Algebra: Solving Simultaneous Equations: Substitution Method

Overview

This presentation will cover the use of the substitution method to solve a system of two equations with two unknown variables.

Simultaneous equations

- A set of simultaneous equations is a set of equations for which common solutions are sought for a number of variables.
- You need at least the same number of equations as variables to be able to find a solution.
- This presentation will only focus on using two equations with two unknown variables.
- There are a number of different ways that you can solve a set of simultaneous equations. All of the methods described are equally valid. It is up to you to choose the method that is easiest for you to use.
- This presentation will cover the substitution method.

Substitution Method

In this method, we substitute one variable from one equation into the other. The steps involved are:

1. Using either of the equations, express one variable in terms of the other.
2. This expression is then substituted into the other equation to form an equation in one variable only.
3. Solve this equation to find the value of one of the variables.
4. Substitute the value of this variable into the equation formed in the first step to find the value of the other variable.
5. Check your answer in both of the original equations.

Example

Let us follow the steps through in an example.

$$
\begin{aligned}
2x + 5y &= 6 \qquad (1)\\
3x + 2y &= -2 \qquad (2)
\end{aligned}
$$

Step 1: Express one variable in terms of the other

Using Equation (1), rearrange to make x the subject.

$$
\begin{aligned}
2x + 5y &= 6\,,\\
2x &= 6 - 5y\,,\\
x &= \frac{6-5y}{2}\,.
\end{aligned}
$$

Note: you could have also made y the subject, or used Equation (2) to make either x or y the subject. It is always good to see which equation will be the simplest to rearrange.

Step 2 and 3: Substitute this into Equation (2) and solve for the single variable

$$
\begin{aligned}
3x + 2y &= -2\,,\\
3\left(\frac{6-5y}{2}\right) + 2y &= -2\,,\\
3(6-5y) + 4y &= -4\,,\\
18 - 15y + 4y &= -4\,,\\
-11y &= -4 - 18\,,\\
-11y &= -22\,,\\
y &= 2\,.
\end{aligned}
$$

Step 4: Substitute this value into the other equation.

$$
\begin{aligned}
2x + 5y &= 6\,,\\
2x + 5 \times 2 &= 6\,,\\
2x &= 6 - 10\,,\\
2x &= -4\,,\\
x &= -2\,.
\end{aligned}
$$

From our calculations our answers are $x = -2$ and $y = 2$.

Step 5: Check the answer in both of the original equations.

Check: $x = -2$ and $y = 2$.

Equation (1):

$$2x + 5y = 6,$$

$$\begin{aligned} \text{LHS} &= 2 \times -2 + 5 \times 2 \\ &= 6 \\ &= \text{RHS}. \end{aligned}$$

Equation (2):

$$3x + 2y = -2,$$

$$\begin{aligned} \text{LHS} &= 3 \times -2 + 2 \times 2 \\ &= -2 \\ &= \text{RHS}. \end{aligned}$$

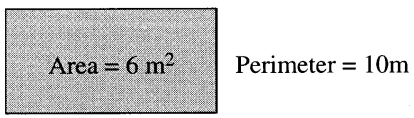
Both values substitute correctly into both equations so the answer must be correct.

Worded example (non-linear equations)

We may need to find the dimensions of a rectangle (perhaps a field or a room) whose area is 6 square units and perimeter is 10 units.

Solution:

Firstly, draw a diagram.



Secondly, define the unknowns:
dimensions: length $= x$, width $= y$.

$$x = ?, \quad y = ?$$

Solution (continued)

Let's set up the equations.

Area must be 6.

$$xy = 6.$$

Perimeter must be 10.

$$2x + 2y = 10.$$

The resulting equations can only be solved by the substitution method.

$$xy = 6. \quad (3)$$

$$2x + 2y = 10. \quad (4)$$

Solution (continued)

Rearrange one of the equations to express one of the variables in terms of the other.

From Equation (3),

$$y = \frac{6}{x}. \qquad (5)$$

Solution (continued)

Substitute this expression into the other equation to form an equation in one variable.

Substituting into Equation (4),

$$2x + 2 \times \frac{6}{x} = 10\,.$$

Solve this equation:

$$\begin{aligned} 2x^2 + 12 &= 10x\,, \\ 2x^2 - 10x + 12 &= 0\,, \\ x^2 - 5x + 6 &= 0\,, \\ (x-2)(x-3) &= 0\,, \\ x = 2\,, &\text{ or } \; x = 3\,. \end{aligned}$$

Solution (continued)

Use this equation to find the first variable.

