Module 6

INTEGRATION

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Objectives

You are required to:

- use the table of integrals to find the integrals of basic functions;
- use the method of substitution to find the integrals of some more difficult functions;
- evaluate definite integrals; and
- find the area beneath functions using definite integrals.

Use either the Trapezoidal or Simpson's Rule to find the area beneath a function between two limits.

6.1 Integration of Basic Functions

If we differentiate a simple function such as $y = x^3$, we get $\frac{dy}{dx} = 3x^2$. Let us consider this process in reverse, i.e. given $\frac{dy}{dx} = 3x^2$, we may show that $y = x^3$. This is called **integration** and the process is denoted by the symbol: $\int$.

If $\frac{dy}{dx} = 3x^2$, then we may rewrite this as:

$$dy = 3x^2\,dx$$

Thus $y = \int 3x^2 dx$ and we say that y is the **integral** or antiderivative with respect to x of the **integrand** $3x^2$.

The integral sign indicates that we want the function whose derivative is $3x^2$ and the 'dx' symbol indicates that the variable concerned is x. Thus we have:

$$y = \int 3x^2 dx = x^3$$

However, the derivatives of $(x^3 + 3)$ or $(x^3 + 7)$ are also $3x^2$. In fact the derivative of $(x^3 + C)$ where C is any constant is $3x^2$. Therefore, we must say:

$$y = \int 3x^2 dx = x^3 + C$$

The constant term C is called the 'constant of integration' and it must be included in any integration. Unless the values of x and y are known at some point, C cannot be determined. Such integrals are known as 'indefinite integrals'.

By using the process of reverse differentiation we can construct a table of basic functions and their integrals. Note that this table is essentially the same as the Table of Derivatives in module 9 with the columns reversed.

Table of Integrals of Basic Functions

Function	**Integral**
0	C
a (constant)	$ax + C$
x^n $(n \neq -1)$	$\dfrac{x^{n+1}}{n+1} + C$
$\dfrac{1}{x}$	$\ln x + C$ $(x > 0)$
e^n	$e^n + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$

Exercise 6.1

Integrate the following functions with respect to x:

(a) $x^4 + 3\sqrt{x} + \dfrac{4}{x^2}$

(b) $4e^x - \dfrac{5}{x} + 2$

(c) $1 + \tan^2 x$

(d) $\left(x^2 - \dfrac{1}{x}\right)(3 + x)$

(e) $\dfrac{x^2 + 3x - 2}{x}$

(f) $\sqrt{x^3} + x^{\frac{2}{3}}$

Solution

(a) $\displaystyle\int\left(x^4 + 3x^{\frac{1}{2}} + 4x^{-2}\right)\,dx = \frac{x^5}{5} + \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{4x^{-1}}{-1} + C = \frac{x^5}{5} + 2x^{\frac{3}{2}} - \frac{4}{x} + C$

(b) $\displaystyle\int\left(4e^x - \frac{5x}{x} + 2\right)\,dx = 4e^x - 5\,ln\,x + 2x + C$

(c) $\displaystyle\int(1 + \tan^2 x)\,dx = \int\sec^2 x\,dx = \tan x + C$

(d) $\displaystyle\int\left(x^2 - \frac{1}{x}\right)(3 + x)\,dx = \int\left(3x^2 + x^3 - \frac{3}{x} - 1\right)\,dx$

$\displaystyle = \frac{3x^3}{3} + \frac{x^4}{4} - 3\,lnx - x + C$

$\displaystyle = x^3 + \frac{x^4}{4} - 3\,lnx - x + C$

(e) $\displaystyle\int\frac{x^2 + 3x - 2}{x}\,dx = \int\left(x + 3 - \frac{2}{x}\right)\,dx = \frac{x^2}{2} + 3x - 2\,lnx + C$

(f) $\displaystyle\int\left(x^{\frac{3}{2}} + x^{\frac{2}{3}}\right)\,dx = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + C = \frac{2x^{\frac{5}{2}}}{5} + \frac{3x^{\frac{5}{3}}}{5} + C$

