the derivative function, we could find the original function by 'undoing' the derivative. This is a very important concept, and may take a while for you to be comfortable with it. At this level of mathematics, you don't have to prove this, but to see why this is true, let's look at a few functions.

Take the function $y = 2x$.

Figure 7.15: $y = 2x$



Find areas under the curve from $x = 0$ and complete the following table (as the areas are triangles, this is relatively easy to do):

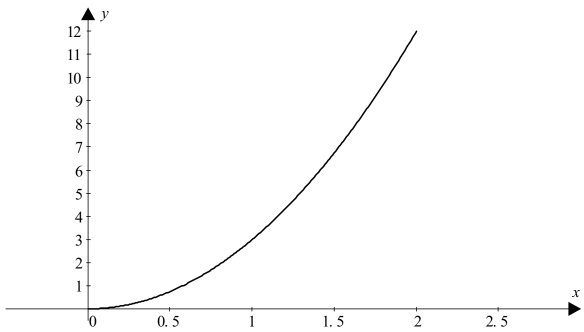
x	Total area
0	0
1	1
2	4
3	
4	
5	
6	
7	

For the value of x from 0 to 7 you should have the following areas:

x	0	1	2	3	4	5	6	7
Area $F(x)$	0	1	4	9	16	25	36	49

Can you recognise the relation as the function $F(x) = x^2$? (see figure 7.16) Can you see the pattern would work even if we increased our domain?

Figure 7.16: $F(x) = x^2$



So we could say:

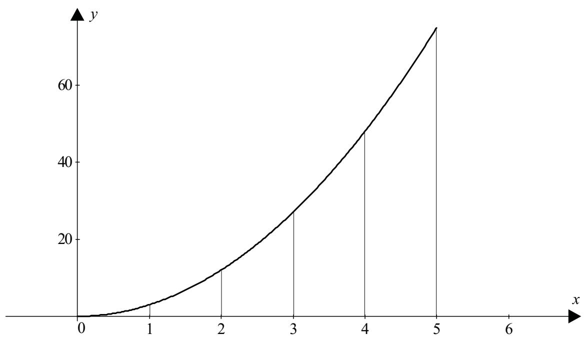
$$\int_0^a 2x\,dx = a^2$$

Let's try this again with another function.

Example

Sketch the function $f(x) = 3x^2$ in the domain $0 \le x \le 5$. Using an appropriate method, find approximate areas under the curve starting at 0. Sketch the function of these areas, $F(x)$.

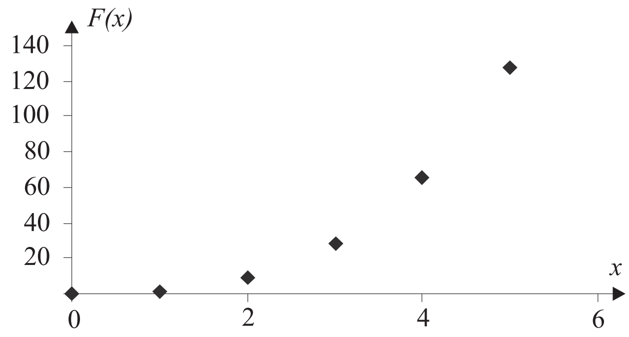
Figure 7.17: $f(x) = 3x^2$



x	0	1	2	3	4	5
$f(x)$	0	3	12	27	48	75
Interval area (one trapezium)	0	1.5	7.5	19.5	37.5	61.5
$F(x) \approx$	0	1.5	9	28.5	66	127.5

Can you guess what the graph of the function will be?

Figure 7.18: Graph of the areas under the curve of the function $f(x) = 3x^2$



It's a bit difficult to guess this one but if we took larger values in the domain or more exact areas, then we would get a more accurate picture. We can use Graphmatica to help us here. Using the integrate option we can integrate using rectangles or trapeziums. Look up the help section in the introductory book. In the above example, using Graphmatica and more trapeziums, the area under the curve from 0 to 5 gets closer and closer to 125.

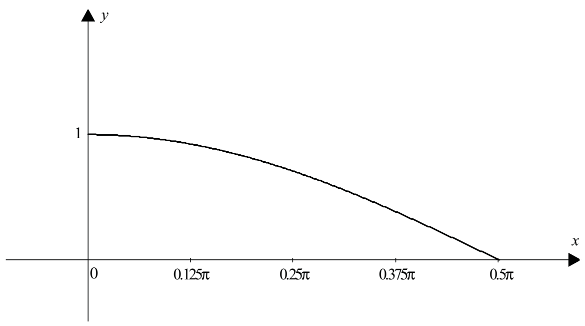
It looks as though the area function is about $F(x) = x^3$. So we could say that:

$$\int_0^a 3x^2\ dx = a^3$$

Let's take one more curve. Look at the area under the curve of the function,

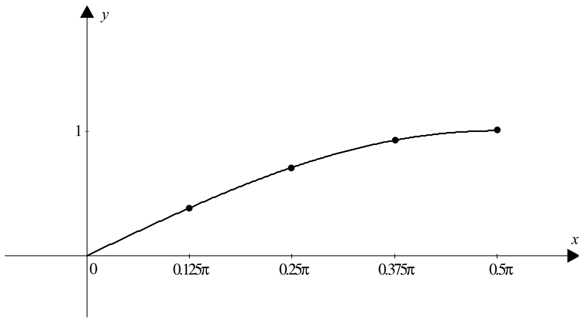
$f(x) = \cos x;\ 0 \le x \le \frac{\pi}{2}$ at intervals of $\frac{\pi}{8}$.

Figure 7.19: $f(x) = \cos x; 0 \le x \le \frac{\pi}{2}$



x	0	0.125π	0.25π	0.375π	0.5π
$f(x)$	1	0.9239	0.707	0.3827	0
Interval area	0	0.3778	0.3202	0.2140	0.0751
$F(x) \approx$	0	0.3778	0.6980	0.9120	0.9871

Figure 7.20: Graph of the areas under the curve of the function $f(x) = \cos x; 0 \le x \le \frac{\pi}{2}$



Can you guess what this function will be?

It looks as though $F(x) = \sin x$

$$\int_0^{\frac{\pi}{2}} \cos x \, dx = \sin x \ (0 \le x \le \frac{\pi}{2})$$

Let's have a look at the curves so far and their areas for the interval from 0.

Function	Area
$f(x) = 2x$	$F(x) = x^2$
$f(x) = 3x^2$	$F(x) = x^3$
$f(x) = \cos x$	$F(x) = \sin x$ (for 0 to $\frac{\pi}{2}$)

Can you see a relationship between the area function and the original function?

Recall the derivatives from the last module. Remember we said:

if $f(x) = x^2$, then $f'(x) = 2x$ and if $f(x) = x^3$, then $f'(x) = 3x^2$.

So it appears that to find the area under the derivative function to a value, $x = a$, we find the integral of a derivative which is the original function i.e., $F(x) = \int_0^a F'(x) \, dx$.

This will work for the functions we have dealt with in the previous module and will work for any part of the domain. But before we consider this, let's look at this aspect of the relationship between the integral and the derivative in more detail.

7.3 The antiderivative

If we had a derivative function $f'(x) = 2x$, then what is the function from which this derives?

From the previous section and the previous module, we could have said that it would be $f(x) = x^2$.

However we could have also said the function could have been $f(x) = x^2 + 1$ or $f(x) = x^2 - 4$, or any function of the form $f(x) = x^2 + C$, where C is any constant, since each of these functions would differentiate to $f'(x) = 2x$.

We call the family of antiderivatives that look like the definite integral without the boundaries, **indefinite integrals**. The notation for this is: $\int f(x)\ dx = F(x) + C$. Note the difference between this and the definite integral. The definite integral is a number since we are taking an actual area within boundaries, but the indefinite integral is a family of functions. We can find the indefinite integral as before, except, as there are no boundaries, we must add some unknown constant *C*.

7.4 Steps in integration

7.4.1 Using standard rules of integration

From the previous module we developed some standard differentiation formulas. We can use these to state families of functions in integral notation. If you read these from right to left then you can see the similarity to the table of standard derivatives you developed in section 6.3.5.

Recall that if $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$. In the table below, instead of using y, the function itself is used.

Derivative	**Integral**
$\frac{d(x^n)}{dx} = nx^{n-1}$	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
$\frac{d(\ln x)}{dx} = \frac{1}{x}$	$\int \frac{1}{x} \, dx = \ln x + C, \ x > 0$
$\frac{d(e^x)}{dx} = e^x$	$\int e^x \, dx = e^x + C$
$\frac{d(\sin x)}{dx} = \cos x$	$\int \cos x \, dx = \sin x + C$
$\frac{d(\cos x)}{dx} = -\sin x$	$\int \sin x \, dx = -\cos x + C$

Example

Find $\int x^{-4} dx$

$$\begin{aligned}\int x^{-4} dx &= \frac{x^{-4+1}}{-4+1} + C \\ &= \frac{x^{-3}}{-3} + C \\ &= -\frac{1}{3x^3} + C\end{aligned}$$

Activity 7.3

1. Find

(a) $\int x^{-2} dx$ (b) $\int \frac{dx}{x}$ (Hint: $\int \frac{dx}{x} = \int \frac{1}{x} dx$)

(c) $\int \sin\theta \ \ d\theta$ (d) $\int e^t dt$ (e) $\int \sqrt{x} \ dx$ (Hint $\sqrt{x} = x^{\frac{1}{2}}$)

2. In the table of standard derivatives and integrals, why is $x > 0$ in the integral $\int \frac{1}{x} \, dx = \ln x + C$.