Substitute these values into Equation (5).

$$\begin{aligned} y &= \frac{6}{2} = 3\,, \text{ or} \\ y &= \frac{6}{3} = 2\,. \end{aligned}$$

Thus the two solutions are $(x = 2\,,\ y = 3)$, or $(x = 3\,,\ y = 2)$.

Solution (continued)

Check this solution.
In Equation (3) LHS $= 2 \times 3 = 6 =$ RHS
In Equation (4) LHS $= 2 \times 2 + 2 \times 3 = 10 =$ RHS.
As an exercise you should check the alternative solution $x = 3\,,$ and $y = 2$.

Exercise

Solve the following sets of simultaneous equations.

1.

$$\begin{aligned} 3x - y &= 12\,, \\ x + y &= 8\,. \end{aligned}$$

2.

$$\begin{aligned} 3x - 4y &= 5\,, \\ 5x - 12y &= 3\,. \end{aligned}$$

Solution: Question 1

$$3x - y = 12\,, \qquad (6)$$

$$x + y = 8\,. \qquad (7)$$

Rearranging Equation (7):

$$x = 8 - y \qquad (8)$$

Solution: Question 1 (continued)

Substituting Equation (8) into Equation (6) gives:

$$\begin{aligned} 3x - y &= 12 \\ 3(8-y) - y &= 12 \\ 24 - 3y - y &= 12 \\ 24 - 4y &= 12. \\ -4y &= 12 - 24 \\ -4y &= -12 \\ y &= \frac{-12}{-4} \\ y &= 3\,. \end{aligned} \qquad (9)$$

Solution: Question 1 (continued)

Substituting Equation (9) into Equation (8) gives:

$$\begin{aligned} x &= 8 - y \\ &= 8 - 3 \\ &= 5\,. \end{aligned}$$

Therefore the solution is $x = 5$ and $y = 3$.

Solution: Question 1 (continued)

Check: $x = 5$ and $y = 3$.

Equation (6):

$$3x - y = 12\,,$$

$$\begin{aligned} \text{LHS} &= 3 \times 5 - 3 \\ &= 12 \\ &= \text{RHS}. \end{aligned}$$

Equation (7):

$$x + y = 8\,,$$

$$\begin{aligned} \text{LHS} &= 5 + 3 \\ &= 8 \\ &= \text{RHS}. \end{aligned}$$

Both values substitute correctly into both equations so the answer must be correct.

Solution: Question 2

$$\begin{aligned} 3x - 4y &= 5\,, && (10) \\ 5x - 12y &= 3\,. && (11) \end{aligned}$$

Rearranging Equation (10) gives:

$$\begin{aligned} 3x - 4y &= 5 \\ 3x &= 5 + 4y \\ 3x - 5 &= 4y \\ y &= \frac{3x - 5}{4}\,. && (12) \end{aligned}$$

Solution: Question 2 (continued)

Substituting Equation (12) into Equation (11) gives:

$$\begin{aligned} 5x - 12y &= 3 \\ 5x - 12\left(\frac{3x - 5}{4}\right) &= 3 \\ 5x - 3(3x - 5) &= 3 \\ 5x - 9x + 15 &= 3 \\ -4x &= 3 - 15 \\ -4x &= -12 \\ x &= 3\,. && (13) \end{aligned}$$

Solution: Question 2 (continued)

Substituting Equation (13) into Equation (12) gives:

$$\begin{aligned} y &= \frac{3x - 5}{4} \\ &= \frac{3 \times 3 - 5}{4} \\ &= 1\,. \end{aligned}$$

Therefore the solution is $x = 3$ and $y = 1$.

Solution: Question 2 (continued)

Check: $x = 3$ and $y = 1$.

Equation (10):

$$3x - 4y = 5\,,$$

$$\begin{aligned} \text{LHS} &= 3 \times 3 - 4 \times 1 \\ &= 5 \\ &= \text{RHS}. \end{aligned}$$

Equation (11):

$$5x - 12y = 3\,,$$

$$\begin{aligned} \text{LHS} &= 5 \times 3 - 12 \times 1 \\ &= 3 \\ &= \text{RHS}. \end{aligned}$$

Both values substitute correctly into both equations so the answer must be correct.

Summary

This presentation covered the use of the substitution method to solve a system of two equations with two unknown variables.