Additional Exercises

Find the following integrals:

(a) $\int 3x^7 dx$

(b) $\int 4\sqrt{x}\ dx$

(c) $\int (x^3 - 5x^2 + 7x - 11)\ dx$

(d) $\int \frac{t^4 + 3t^2 + 1}{t^3}\ dt$

(e) $\int 5v^2\ dv$

(f) $\int (4x^2 - 5x + 1)\ dx$

(g) $\int (15a^2 + 3a + 1)\ da$

(h) $\int \frac{dx}{x^2}$

(i) $\int \left(x^2 + 1 + x^{\frac{1}{3}}\right)\ dx$

(j) $\int (\cos x - e^x)\ dx$

(k) $\int (\sin x + \sec^2 x)\ dx$

(l) $\int (3x^2 - 3e^x + 2\sin x)\ dx$

(m) If $\frac{dy}{dx} = 3x^2$, find y.

(n) If $\frac{dy}{dx} = 6\sqrt{x} + \frac{3}{\sqrt{x}}$, find y.

Answers to Additional Exercises

(a) $\frac{3x^8}{8} + C$

(b) $\frac{8x^{\frac{3}{2}}}{3} + C$

(c) $\frac{x^4}{4} - \frac{5x^3}{3} + \frac{7x^2}{2} - 11x + C$

(d) $\frac{t^2}{2} + 3\ln t - \frac{1}{2t^2} + C$

(e) $\frac{5v^3}{3} + C$

(f) $\frac{4x^3}{3} - \frac{5x^2}{2} + x + C$

(g) $5a^3 + \frac{3a^2}{2} + a + C$

(h) $-\frac{1}{x} + C$

(i) $\frac{x^3}{3} + x + \frac{3x^{\frac{4}{3}}}{4} + C$

(j) $\sin x - e^x + C$

(k) $-\cos x + \tan x + C$

(l) $x^3 - 3e^x - 2\cos x + C$

(m) $y = x^3 + C$

(n) $y = 4x^{\frac{3}{2}} + 6\sqrt{x} + C$

6.2 Integration by Guess and Check

The examples we have seen succeeded because our experience with derivatives allowed us to determine the answers exactly using the table of standard integrals. However, often the integrand is more complicated and we cannot determine the answer exactly at once. Fortunately, there is a procedure that assumes that we have enough experience to make a reasonable guess at the answer, but it does not require us to guess exactly right the first time.

The steps are:

1. Write down a guess.
2. Find its derivative.
3. Compare its derivative with the integrand.
4. If necessary modify the guess and return to step 2.
5. Add the constant of integration.

Exercise 6.2

Evaluate the following integrals:

(a) $\int \sqrt{2x+1}\, dx$ (b) $\int -\cos(4x)\, dx$

Solution

(a) We seek F(x) such that $F'(x) = f(x) = (2x+1)^{\frac{1}{2}}$

1. Guess $F_0(x) = (2x+1)^{\left(\frac{1}{2}\right)+1} = (2x+1)^{\frac{3}{2}}$

2. Differentiate guess $\therefore$ $F'_0(x) = \frac{3}{2}(2x+1)^{\frac{1}{2}} \cdot 2 = 3(2x+1)^{\frac{1}{2}}$

3. Compare with $f(x) = (2x+1)^{\frac{1}{2}}$: $F'_0(x)$ is 3 times larger than $f(x)$

4. Modify $F_0(x)$ by dividing by 3 $\rightarrow F_1(x) = \frac{1}{3}(2x+1)^{\frac{3}{2}}$

Back to 2. Differentiate guess $\therefore$ $F'_1(x) = \frac{1}{3} \cdot \frac{3}{2}(2x+1)^{\frac{1}{2}} \cdot 2 = (2x+1)^{\frac{1}{2}}$

3. Compare with $f(x) = (2x+1)^{\frac{1}{2}}$: $F'_1(x) = f(x)$

5. $\Rightarrow$ $\int (2x+1)^{\frac{1}{2}} dx = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C$