7.4.2 Integrals of functions with constant multiples

This formula is very similar to the one for differentiation.

$$\int ax^n\,dx = a\int x^n\,dx$$

Example

$$\int 4x^3\,dx = 4\int x^3\,dx = 4\times\frac{x^4}{4}+C = x^4+C$$

Activity 7.4

1. Use integration formulas to find the following indefinite integrals:

(a) $\int 3g^2\ dg$ (b) $\int -2e^x\ dx$ (c) $\int \frac{1}{2}\cos x\ \ dx$

(d) $\int \frac{3}{y^4}\,dy$ (e) $\int \frac{1}{2x}\,dx$

2. A student wrote $\int 3x^2 dx = 3x\frac{x^3}{3}+C$. Explain why the answer is wrong.

7.4.3 Integrals of sum and difference functions

Recall in the previous module that you could differentiate sums and differences fairly easily. Well the same occurs for integration.

$$\int [f(x)\pm g(x)]\ dx = \int f(x)\ dx \pm \int g(x)\ dx$$

Example

$$\begin{aligned}\int (x^2-2x+1)\ dx &= \frac{x^3}{3}-2\frac{x^2}{2}+x+C\\ &= \frac{x^3}{3}-x^2+x+C\end{aligned}$$

Example

$$\begin{aligned}\int\frac{2x^3+3x+1}{x}\,dx &= \int\left(\frac{2x^3}{x}+\frac{3x}{x}+\frac{1}{x}\right)dx\\ &= \int\left(2x^2+3+\frac{1}{x}\right)dx\\ &= \frac{2x^3}{3}+3x+\ln x+C\end{aligned}$$

Activity 7.5

1. Integrate with respect to x

(a) x^3+3x^2-4 (b) $x(2x-7)$ (c) $(x+2)(x-2)$

(d) $\frac{1}{x^2}(x^3-3x-2)$ (e) $\frac{x^2+6x+9}{x+3}$ (f) $\frac{xe^x-x\sin x}{x}$

7.5 More areas

Let's go back to our area under the curve with different domains and use the information from the last section to find the areas under the function which can be described using algebraic symbols.

If we integrate $f(x)=4x^2$ we get $F(x)=\frac{4x^3}{3}+C$.

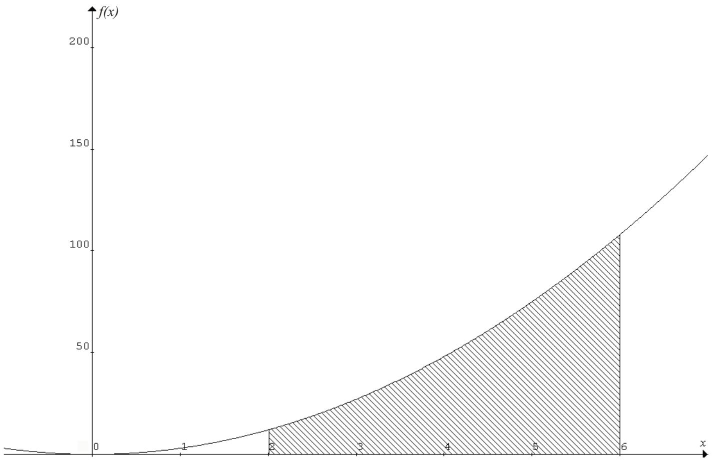
Now let $x=3$,

$$\begin{aligned}F(3) &= \frac{4\times 3^3}{3}+C\\ &= 36+C\end{aligned}$$

This is the same as finding the area under the curve of the function $f(x)=4x^2$ from $x=0$ to $x=3$, except for the value of *C*. Let's examine this further to see why *C* is not needed when looking at the definite integral.

Consider the function $f(x)=4x^2$ from $x=2$ to $x=6$. The area we are considering is shown in figure 7.21.

Figure 7.21: $f(x) = 4x^2$ from $x = 2$ to $x = 6$



To find the shaded area you could again divide it up into rectangles or trapeziums from 2 to 6 and add the figures, or you could find the area from $x = 0$ to $x = 6$ and subtract the area from $x = 0$ to $x = 2$. In integral notation this is:

$$\int_2^6 4x^2\,dx = \int_0^6 4x^2\,dx - \int_0^2 4x^2\,dx$$
$$= \frac{4\times 6^3}{3} - \frac{4\times 2^3}{3}$$
$$= \frac{4\times 208}{3}$$
$$= 277\frac{1}{3}$$

We could also consider it as $\int_2^6 4x^2 dx = F(6) - F(2)$

$$\int_2^6 (4x^2)dx = \frac{4\times 6^3}{3} + C - \left(\frac{4\times 2^3}{3} + C\right)$$
$$= \frac{4\times 6^3}{3} - \frac{4\times 2^3}{3}$$
$$= \frac{4\times 208}{3}$$
$$= 277\frac{1}{3}$$

Note $C - C = 0$. This will always occur when finding an area under the curve.

Generally speaking if a function *f* is continuous then $\int_a^b f(x)\,dx$ is the definite integral from *a* to *b* of *f* of *x dx*, and the numbers *a* and *b* are called the limits of integration. Also

$\int_a^b F'(x)dx = F(b) - F(a)$ often abbreviated to $\left[F(x)\right]_a^b$

This last formula is called the **fundamental theorem of calculus**. It is most important because it links the derivative with the definite integral and has wide applications in mathematics.

We will not prove it at this level of mathematics, but in the following examples, you will see how it works for areas under the curve and for different parts of the Cartesian plane.

Look at the example we used just before activity 7.2 (figure 7.9). The equation to the function was $f(x) = x^2$. We were looking for the area under the curve from –2 to 2.

$$\int_{-2}^{2} x^2 dx = \left[\frac{x^{2+1}}{2+1}\right]_{-2}^{2}$$

Integrate the function. The limits are –2 and 2, so put them beside the large square brackets.

$$= \left[\frac{x^3}{3}\right]_{-2}^{2}$$

$$= \frac{8}{3} - \frac{-8}{3}$$

Substitute 2 for *x* and then substitute – 2 for *x* and subtract. (Always start with the number at the top.)

$$= \frac{16}{3}$$

$$= 5\frac{1}{3}$$

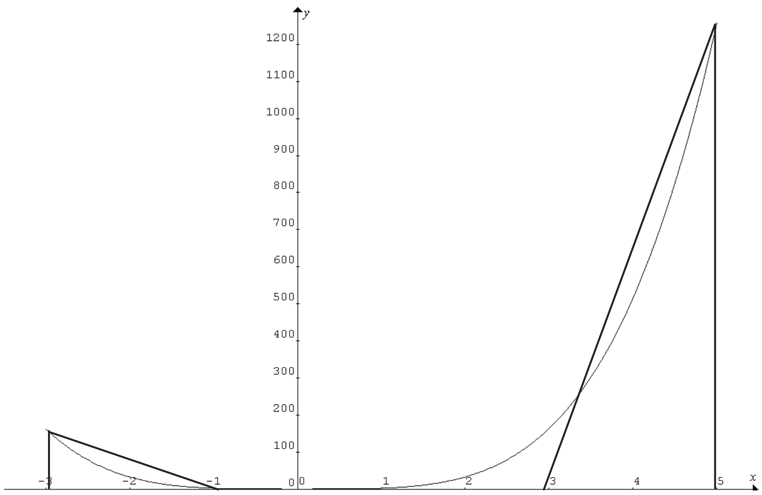
When we used trapeziums to find this area, we obtained 6, which was too large since the trapeziums were larger than the actual area.

Example

For the function $f(x) = 2x^4 (-3 \le x \le 5)$,

- sketch the graph of the curve
- find the area under the curve using the fundamental theorem
- using an appropriate geometric technique, check the area is approximately the one you calculated.

Figure 7.22: $f(x) = 2x^4 (-3 \le x \le 5)$



$$\int_{-3}^{5} 2x^4 dx = \left[\frac{2x^5}{5}\right]_{-3}^{5}$$
$$= \frac{2 \times 5^5}{5} - \left(\frac{2(-3)^5}{5}\right)$$
$$= 1250 + 97.2$$
$$= 1347.2$$

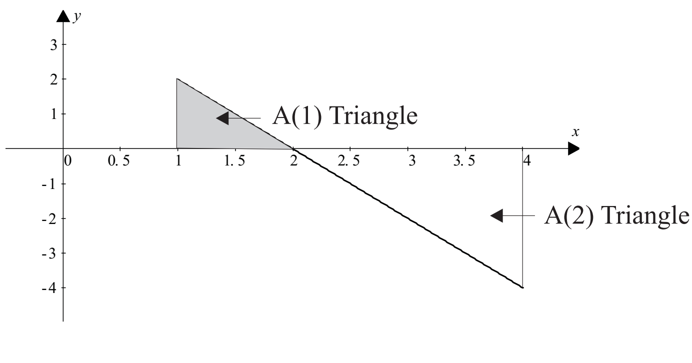
Using two triangles, the area is approximately 54 + 1250 = 1304. This is fairly close to the definite integral area calculated (1347.2).

How will it work for areas below the x-axis?

Example

Find the area between the curve and the x-axis of the function $f(x) = -2x + 4,\ (1 \le x \le 4)$

Figure 7.23: $f(x) = -2x + 4, (1 \le x \le 4)$



$\int_{1}^{4}(-2x+4)\ dx$ could now be thought of as two triangles, so the area would be

$$\left(\frac{1\times 2}{2}+\frac{2\times 4}{2}\right)=5$$

If we just integrated it, using the method just learned we would get

$$\begin{aligned}\int_{1}^{4}(-2x+4)dx &= \left[\frac{-2x^2}{2}+4x\right]_1^4\\ &= \left[-x^2+4x\right]_1^4\\ &= (-16+16)-(-1+4)\\ &= -3\end{aligned}$$

If you use the Graphmatica integration tool, you will get the same answer. This is clearly not correct if we are trying to find a physical area.

While it is true to say that the definite integral $\int_{-2}^{4}(-2x+4)dx = -3,$ when we are trying to find the physical area between the curve and the x-axis and part of the curve is below the x-axis, we have to break the curve into parts and add the **absolute value** of the parts. In the case above, we would say:

$$\begin{aligned}\int_{1}^{4}(-2x+4)\ dx &= \left|\int_{1}^{2}(-2x+4)\ dx\right|+\left|\int_{2}^{4}(-2x+4)\ dx\right|\\ &= \left|\left[-x^2+4x\right]_1^2\right|+\left|\left[-x^2+4x\right]_2^4\right|\\ &= \left|(-4+8)-(-1+4)\right|+\left|(-16+16)-(-4+8))\right|\\ &= \left|4-3\right|+\left|0-4\right|\\ &= 1+4\\ &= 5\end{aligned}$$

Note the position of the boundaries of integration, with the lower boundary on the bottom

Note it's $F(2)-F(1)$ and $F(4)-F(2)$

Take care with negatives and the absolute value – do one step at a time.