(b) $\int -\cos(4x)\, dx$

1. Guess $F_0(x) = \sin(4x)$

2. Differentiate guess $\therefore$ $F'_0(x) = \cos(4x) . 4 = 4\cos(4x)$

3. Compare with $f(x) = -\cos(4x)$: $F'_0(x)$ is (-4) times larger than $f(x)$

4. Modify $F_0(x)$ by dividing by $(-4) \rightarrow F_1(x) = -\frac{1}{4}\sin(4x)$

Back to 2. Differentiate guess $\therefore$ $F'_1(x) = -\frac{1}{4} \cdot \cos(4x) \cdot 4 = -\cos(4x)$

3. Compare with $f(x) = -\cos(4x)$: $F'_1(x) = f(x)$

5. $\Rightarrow$ $\int -\cos(4x)\, dx = -\frac{1}{4}\sin(4x) + C$

6.3 Integration by Substitution

Many integrals may be simplified by making an appropriate substitution. This means that a new variable replaces the original variable and the integral of the new function is found using the table of basic functions. For instance, consider the integral:

$$\int (x^3+4)^2 \cdot 3x^2\, dx$$

$$\begin{aligned}
\text{If we put } u &= x^3+4 \text{ we have:}\\
\frac{du}{dx} &= 3x^2\\
\therefore\quad du &= 3x^2\, dx\\
dx &= \frac{du}{3x^2}
\end{aligned}$$

Thus:

$$\begin{aligned}
\int (x^3+4)^2 \cdot 3x^2\, dx &= \int (x^3+4)^2 \cdot 3x^2 \cdot \frac{du}{3x^2} = \int (x^3+4)^2 du = \int u^2 du\\
&= \frac{u^3}{3} + C\\
&= \frac{1}{3}(x^3+4)^3 + C
\end{aligned}$$

Note that in selecting a function u to substitute, we chose one that not only simplified the integrand but also one whose **derivative**, $3x^2$, existed in the integrand. This derivative together with dx becomes du and we end up with a simple function in u to integrate *w.r.t. u.*

Not all integrations can be performed by substitution and in fact there are numerous techniques required to solve the more difficult integrations. These techniques are beyond the scope of this course but they may be found in any calculus textbook.

Exercise 6.3

Evaluate the following integrals by selecting suitable substitutions.

(a) $\displaystyle\int \frac{2x+3}{x^2+3x-4}\,dx$

(b) $\displaystyle\int \frac{ln^2 x - 4\,ln x}{x}\,dx$

(c) $\displaystyle\int \cos x \ \ e^{\sin x}\,dx$

(d) $\displaystyle\int \frac{x+3}{\sqrt{x-1}}\ dx$

(e) $\displaystyle\int \tan x \ \ dx$

Solution

(a) Let $\quad u = x^2 + 3x - 4 \quad \therefore \quad \dfrac{du}{dx} = 2x + 3 \quad \therefore \quad dx = \dfrac{du}{2x+3}$

$$\int \frac{(2x+3)}{x^2+3x-4}\,dx = \int \frac{2x+3}{u} \cdot \frac{du}{2x+3} = \frac{1}{u}du = ln\,u + C$$

$$= ln\,(x^2 + 3x - 4) + C$$

(b) Let $\quad u = ln\,x \quad \therefore \quad \dfrac{du}{dx} = \dfrac{1}{x} \quad \therefore \quad dx = x\,du$

$$\int \frac{ln^2 x - 4\,ln x}{x}\,dx = \int \frac{(u^2 - 4u)}{x}x \ \ du = \int u^2 - 4u \ \ du$$

$$= \frac{u^3}{3} - 2u^2 + C = \frac{ln^3 x}{3} - 2 \ \ ln^2 x + C$$

(c) Let $\quad u = \sin x \quad \therefore \quad \dfrac{du}{dx} = \cos x \quad \therefore \quad dx = \dfrac{du}{\cos x}$

$$\int \cos x \ \ e^{\sin x}\,dx = \int \cos x \ \ e^u \cdot \frac{du}{\cos x} = \int e^u \ \ du$$

$$= e^u + C = e^{\sin x} + C$$

(d) Let $u = x - 1 \quad \therefore \quad \dfrac{du}{dx} = 1 \quad \therefore \quad dx = du$

$$\int \frac{x+3}{\sqrt{x-1}}\ dx = \int \frac{x-1+4}{\sqrt{x-1}}\ dx = \int \frac{u+4}{\sqrt{u}}\ du$$

$$= \int \left(u^{\frac{1}{2}} + 4u^{-\frac{1}{2}}\right) du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + \frac{4u^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{2}{3}(x-1)^{\frac{3}{2}} + 8\sqrt{x-1} + C$$

(e) $$\int \tan x\ dx = \int \frac{\sin x}{\cos x}\ dx$$

Let $u = \cos x \quad \therefore \quad \dfrac{du}{dx} = -\sin x \quad \therefore \quad dx = -\dfrac{du}{\sin x}$

$$\int \frac{\sin x}{\cos x}\ dx = \int \frac{\sin x}{\cos x} \cdot -\frac{du}{\sin x} = -\int \frac{du}{u} = -1\, ln\, u + C$$

$$= -\,ln\,(\cos x) + C$$

Exercise 6.4

1. The rate at which air is breathed in or out of a person's lungs is given by:

$$\frac{dV}{dt} = 0.6\sin\left[\frac{\pi t}{2}\right]$$

where V is the volume in litres and t the time in seconds.

If V is zero at time zero, find V in terms of t.

Solution

$$\frac{dV}{dt} = 0.6\sin\left(\frac{\pi t}{2}\right)$$

$$dV = 0.6\sin\left(\frac{\pi t}{2}\right)\ dt$$

$$V = \int 0.6\sin\left(\frac{\pi t}{2}\right)\ dt$$

$$\text{Let}\quad u = \frac{\pi t}{2} \quad \therefore \quad \frac{du}{dt} = \frac{\pi}{2} \quad \therefore \quad dt = \frac{2du}{\pi}$$

$$\therefore \quad V = \int 0.6\sin u \times \frac{2du}{\pi} = \int \frac{1.2}{\pi}\sin u\ du$$

$$= -\frac{1.2}{\pi}\cos u + C = -\frac{1.2}{\pi}\cos\left(\frac{\pi t}{2}\right) + C$$

$$\text{When}\ t = 0,\ V = 0 \quad \therefore \quad 0 = -\frac{1.2}{\pi}\cos 0 + C = -\frac{1.2}{\pi} + C$$

$$\therefore \quad C = \frac{1.2}{\pi}$$

$$\therefore \quad V = \frac{1.2}{\pi}\left[1 - \cos\left(\frac{\pi t}{2}\right)\right]\ \text{litres}$$

Additional Exercises

Find the following integrals using Guess and Check or substitution.

1. $\int 3e^{2x}\ dx$
2. $\int e^{3x-1}\ dx$
3. $\int (2 + e^{-x})\ dx$
4. $\int \frac{dx}{\sqrt{x+2}}$
5. $\int \frac{dx}{x+2}$
6. $\int \frac{2x-1}{2x+3}\ dx$

7. $\int \frac{dx}{x+3}$

8. $\int \frac{dx}{3x-2}$

9. $\int \sqrt{2x+3}\ dx$

10. $\int \frac{dx}{\sqrt{1-x}}$

11. $\int \frac{x\ dx}{x^2-1}$

12. $\int \frac{e^{3x}}{e^{3x}+6}\ dx$

13. $\int 3x^2(x^3+2)^4\ dx$

14. $\int x(x^2+4)^3\ dx$

15. $\int (3x^2-2x)(2x^3-2x^2-3)^2\ dx$

16. $\int \sin x\ e^{\cos x}\ dx$

17. $\int e^x \sin(e^x)\ dx$

18. $\int (2x-1)e^{x^2-x+1}\ dx$

19. $\int \frac{dx}{x\, ln x}$

20. $\int \frac{\sin(ln x)}{x}\ dx$

Answers to Additional Exercises (Substitutions are shown in brackets)