In general:

When the function is above and below the x-axis we need to:

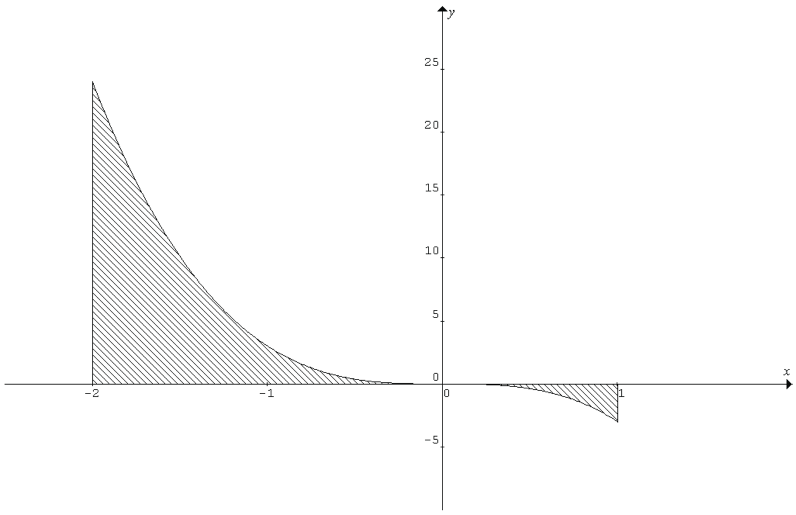
- sketch the graph;
- find where the function cuts the x-axis;
- add the absolute values of the integrals above and below the x-axis.

Example

Find the area between the curve and the x-axis of the function $f(x)=-3x^3$ from $-2\le x\le 1$.

From your knowledge of functions and their graphs, this cubic function will cut the x-axis at the origin. The area will fall above and below the x-axis. We need to sketch the graph to get a better view of the required area. When we have an area above and below the x-axis, we need to divide the area into two parts, i.e., between –2 and 0 and between 0 and 1.

Figure 7.24: $f(x) = -3x^3$ from $-2 \le x \le 1$



$$\int_{-2}^{1} f(x)\,dx = \int_{-2}^{1} (-3x^3)\,dx$$
$$= \left|\int_{-2}^{0} (-3x^3)\,dx\right| + \left|\int_{0}^{1} (-3x^3)\,dx\right|$$
$$= \left|\left[\frac{-3x^4}{4}\right]_{-2}^{0}\right| + \left|\left[\frac{-3x^4}{4}\right]_{0}^{1}\right|$$
$$= \left|\left(0 - \frac{-3\times(-2)^4}{4}\right)\right| + \left|\frac{-3}{4} - 0\right|$$
$$= 12 + \left|\frac{-3}{4}\right|$$
$$= 12\frac{3}{4}$$
$$= 12.75$$

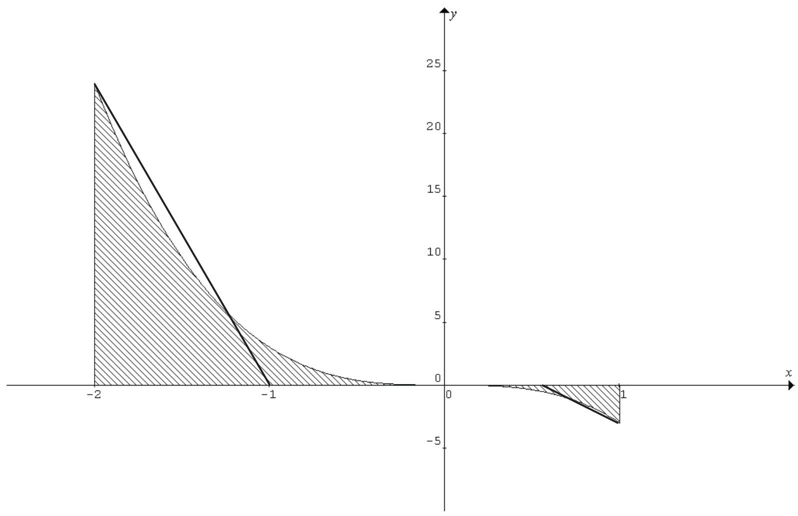
If we again checked by taking two triangles as in figure 7.25, the area is approximately:

$$\frac{1\times 24}{2} + \frac{0.5\times 4}{2}$$
$$\approx 13$$

This is fairly close to our calculated area.

Note: $\int_{-2}^{1} f(x)dx = 12 - \frac{3}{4} = 11\frac{1}{4}$, if we were not considering the physical area under the curve.

Figure 7.25: Finding the area under the curve $f(x) = -3x^3$ from $-2 \le x \le 1$, using two triangles

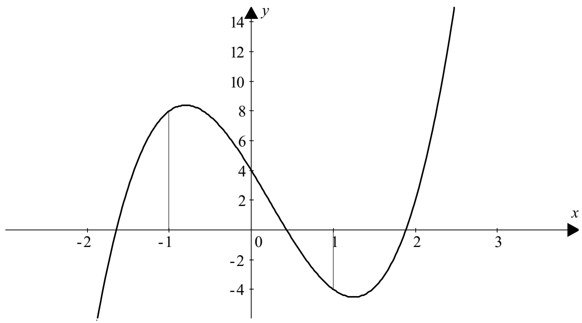


Example

Find the area between the curve and the x-axis of the function $f(x) = 3x^3 - 2x^2 - 9x + 4$ between $-1 \le x \le 1$

From your knowledge of functions and their graphs, cubic functions may cut the x-axis at up to three points. The area may fall above and below the x-axis. We need to sketch the graph to get a better view of the required area. When we have an area above and below the x-axis, we need to divide the area into two parts. We must find the point/s where the curve cuts the x-axis. This can be done algebraically, but is beyond the scope of this unit, so we will use Graphmatica to find that the middle x-intercept is about 0.4. So we need to find the area between –1 and about 0.4 and between 0.4 and 1. (Using Calculus/critical points feature of Graphmatica gives the more exact value of 0.4299.)

Figure 7.26: $f(x) = 3x^3 - 2x^2 - 9x + 4$ from $-1 \le x \le 1$



The area will be:

$$\int_{-1}^{0.4} f(x)dx + \left|\int_{0.4}^{1} f(x)dx\right|$$

$$= \left|\int_{-1}^{0.4} (3x^3 - 2x^2 - 9x + 4)dx\right| + \left|\int_{0.4}^{1} (3x^3 - 2x^2 - 9x + 4)dx\right|$$

$$= \left|\left[\frac{3x^4}{4} - \frac{2x^3}{3} - \frac{9x^2}{2} + 4x\right]_{-1}^{0.4}\right| + \left|\left[\frac{3x^4}{4} - \frac{2x^3}{3} - \frac{9x^2}{2} + 4x\right]_{0.4}^{1}\right|$$

$$= \left|\left(\frac{3(0.4)^4}{4} - \frac{2(0.4)^3}{3} - \frac{9(0.4)^2}{2} + 4(0.4)\right) - \left(\frac{3(-1)^4}{4} - \frac{2(-1)^3}{3} - \frac{9(-1)^2}{2} + 4(-1)\right)\right| +$$

$$\left|\left(\frac{3(1)^4}{4} - \frac{2(1)^3}{3} - \frac{9(1)^2}{2} + 4(1)\right) - \left(\frac{3(0.4)^4}{4} - \frac{2(0.4)^3}{3} - \frac{9(0.4)^2}{2} + 4(0.4)\right)\right|$$

$$\approx \left|(0.0192 - 0.0427 - 0.72 + 1.6) - (0.75 + 0.6667 - 4.5 - 4)\right| +$$

$$\left|(0.75 - 0.6667 - 4.5 + 4) - (0.0192 - 0.0427 - 0.72 + 1.6)\right|$$

$$\approx \left|0.8565 + 7.0833\right| + \left|-0.4167 - 0.8565\right|$$

$$\approx 7.9398 + 1.2722$$

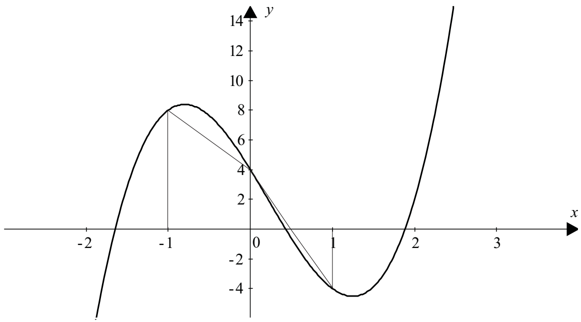
$$\approx 9.2120$$

The shaded area is about 9.212.

Creating a trapezium and two triangles as in figure 7.27 should give an answer a bit less than 9 square units.

Figure 7.27: Approximating the area under $f(x)$ by using a trapezium and two triangles

$$\text{Area} = \frac{(8+4)\times 1}{2} + \frac{0.5\times 4}{2} + \frac{0.5\times 4}{2} = 6+2 = 8 \text{ sq units}$$



Activity 7.6

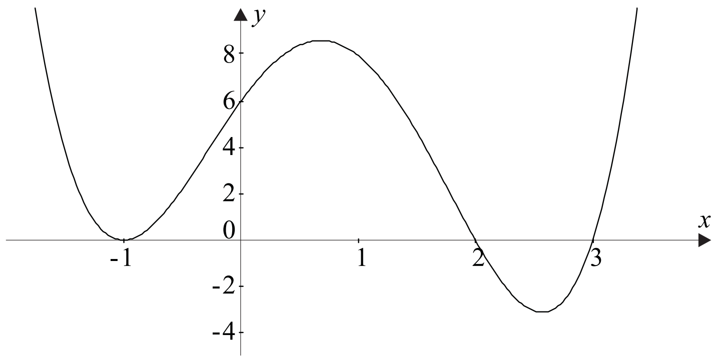
1. For the function $f(x) = (x-1)^3$, $(1 \le x \le 4)$:

   Sketch the graph of $f(x)$.