1. $\frac{3}{2}e^{2x}+C, (u = 2x)$

2. $\frac{1}{3}e^{3x-1}+C, (u = 3x-1)$

3. $2x-e^{-x}+C, (u = -x)$

4. $2\sqrt{x+2}+C, (u = x+2)$

5. $ln(x+2)+C, (u = x+2)$

6. $x-2\,ln(2x+3)+C, (u = 2x+3)$

7. $ln(x+3)+C, (u = x+3)$

8. $\frac{1}{3}ln(3x-2)+C, (u = 3x-2)$

9. $\frac{(2x+3)^{\frac{3}{2}}}{3}+C, (u = 2x+3)$

10. $-2\sqrt{1-x}+C, (u = 1-x)$

11. $\frac{ln(x^2-1)}{2}+C, (u = x^2-1)$

12. $\frac{ln(e^{3x}+6)}{3}+C, (u = e^{3x}+6)$

13. $\frac{(x^3+2)^5}{5}+C, (u = x^3+2)$

14. $\frac{(x^2+4)^4}{8}+C, (u = x^2+4)$

15. $\frac{(2x^3-2x^2-3)^3}{6}+C, (u = 2x^3-2x^2-3)$

16. $-e^{\cos x}+C, (u = \cos x)$

17. $-\cos(e^x)+C, (u = e^x)$

18. $e^{x^2-x+1}+C, (u = x^2-x+1)$

19. $ln(ln x)+C, (u = ln x)$

20. $-\cos(ln x)+C, (u = ln x)$

By using substitution methods with $u = kx$ or $u = f(x)$, the following integrals can be developed. **There is no need to learn this table.**

Function	**Integral**
e^{kx}	$\frac{1}{k} e^{kx} + C$
$\sin kx$	$-\frac{1}{k} \cos kx + C$
$\cos kx$	$\frac{1}{k} \sin kx + C$
$\sec^2 kx$	$\frac{1}{k} \tan kx + C$
$f'(x)\, e^{f(x)}$	$e^{f(x)} + C$
$\frac{f'(x)}{f(x)}$	$ln\, f(x) + C$
$f'(x) \sin f(x)$	$-\cos f(x) + C$
$f'(x) \cos f(x)$	$\sin f(x) + C$
$f'(x) \sec^2 f(x)$	$\tan f(x) + C$

For example:

1. $\int e^{2x} dx = \frac{1}{2} e^{2x} + C$

2. $\int \sin 4x \; dx = -\frac{1}{4} \cos 4x + C$

3. $\int \cos 6x \; dx = \frac{1}{6} \sin 6x + C$

4. $\int \sec^2 3x \; dx = \frac{1}{3} \tan 3x + C$

5. $\int 3x^2 e^{x^3} \; dx = e^{x^3} + C$

6. $\int \frac{2x+1}{x^2+x-1} \; dx = ln(x^2 + x - 1) + C$

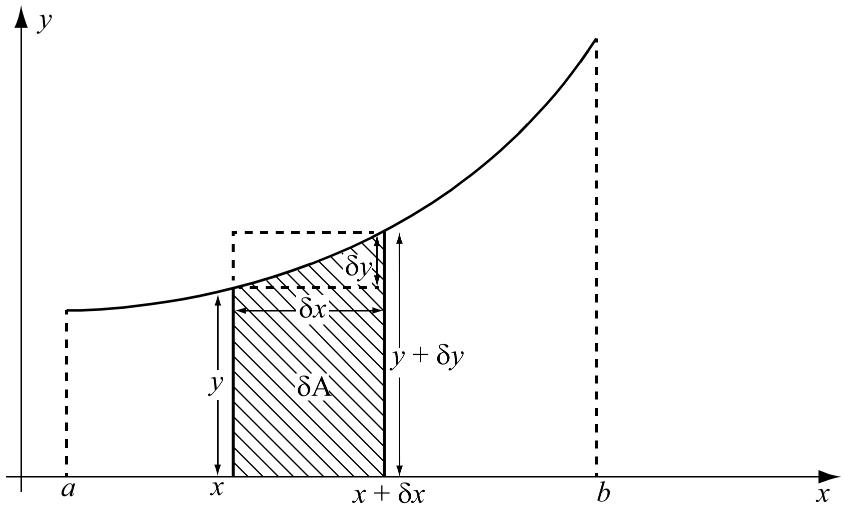
7. $\int (3x^2 + 2)\sin(x^3 + 2x + 4)\ \ dx = -\cos(x^3 + 2x + 4) + C$