   Use the fundamental theorem of calculus to find the area under the curve.

   Check your answer by using an appropriate geometrical approximation.

2. The graph of $f(x) = x^4 - 3x^3 - 3x^2 + 7x + 6$ is shown below:



   Check the function cuts the x-axis at -1, 2 and 3.

   Find the area between the curve and the x-axis of the function for $-1 \le x \le 3$.

   Check by geometrical means that your answer is approximately correct.

3. Given $f(x) = 2e^x$, find the area bounded by $f(x)$, the x-axis, and the two lines $x = -1$ and $x = 2$.

4. (a) Find $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \quad dx$

(b) Find the area between the curve and the x-axis of the function in (a).

(c) Explain the relationship between your answer to (a) and (b).

7.6 Applications of integral calculus

In the previous module and in some examples in this module, we looked at the velocity function. Let's have a closer look at velocity and its relationship to displacement and acceleration.

Recall from the last module that if s is the displacement of an object, v is the velocity and a is the acceleration, then:

$$v = \frac{ds}{dt} \text{ and } a = \frac{dv}{dt}$$

Using our knowledge of integration, we can now say that:

$$v = \int a \ dt \text{ and } s = \int v \ dt$$

Suppose a car is travelling along the road. At a particular point in time it is accelerating at 8 ms^{-2}. What can we say about the velocity at this time?

Since $v = \int a \ dt$ then

$$v = \int 8 \ dt \ = 8t + C$$

This does not tell us how fast the car is travelling at any given point in time since we don't know the value of C. It does, however, tell us that the velocity is increasing at a rate of 8 metres/second. If the driver looked at the speedometer when the accelerator was pressed and saw the speed was 40 kmh^{-1} (i.e. about 11.1 ms^{-1}), then we could substitute 40 for v and 0 for t.

$$11.1 \approx \int 8 \ dt$$
$$11.1 \approx 8 \times 0 + C$$
$$C \approx 11.1$$

Then $v = 8t + 11.1$

What this last equation is saying, is that when the accelerator was pressed (at $t = 0$), the velocity was 11.1 ms^{-1}, and the velocity increased 8 metres per second every second after that.

What can this tell us about the distance travelled?

$$\begin{aligned}
s &= \int v\ dt \\
&= \int 8t + 11.1\ dt \\
&= \frac{8t^2}{2} + 11.1t + C
\end{aligned}$$

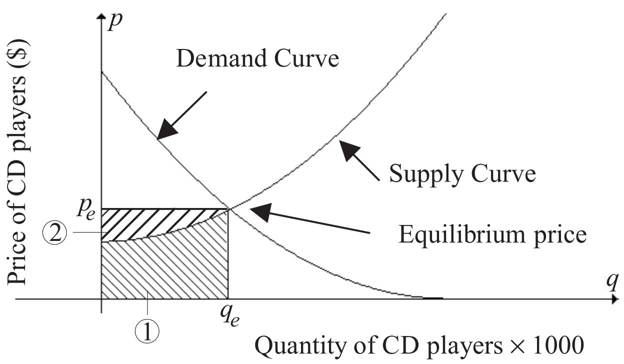
Again this does not tell us about the total distance travelled, but if the accelerator was pressed for 5 seconds, then we are looking at the total distance from $t = 0$ to $t = 5$.

$$\begin{aligned}
s &= \int_0^5 v\ dt \\
&= \int_0^5 (8t + 11.1)\ dt \\
&= \left[\frac{8t^2}{2} + 11.1t\right]_0^5 \\
&= \left(\frac{8 \times 5^2}{2} + 11.1 \times 5\right) - \left(\frac{8 \times 0^2}{2} + 11.1 \times 0\right) \\
&= 100 + 55.5 - 0 \\
&= 155.5 \text{ metres}
\end{aligned}$$

(When $t = 5$, the velocity of the car would be about 184 kmh^{-1} – a little excessive!)

Example

Figure 7.28: The supply and demand curves for CDs



The supply curve is relationship between the price of the CDs and the quantity of CDs supplied. The higher the price, the greater the quantity the suppliers are willing to supply. The demand curve is the relationship between the price and the quantity of CDs demanded. The lower the price, the greater the quantity that will be demanded by consumers. The equilibrium price is where the quantities demanded and supplied are exactly equal. If all CDs are sold at this price, the total price for all the CDs will be the area of the rectangle $p_e \times q_e$. The shaded area 1 is the total income the producers would have received if they sold the CDs at the price they were willing to accept. The shaded area 2 is the **producer surplus** if the CDs were all sold at the equilibrium price.

In the example above, suppose the supply curve was modelled by the function $p = q^2 + 4$.

And the demand curve by the function $p = (q-4)^2$. What is the producer surplus?

The producer surplus is the area of the rectangle whose sides are the coordinates of the equilibrium point minus the area under the supply curve from $q = 0$ to the equilibrium point, i.e.,

$$\text{Producer surplus} = q_e \times p_e - \int_0^{q_e} (q^2 + 4)\, dq.$$

First we have to find the equilibrium point. This is the point where the curves intersect.

$$q^2 + 4 = (q-4)^2$$
$$q^2 + 4 = q^2 - 8q + 16$$
$$8q = 12$$
$$q = 1.5$$

To find p, we substitute this value of q into the equation.

$$\begin{aligned} p &= q^2 + 4 \\ &= 2.25 + 4 \\ &= 6.25 \end{aligned}$$

So the equilibrium price is \$6.25 and the equilibrium quantity is 1.5 (×1000).

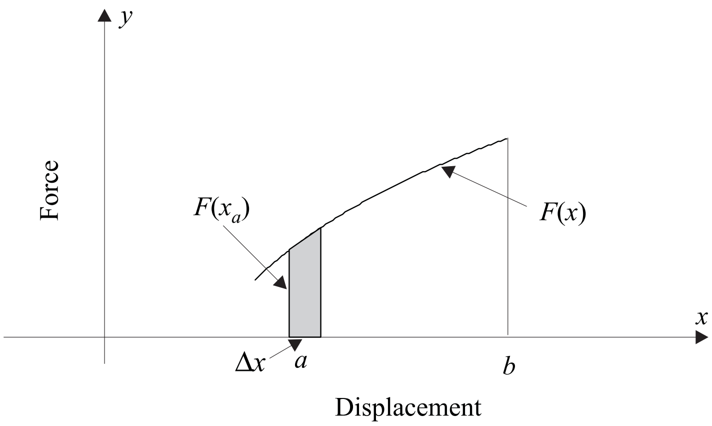
$$\begin{aligned} \text{Producer surplus} &= 1.5 \times 6.25 - \int_0^{1.5} (q^2 + 4)\, dq \\ &= 9.375 - \left[\frac{q^3}{3} + 4q\right]_0^{1.5} \\ &= 9.375 - \left(\frac{1.5^3}{3} + 4 \times 1.5\right) - 0 \\ &= 9.375 - 7.125 \\ &= 2.25 \end{aligned}$$

So the producer surplus is \$2.25 (×1000) = \$2250.

Example

If a constant force (F) is applied to an object over a certain distance (x), then the work done on the object (W) is force times the displacement of the object ($W = Fx$). The unit of work is the joule (J), the force is in newtons (N) and the displacement is in metres (m). If however, the force varies with displacement then the work done is equivalent to the area under the definite integral of the force with respect to the displacement. Figure 7.29 shows this in terms of rectangles. The work done will be the sum of rectangles as before and in the limit, the work done will be $W = \int_a^b F\, dx$.

Figure 7.29: Force – displacement graph



Determine the work done on stretching a spring from rest to stretching by 2 cm, if the force applied changes according to the equation $F = 3x$.

$$\text{Work} = \int_0^2 3x\,dx$$

$$= \left[\frac{3x^2}{2}\right]_0^2$$

$$= \frac{3\times 4}{2} - 0$$

$$= 6$$

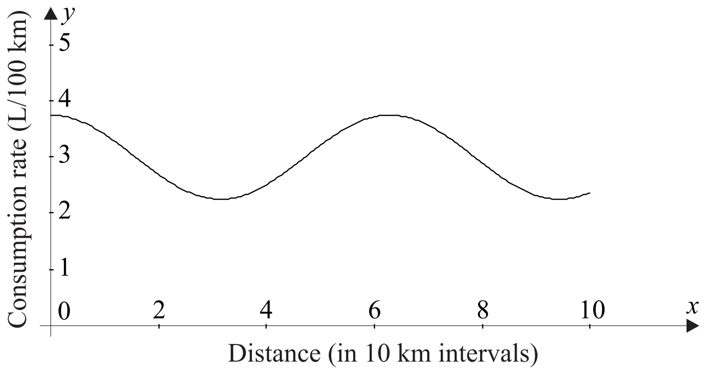
So the work done is 6 Joules.

Activity 7.7

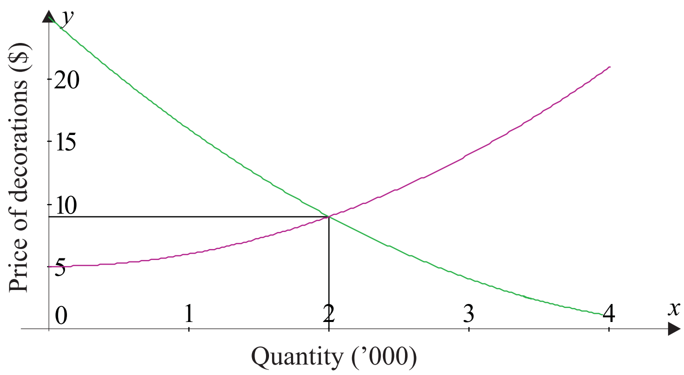
1. A remote controlled toy has acceleration of $(3t - 2)$ ms^{-2}. Initially the toy was displaced 40 m due east of the controller. At the time the toy was moving at 3 ms^{-1} in a westerly direction. After 4 seconds the controller thought the velocity is about 15 ms^{-1} eastward and the displacement about 45 m east. Do you agree? Calculate velocity and displacement exactly.

2. Because of the nature of the road, the fuel consumption of a truck varies according to $y = 0.75\cos x + 3$ where y is the consumption rate (L per 10 km) and x is the distance (in 10 km intervals) along the 100 km road.

   Use integration to calculate the number of litres used in the 100 km trip.



3. The equations $p = q^2 + 5$ and $p = (q - 5)^2$ respectively represent the supply and demand curves for a Christmas decoration, p representing the price ($) of each decoration and q the quantity (in thousands).



(a) Find the equilibrium price (the price when the curves intersect).

(b) What (in dollars) is the Producer Surplus?

4. A variable force (F Newtons) modelled by the equation F = 2x + 1 is applied over a certain distance (x).

What is the work (W joules) done in moving the object from a displacement of 2 m to a displacement of 4 m. (Note: W = Fx)

Something to talk about…

As you go on to do more mathematics, you will find integral calculus an important part of your mathematical tool kit. You may be able to see its application in your work or other experiences. Try to develop a question of your own that uses integral calculus. Discuss it with your fellow students in the discussion group to make sure it works.

Before we finish this module on integral calculus, a final piece of information. At the beginning of this module, we talked about Kepler and his method of finding areas of sectors of an ellipse. It turns out his method is fairly sound. Modern calculus techniques cannot come to the rescue. There is no exact method of determining the area of a sector of an ellipse.

That's the end of this module and the unit. How do you feel now? Tell us about you mathematics learning in the learning diary.

But before you have really finished you should so a number of things.

1. You should be getting close to your final revision. Have a close look at your action plan to prepare for your final assessment. Are you on schedule? Do you need to restructure your action plan or contact your tutor to discuss any delays or concerns?
2. Make a summary of the important points in this module, noting your strengths and weaknesses. Add any new words to your notebook. This will help in future revision.
3. Have you started you're A4 sheet you can take into the exam? You could start to structure this.
4. Practice some more problems in 'A taste of things to come'.
5. Check your skill level by attempting the post-test.
6. When you are ready, complete and submit your assignment.

7.7 A taste of things to come

Example

Let's have a look at the application of integral calculus in chemistry. The following example is taken from Hughes,-Hallett, Gleason, McCallum, et al. 1998, *Calculus. Single and Multivariable*, Wiley, New York.

The energy required to separate two charged particles, originally a units apart, to a distance b, is given by the integral:

$$E = \int_a^b \frac{kq_1q_2}{r^2} dr$$

where q_1 and q_2 are the magnitude of the charges and k is a constant.

$q_1$and q_2 are in coulombs, a and b are in metres, and E is in joules, the value of the constant is 9×10^9.

A hydrogen atom consists of a proton and an electron, with opposite charges of magnitude 1.6×10^{-19} coulombs. Find the energy required to take a hydrogen atom apart (that is to move the electron from its orbit to an infinite distance from the proton). Assume the initial distance between the electron and the proton is 5.3×10^{-11} metres (this is called the Bohr radius).

$$E = \int_{5.3\times10^{-11}}^{\infty} \frac{kq_1q_2}{r^2}dr$$

$$= kq_1q_2 \int_{5.3\times10^{-11}}^{\infty} \frac{1}{r^2}dr$$ Since k, q_1 and q_2 are constants.

$$= kq_1q_2 \left| (-\frac{1}{r}) \right|_{5.3\times10^{-11}}^{\infty}$$ Integrating $\frac{1}{r^2}$.

$$= kq_1q_2 \lim_{b\to\infty} \left| (-\frac{1}{r}) \right|_{5.3\times10^{-11}}^{b}$$ Since one of the boundaries is infinity, we call this an improper integral, and as the boundary b tends to infinity, we can express it as a limit.

$$= kq_1q_2 \lim_{b\to\infty} \left(-\frac{1}{b} + \frac{1}{5.3\times10^{-11}} \right)$$ Substituting the boundaries in the fraction $-\frac{1}{r}$.

$$= kq_1q_2 \times \frac{1}{5.3\times10^{-11}}$$ As b approaches infinity $-\frac{1}{b}$ approaches zero.

$$= \frac{9\times10^9 \times \left(1.6\times10^{-19}\right)^2}{5.3\times10^{-11}}$$ The values of q_1 and n q_2 are the same.

$$\approx 4.35\times10^{-18} \text{ joules}$$

This is about the amount of energy required to lift a speck of dust 0.000000625 cm off the ground!

Now try this activity.

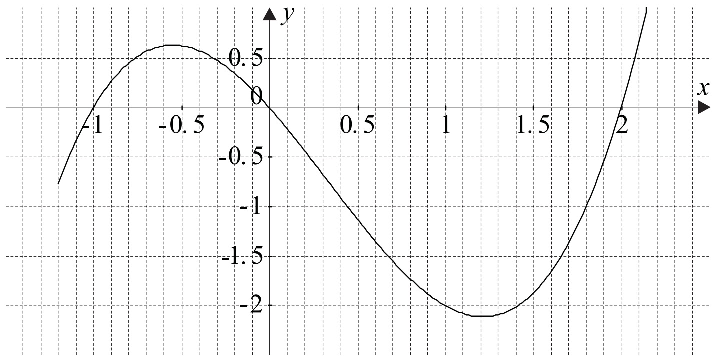
What is the energy required to separate opposite electric charges of magnitude 0.05 coulomb. The charges are initially 20 cm apart and one of the charges moves infinitely away from the other.

7.8 Post-test

1. Sketch the graph (Speed against Time). Using appropriate geometrical shapes, find the distance covered by the following journey:

   Jay accelerates constantly from rest for 30 seconds until he reaches a speed of 20 ms^{-1}. He then decreases his speed constantly to 15 ms^{-1} over the next 20 seconds and continues at 15 ms^{-1} for 10 more seconds. It takes him 15 seconds to come to a stop decreasing at a constant rate.

2.



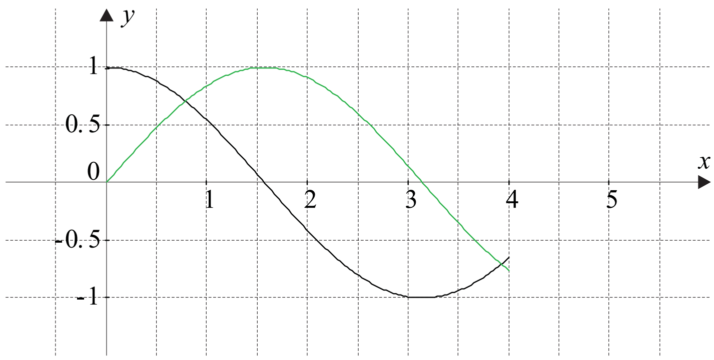
By calculating the areas of trapeziums at 0.5 unit intervals, find the approximate area under the curve from $x = -1$ to $x = 2$.

3. Use integration rules to find the following indefinite integrals:

(a) $\int \frac{1}{2} x^{-3} dx$ (b) $\int 4e^t dt$ (c) $\int ky^2 dy$

(d) $\int (\sin\theta + 2) d\theta$ (e) $\int \frac{1}{x}(x^2 + 1) dx$

4.



Above are the graphs of $y = \sin x$ and $y = \cos x$.

(i) Verify that the graphs intersect at x $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$

(ii) Find the area completely enclosed by these curves.

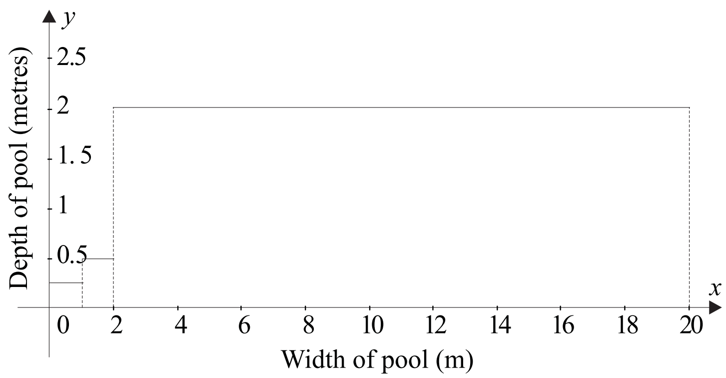
5. $F = \frac{1}{x}$ is the variable force (Newtons) applied to slide an object across a room. What work is done in displacing the object from 0.5 metres to 3 metres? (Note: $W = Fx$)

7.9 Solutions

Solutions to activities

Activity 7.1

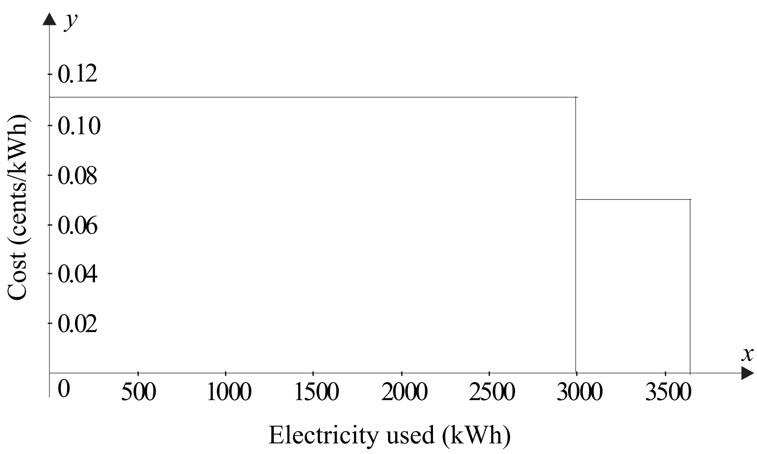
1. (a) Cross-section of Inground Swimming Pool



(b) Area = (1 × 0.25) + (1 × 0.5) + (18 × 2)
= 36.75 m^2
Volume = 36.75 × 7
= 257.25 m^3

The capacity of the pool is 257.25 kL ($1m^3 = 1kL$)