8. $\int \sec^2 x\cos(\tan x)\ \ dx = \sin(\tan x) + C$

9. $\int (x + 2)\sec^2(x^2 + 4x)\ \ dx = \frac{1}{2}\tan(x^2 + 4x) + C$

6.4 Definite Integration

Consider the curve $y = f(x)$ which is sketched below and the small element of area, δA indicated by the shaded area.



The shaded area δA is greater than that of the rectangle of height y but less than that of the rectangle of height $y + \delta y$. Thus:

$$y\,.\,\delta x \ <\ \delta A\ <\ (y + \delta y)\,.\,\delta x$$

$$\therefore\ \ y\ <\ \frac{\delta A}{\delta x}\ <\ y + \delta y$$

As δx approaches zero, $\frac{\delta A}{\delta x}$ becomes $\frac{dA}{dx}$ and δy approaches zero.

Thus:

$$\frac{dA}{dx} = y$$

$$\therefore \quad A = \int y\,dx = \int f(x)\,dx = F(x) + C$$

where F(x) represents the integral or antiderivative of the function $f\,(x)$. This result provides the formula for the area A, in terms of any value of x and the constant C. Suppose the required area lies between $x = a$ and $x = b$.

When $x = a$, the shaded area would be zero.

$$\therefore \quad F(a) + C = 0$$

$$\therefore \quad C = -F(a)$$

When $x = b$, $A = F(b) - F(a)$.

This represents the total area beneath the curve $y = f(x)$ between the values $x = a$ and $x = b$. This is written as:

$$\text{Area} = \int_a^b f(x)\,dx = F(b) - F(a)$$

and is known as THE FUNDAMENTAL THEOREM OF CALCULUS

The constants a and b are called the lower and upper limits of integration and since the result contains no integration constant C, it is called a definite integral. Note that a definite integral is a number rather than an algebraic expression and it is obtained by integrating the function, evaluating it for the two limits and taking the difference of the two results. The layout for this procedure is demonstrated in the example below.

Exercise 6.5

Find the area under the curve $y = x^2 + 3x + 2$ between $x = 2$ and $x = 4$.

Solution

$$\text{Area} = \int_2^4 (x^2 + 3x + 2)\ dx$$

$$= \left[\frac{x^3}{3} + \frac{3x^2}{2} + 2x\right]_2^4$$

$$= \left(\frac{4^3}{3} + 3 \times \frac{4^2}{2} + 2 \times 4\right) - \left(\frac{2^3}{3} + \frac{3 \times 2^2}{2} + 2 \times 2\right)$$

$$= 53\frac{1}{3} - 12\frac{2}{3} = 40\frac{2}{3} \text{ units}^2$$

This procedure may be summarised as follows:

(a) Write down the integral with the larger x-value as the upper limit,

i.e. $\int_a^b f(x)\ dx$ where $b > a$

(b) Find the integral of the function omitting the integration constant C. Place the result in square brackets with the limits at the right,

i.e. $\int_a^b f(x)\ dx = [\mathrm{F}(x)]_a^b$

(c) Substitute the upper limit in the integrated function and subtract the result obtained by substituting the lower limit, i.e. $\mathrm{F}(b) - \mathrm{F}(a)$. Note that the final result must be a single number.