2. Household Electricity Account



Cost = (3000 × \$0.11) + (600 × \$0.07)
= \$330 + \$42
= \$372

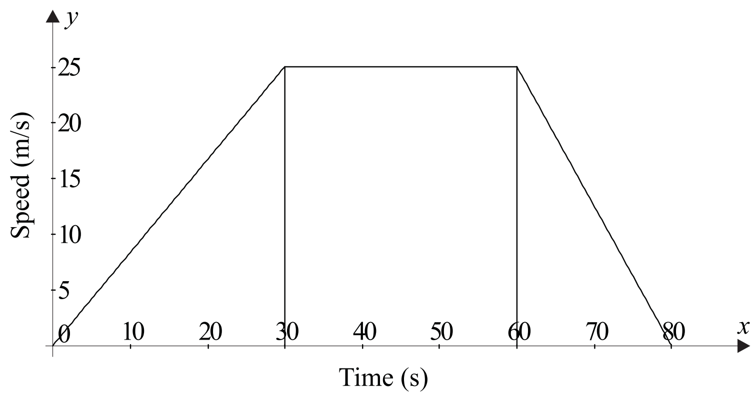
The cost of the electricity account will be \$372

3. Real Estate Commission

Commission (\$,000) = (18 × 0.055) + (147 × 0.0275) (18 + 147 = 165)
= 0.99 + 4.0425
= 5.0325

$\therefore$ Total commission is \$5 032.50

4. Motor Cyclist's Journey

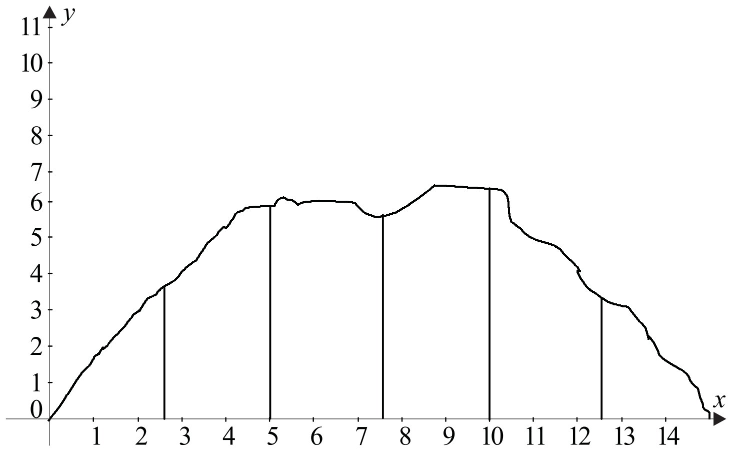


$$\text{Distance covered} = \frac{30\text{s} \times 25\text{m/s}}{2} + 30\text{s} \times 25\text{m/s} + \frac{20\text{s} \times 25\text{m/s}}{2}$$
$$= 1375\text{m}$$

$\therefore$ Total distance covered is 1375m or 1.375lm.

Activity 7.2

1.

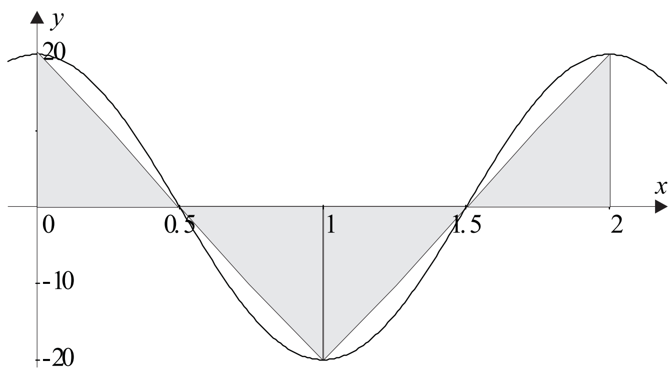


Taking 6 trapeziums means the bases will be 2.5 metres long.

$$\text{Area } \approx 2.5\left(\frac{0+3.6}{2}\right)+2.5\left(\frac{3.6+5.8}{2}\right)+2.5\left(\frac{6.2+5.5}{2}\right)$$
$$+2.5\left(\frac{6+6.2}{2}\right)+2.5\left(\frac{6.2+3.3}{2}\right)+2.5\left(\frac{3.3+0}{2}\right)$$
$$= 4.5+11.75+14.625+15.25+11.875+4.125$$
$$= 62.125$$

$\therefore$ The area of the cross-section of the dam is approximately 62.125 square metres.

2. Velocity of a Pendulum



Distance = velocity × time (cm s^{-1} × s)

Area from 0 to 2 ≈ Area of 4 triangles each with base 0.5 units and height 20 units.

Area of one triangle $\approx \left(\dfrac{0.5 \times 20}{2}\right) = 5$ square units.

Total area ≈ 4 × 5 = 20 square units.

∴ The pendulum swings approximately 20 m in 2 seconds.

3. Since there are 8 trapeziums, they will be at 100 m intervals.

   Reading the following approximate coordinates from the graph at 100 m intervals:
   (0, 0), (100, 8), (200, 10), (300, 10), (400, 12), (500, 13), (600, 12), (700, 8), (800, 2)

   Area of cross section ≈ Area of 8 trapeziums

$$\begin{aligned}
&= 100\left[\begin{array}{l}\left(\dfrac{0+8}{2}\right)+\left(\dfrac{8+10}{2}\right)+\left(\dfrac{10+10}{2}\right)+\left(\dfrac{10+12}{2}\right)\\ +\left(\dfrac{12+13}{2}\right)+\left(\dfrac{13+12}{2}\right)+\left(\dfrac{12+8}{2}\right)+\left(\dfrac{8+2}{2}\right)\end{array}\right]\\
&= 100\left[4+9+10+11+12.5+12.5+10+5\right]\\
&= 100 \times 74\\
&= 7\,400
\end{aligned}$$

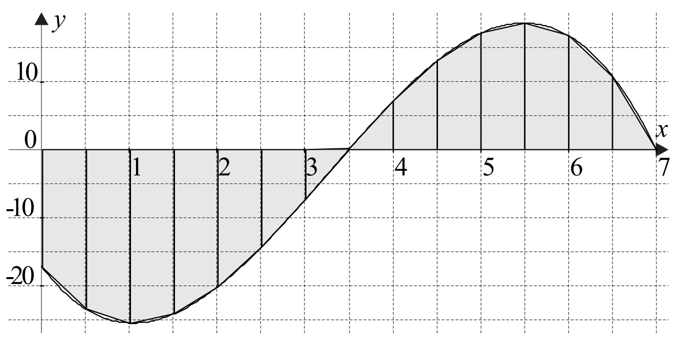
   ∴ The cross-section has an approximate area of 7 400 m^2.

4. Area of surface water ≈ Area of 6 trapeziums

$$\begin{aligned}
&= 5\left[\frac{14.2+14.8}{2}+\frac{14.8+12}{2}+\frac{12+13.6}{2}+\frac{13.6+13.6}{2}+\frac{13.6+13.6}{2}+\frac{13.6+12}{2}\right]\\
&= 5\left[14.5+13.4+12.8+13.6+13.6+12.8\right]\\
&= 5 \times 80.7\\
&= 403.5
\end{aligned}$$

   ∴ The surface area of the lake is approximately 403.5 square metres.

5.



Area of each farm ≈ Area of 7 trapeziums

$$\text{Area of farm (west)} \approx 0.5\left[\frac{16+23}{2}+\frac{23+25}{2}+\frac{25+24}{2}+\frac{24+20}{2}+\frac{20+15}{2}+\frac{15+7}{2}+\frac{7+0}{2}\right]$$
$$= 0.5[19.5+24+24.5+22+17.5+11+3.5]$$
$$= 0.5\times 122$$
$$= 61 \text{ km}^2$$
$$= 6\ 100 \text{ ha} \qquad \text{(Approximate area of farm to west)}$$

$$\text{Area of farm (east)} \approx 0.5\left[\frac{0+6}{2}+\frac{6+13}{2}+\frac{13+17}{2}+\frac{17+19}{2}+\frac{19+17}{2}+\frac{17+10}{2}+\frac{10+0}{2}\right]$$
$$= 0.5[3+9.5+15+18+18+13.5+5]$$
$$= 0.5\times 82$$
$$= 41 \text{ km}^2$$
$$= 4\,100 \text{ ha} \qquad \text{(Approximate area of farm to east)}$$

∴ The farm on the west is larger by approximately (6 100 – 4 100) = 2 000 ha.

Activity 7.3

1. (a) $\int x^{-2}\,dx = \dfrac{x^{-2+1}}{-1} + C = -x^{-1} + C$

   (b) $\int \dfrac{dx}{x} = \ln x + C$

   (c) $\int \sin\theta \;\; d\theta = -\cos\theta + C$

   (d) $\int e^t\,dt = e^t + C$

   (e) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \dfrac{x^{\frac{1}{2}+1}}{\frac{3}{2}} + C = \dfrac{2x^{\frac{3}{2}}}{3} + C$ or $\dfrac{2\sqrt{x^3}}{3} + C$

2. If $x < 0$, then the integral would be $\ln x$, but you can't take the log of a negative number. For example,

   if $x = -4$, then $\ln(-4) = y$.

   This is the same as saying $e^y = -4$, which cannot be solved.

Activity 7.4

1. (a) $\int 3g^2\,dg = 3\int g^2\,dg$

$$= 3\times\frac{g^{2+1}}{3} + C$$
$$= g^3 + C$$

(b) $\int -2e^x dx = -2\int e^x dx$

$= -2e^x + C$

(c) $\int \frac{1}{2}\cos x \; dx = \frac{1}{2}\int \cos x \; dx$

$= \frac{1}{2}\sin x + C$

(d) $\int \frac{3}{y^4} \; dy = 3\int y^{-4} dy$

$= 3 \times \frac{y^{-4+1}}{-3} + C$

$= -y^{-3} + C$

$= -\frac{1}{y^3} + C$

(e) $\int \frac{1}{2x} \; dx = \frac{1}{2}\int \frac{1}{x} \; dx$

$= \frac{1}{2}\ln x + C$

2. This integral has a constant multiple. You should first take out the multiple 3 from the integral, then integrate

$\int 3x^2 dx = 3\int x^2 dx$

$= \frac{3x^3}{3} + C$

$= x^3 + C$

Activity 7.5

(a) $\int (x^3 + 3x^2 - 4)dx = \int x^3 dx + 3\int x^2 dx - \int 4dx$

$= \frac{x^4}{4} + \frac{3x^3}{3} - 4x + C$

$= \frac{1}{4}x^4 + x^3 - 4x + C$

(b) $\int x(2x - 7)dx = \int (2x^2 - 7x)dx$

$= \frac{2x^3}{3} - \frac{7x^2}{2} + C$

(c) $\int (x + 2)(x - 2)dx = \int (x^2 - 4)dx$

$= \frac{x^3}{3} - 4x + C$

(d) $\int \frac{1}{x^2}(x^3 - 3x - 2)dx = \int (x - \frac{3}{x} - 2x^{-2})dx$

$$= \frac{x^2}{2} - 3\ln x - \frac{2x^{-2+1}}{-1} + C$$

$$= \frac{1}{2}x^2 - 3\ln x + \frac{2}{x} + C$$

(e) $\int \frac{x^2 + 6x + 9}{x+3}\, dx = \int \frac{(x+3)(x+3)}{(x+3)} dx$

$$= \int (x+3)\, dx \quad (x \neq -3)$$

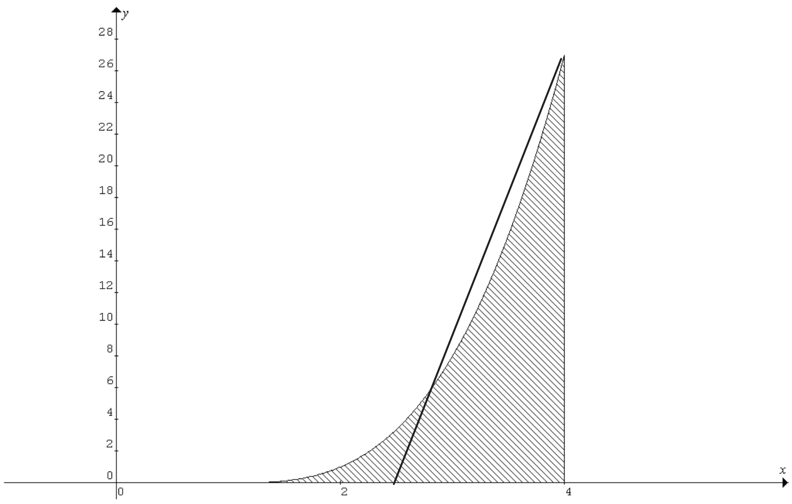
$$= \frac{x^2}{2} + 3x + C$$

(f) $\int \frac{xe^x - x\sin x}{x} dx = \int (e^x - \sin x)dx$

$$= e^x + \cos x + C$$

Activity 7.6

1.



$$\text{Area } = \int_1^4 (x-1)^3 dx = \int_1^4 (x^3 - 3x^2 + 3x - 1)dx$$

$$= \left[\frac{x^4}{4} - x^3 + \frac{3x^2}{2} - x \right]_1^4$$

$$= (64 - 64 + 24 - 4) - \left(\frac{1}{4} - 1 + \frac{3}{2} - 1 \right)$$

$$= \quad 20 \quad - \quad -\frac{1}{4}$$

$= \quad 20.25 \quad$ square units

Approximate area using triangle $= \frac{1.5 \times 27}{2} = 20.25$ (Exact!)

2. $x^4 - 3x^3 - 3x^2 + 7x + 6 = (x+1)^2(x-2)(x-3)$

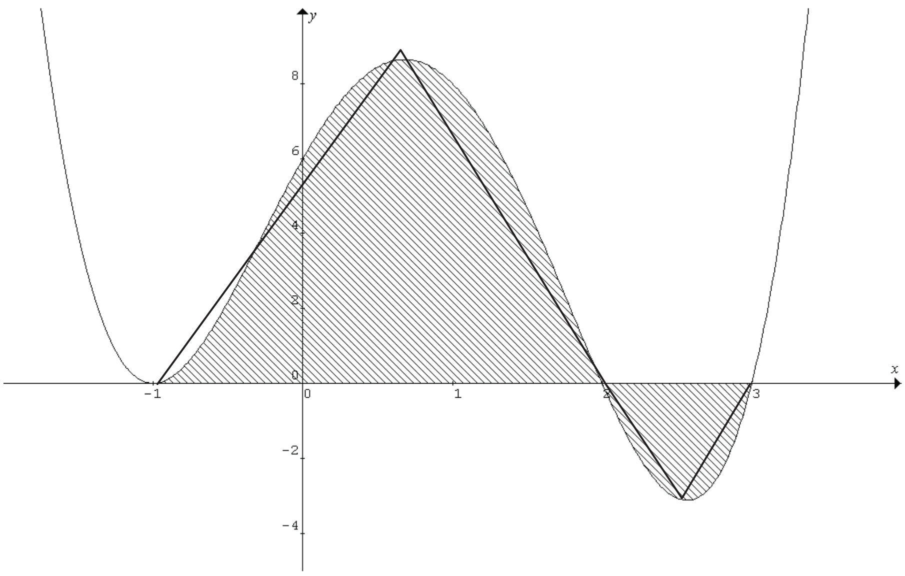
$\therefore f(x) = 0$ when $x = -1,\ 2$ and 3.

Or you could check it by substituting the values of -1, 2 and 3 in the equation.

$$f(-1) = (-1)^4 - 3(-1)^3 - 3(-1)^2 + 7(-1) + 6 = 0$$

$$\begin{aligned} f(2) &= (2)^4 - 3(2)^3 - 3(2)^2 + 7(2) + 6 \\ &= 16 - 24 - 12 + 14 + 6 \\ &= 0 \end{aligned}$$

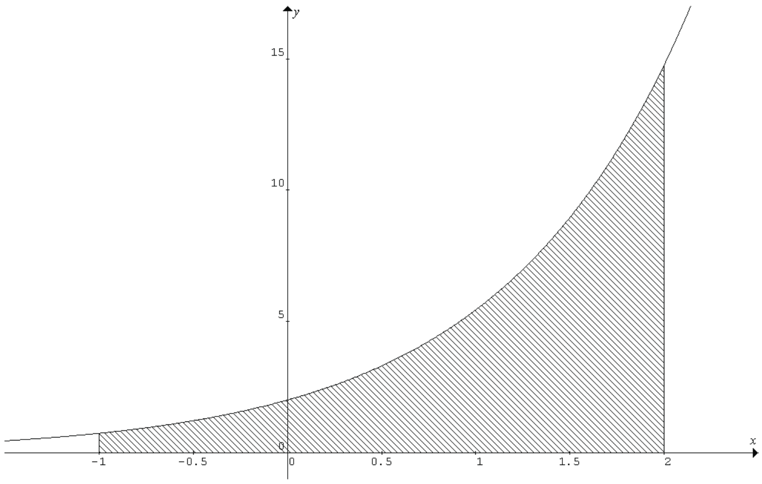
$$\begin{aligned} f(3) &= (3)^4 - 3(3)^3 - 3(3)^2 + 7(3) + 6 \\ &= 81 - 81 - 27 + 21 + 6 \\ &= 0 \end{aligned}$$



Area ≈ Area of two triangles $= \dfrac{3\times 9}{2} + \dfrac{1\times 3.2}{2} = 15.1$ square units (which is clearly smaller than the actual area as shown in the graph).

$$\begin{aligned} \text{Area} &= \int_{-1}^{2} f(x)dx + \int_{2}^{3} f(x)dx \\ &= \left[\frac{x^5}{5} - \frac{3x^4}{4} - x^3 + \frac{7x^2}{2} + 6x\right]_{-1}^{2} + \left|\left[\frac{x^5}{5} - \frac{3x^4}{4} - x^3 + \frac{7x^2}{2} + 6x\right]_{2}^{3}\right| \\ &= \left[\left(\frac{32}{5} - 12 - 8 + 14 + 12\right) - \left(\frac{-1}{5} - \frac{3}{4} + 1 + \frac{7}{2} - 6\right)\right] + \\ &\quad \left|\left(\frac{243}{5} - \frac{243}{4} - 27 + \frac{63}{2} + 18\right) - \left(\frac{32}{5} - 12 - 8 + 14 + 12\right)\right| \\ &= \left(12.4 -{}^{-}2.45\right) + \left|(10.35 - 12.4)\right| \\ &= 14.85 + 2.05 \\ &= 16.9 \end{aligned}$$

3.



$$\text{Area} = \int_{-1}^{2} 2e^x dx$$

$$= 2\int_{-1}^{2} e^x dx$$

$$= 2\left|e^x\right|_{-1}^{2}$$

$$= 2(e^2 - e^{-1})$$

$$\approx 2 \times 7.021$$

$$\approx 14.042$$

4\. (a) $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x dx = [-\cos x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= -\cos\frac{\pi}{2} + \cos\left(-\frac{\pi}{2}\right)$$

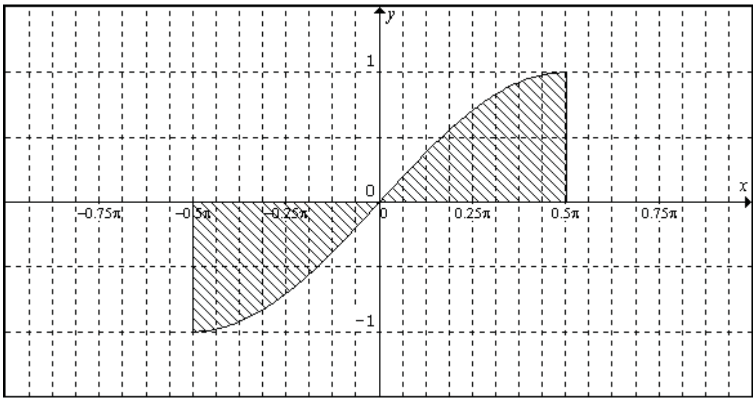
$$= 0$$

(b) $$Area = \left|[-\cos x]_{-\frac{\pi}{2}}^{0}\right| + \left|[-\cos x]_{0}^{\frac{\pi}{2}}\right|$$

$$= \left|{}^{-}1\right| + 1$$

$$= 2$$

(c)



Since the area above the curve is the same as the area below if we add the two integrals we will get zero, since the integral below the *x*-axis will be negative.

Activity 7.7

1. Recall: $s = f(t)$ (displacement)

$$v = \frac{ds}{dt}\ ,\ v = \int a\ dt \qquad \text{(velocity)}$$

$$a = \frac{d^2 s}{dt^2} = \frac{dv}{dt}\ ,\ a = \int v\ dt \quad \text{(acceleration)}$$

Given: $a = 3t - 2,$ and coordinates from the information $(t, v) = (0, -3),$ and $(t, s) = (0, 40)$

So $v = \int (3t - 2)dt$

$$= \frac{3t^2}{2} - 2t + C_1$$

When t is 0, v is -3 so, $-3 = 0 - 0 + C_1$

$$C_1 = -3$$

Hence $v = 1.5t^2 - 2t - 3$ (velocity equation)

From the above equation we can say:

$$s = \int (1.5t^2 - 2t - 3)dt$$

$$= \frac{1.5t^3}{3} - \frac{2t^2}{2} - 3t + C_2$$

When t is 0, s is 40 so, $40 = 0 - 0 - 0 + C_2$

$$C_2 = 40$$

Hence $s = 0.5t^3 - t^2 - 3t + 40$ (displacement equation)

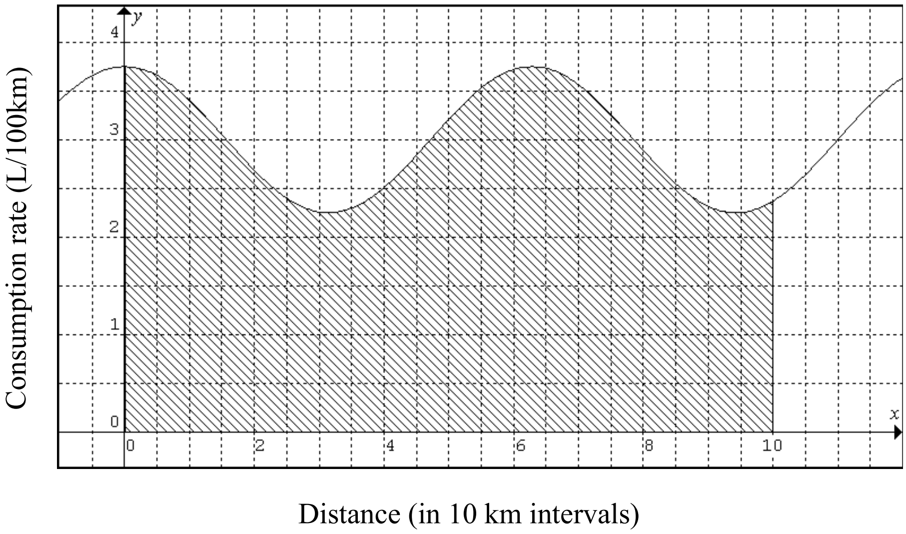
When $t = 4$:

$$v = 1.5(4)^2 - 2(4) - 3 = 13 \text{ m s}^{-1}$$

$$s = 0.5(4)^3 - (4)^2 - 3(4) + 40 = 44 \text{ m}$$

The exact velocity is 13 ms^{-1} in an easterly direction, and the exact displacement is 44 m east.

2.



Number of litres used is the consumption rate times the distance travelled. From the diagram you will see that this will be the area under the curve.

No. of litres used = $\int_0^{10} (0.75\cos x + 3)\,dx$ (10 units = 100 km). Remember to work in radians!

$$= [0.75\sin x + 3x]_0^{10}$$

$$= (0.75\sin 10 + 3(10)) - (0.75\sin 0 + 3(0))$$

$$= (0.75(^{-}0.544) + 30) - (0)$$

$$\approx 29.59$$

3. (a) Equilibrium price:

$$q^2+5=(q-5)^2$$
$$q^2+5=q^2-10q+25$$
$$10q=20$$
$$q=2$$

$$p=q^2+5$$
$$p=2^2+5$$
$$p=9$$

$\therefore$ The equilibrium price is \$9 (and the equilibrium quantity is 2 (× 1000)).

(b) Producer surplus (difference between equilibrium price and supply curve $(0 \le q \le q_e)$).

$$\begin{aligned}\text{Producer surplus } &= (9\times 2) - \int_0^2 (q^2+5)dq \\ &= 18 - \left[\frac{q^3}{3}+5q\right]_0^2 \\ &= 18 - \left(\frac{2^3}{3}+5(2)\right) - 0 \\ &= 18 - 12\frac{2}{3} \\ &= 5.333...\end{aligned}$$

$\therefore$ The producer surplus is $\$5.333..\times 1000 \approx$ \$5 333

4. Given: $F=2x+1$

So
$$\begin{aligned}W &= \int_2^4 (2x+1)dx \\ &= \left[x^2+x\right]_2^4 \\ &= \left(4^2+4\right)-\left(2^2+2\right) \\ &= 20-6 \\ &= 14\end{aligned}$$

Work done is 14J.

Solutions to a taste of things to come

$q_1 = q_2 = 0.05$ coulombs. $a = 20$ cm

$$E = \int_{0.2}^{\infty} \frac{kq_1q_2}{r^2}dr$$

$$= kq_1q_2 \int_{0.2}^{\infty} \frac{1}{r^2}dr$$

$$= kq_1q_2 \left| (-\frac{1}{r}) \right|_{0.2}^{\infty}$$

$$= kq_1q_2 \lim_{b\to\infty} \left| (-\frac{1}{r}) \right|_{0.2}^{b}$$

$$= kq_1q_2 \lim_{b\to\infty} \left(-\frac{1}{b} + \frac{1}{0.2} \right)$$

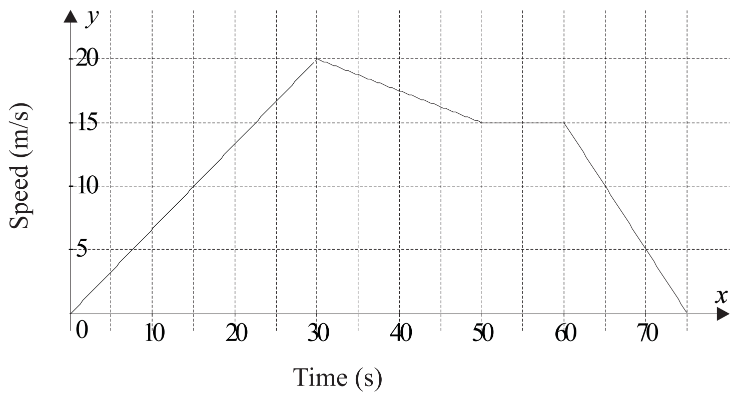
$$= kq_1q_2 \times \frac{1}{0.2}$$

$$= \frac{9\times10^9 \times (0.05)^2}{0.2}$$

$$\approx 1.125\times10^8 \text{ joules}$$

Solutions to post-test

1.



Distance = Total Area (made up of a triangle + a trapezium + a rectangle + another triangle).

$$= \left(\frac{30\times20}{2}\right) + 20\left(\frac{20+15}{2}\right) + (10\times15) + \left(\frac{15\times15}{2}\right)$$

$$= 912.5$$

$\therefore$ The total distance travelled is 912.5 metres.

2. Reading the approximate y-values from the graph at 0.5 unit intervals gives {0, 0.6, 0, −1.2, −2, −1.8, 0}

$$\begin{aligned}\text{Area } &\approx 0.5\left(\frac{0+0.6}{2}\right)+0.5\left(\frac{0.6+0}{2}\right)+0.5\left(\frac{0+1.2}{2}\right)+0.5\left(\frac{1.2+2}{2}\right)+0.5\left(\frac{2+1.8}{2}\right)+0.5\left(\frac{1.8+0}{2}\right)\\ &=0.5\left(0.3+0.3+0.6+1.6+1.9+0.9\right)\\ &=0.5\times 5.6\\ &=2.8\end{aligned}$$

∴ The area under the curve from −1 to 2 is approximately 2.8 square units.

3. (a) $\int \frac{1}{2}x^{-3}dx = \frac{1}{2}\left(\frac{x^{-2}}{-2}\right)+C$

$= -\frac{1}{4}x^{-2}+C$

(b) $\int 4e^t\,dt = 4e^t + C$

(c) $\int ky^2 dy = k\frac{y^3}{3}+C$

$=\frac{k}{3}y^3+C$

(d) $\int (\sin\theta+2)d\theta = -\cos\theta + 2\theta + C$

(e) $\int \frac{1}{x}\left(x^2+1\right)dx = (x+\frac{1}{x})dx$

$=\frac{x^2}{2}+\ln x + C$

$=\frac{1}{2}x^2+\ln x + C$

4. (i) Graphs intersect when $\sin x = \cos x$

Substituting $x=\frac{\pi}{4}$,

$\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

$=\cos\frac{\pi}{4}$

Substituting $x=\frac{5\pi}{4}$,

$\sin\frac{5\pi}{4}=-\frac{1}{\sqrt{2}}$

$=\cos\frac{5\pi}{4}$

$\therefore$ The graphs intersect at $\frac{\pi}{4}$ and $\frac{5\pi}{4}$ (verified).

The area enclosed by the two curves is the difference between the curves.

$$\text{Area} = (\text{Area under the curve } y = \sin x) - (\text{Area under the curve } y = \cos x), \ (\frac{\pi}{4} \le x \le \frac{5\pi}{4})$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x)dx - \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\cos x)dx$$

$$= [-\cos x]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} - [\sin x]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \left(-\cos\frac{5\pi}{4} - \cos\frac{\pi}{4}\right) - \left(\sin\frac{5\pi}{4} - \sin\frac{\pi}{4}\right)$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}$$

$$= \frac{4}{\sqrt{2}}$$

$$\frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} \quad \text{(simplifying the fraction)}$$

$$= 2\sqrt{2} \ \text{ or}$$

$$\approx 2.828$$

5. Work $= W = \int_{0.5}^{3} \frac{1}{x} dx$

$$= [\ln x]_{0.5}^{3}$$

$$= \ln 3 - \ln 0.5$$

$$= 1.791....$$

$\therefore$ The work done in the displacement from 0.5 m to 3 m is approximately 1.8 joules.