Exercise 6.6

Evaluate:

(a) $\displaystyle\int_1^4 \left(\frac{1}{x} + \frac{1}{\sqrt{x}}\right) \ dx$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta) \ d\theta$

Solution

(a) $\displaystyle\int_1^4 \left(\frac{1}{x} + x^{-\frac{1}{2}}\right) \ dx \ = \ [\,ln x + 2\sqrt{x}\,]_1^4$

$$= \ (\,ln 4 + 2\sqrt{4}\,) - (\,ln 1 + 2\sqrt{1}\,)$$

$$= \ (1.386 + 4) - (0 + 2) \ = \ 3.386$$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta) \ d\theta \ = \ [\,\sin\theta - \cos\theta\,]_0^{\frac{\pi}{2}}$

$$= \ \left(\sin\frac{\pi}{2} - \cos\frac{\pi}{2}\right) - (\,\sin 0 - \cos 0\,)$$

$$= \ (1 - 0) - (0 - 1) \ = \ 2$$

Exercise 6.7

A tree fell 5000 years ago. The carbon 14 in the tree decays at the rate:

$$\frac{dc}{dt} = ae^{-0.00012t}$$

where c is the number of carbon 14 atoms present, t is the number of years since the tree fell, and a is a constant. Find the amount of carbon 14 that has decayed in the last 1000 years.

Solution

$$dc = ae^{-0.00012t}\ dt$$

$$\therefore\quad \mathrm{c} = \int ae^{-0.00012t}\ dt$$

From $t = 4000$ to $t = 5000$ gives:

$$\boldsymbol{c} = \int_{4000}^{5000} ae^{-0.00012t}\ dt$$

$$= \left[\frac{ae^{-0.00012t}}{-0.00012}\right]_{4000}^{5000} = \frac{ae^{-0.00012 \times 5000}}{-0.00012} - \frac{ae^{-0.00012 \times 4000}}{-0.00012}$$

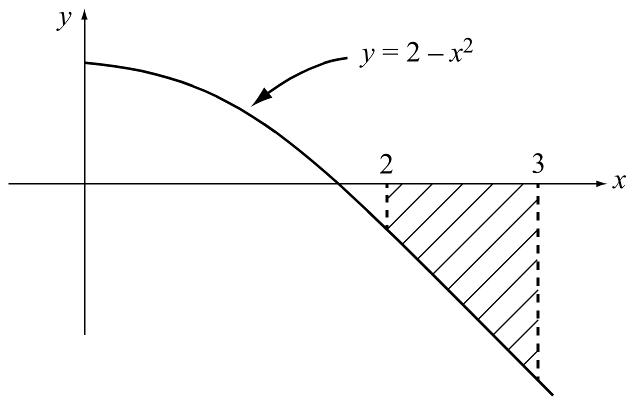
$$= -\frac{a}{0.00012}(e^{-0.6} - e^{-0.48}) = -\frac{a}{0.00012}(0.5488 - 0.6188)$$

$$= \frac{0.07}{0.00012}a = 583a$$

Exercise 6.8

Find the area between the curve $y = 2 - x^2$, the x-axis and the limits $x = 2$ and $x = 3$.

Solution



$$\text{Area} = \int_2^3 (2 - x^2)\ dx = \left[2x - \frac{x^3}{3}\right]_2^3$$

$$= (6 - 9) - \left(4 - \frac{8}{3}\right) = -3 - \frac{4}{3} = -4\frac{1}{3}$$

Area cannot be negative, hence disregard the negative sign. Area $= 4\frac{1}{3}$ units^2.

Note: The negative sign indicates that this area is below the x-axis. This is a general result with all areas above the axis being positive and all areas below the axis being negative. If the total area between the x-axis and a curve which lies above and below the axis is required, then the positive and negative areas must be found separately and the results added together ignoring the negative sign.

This procedure applies only if you are **finding an area**. If you are asked simply to integrate a function between two limits, then negative values (if any) stay negative. An integral may have a number of interpretations apart from area. These practical applications will appear in other units in your course.

Exercise 6.9

Find the area enclosed between the curve $y = (x-1)(x-2)(x+2)$ and the x-axis.

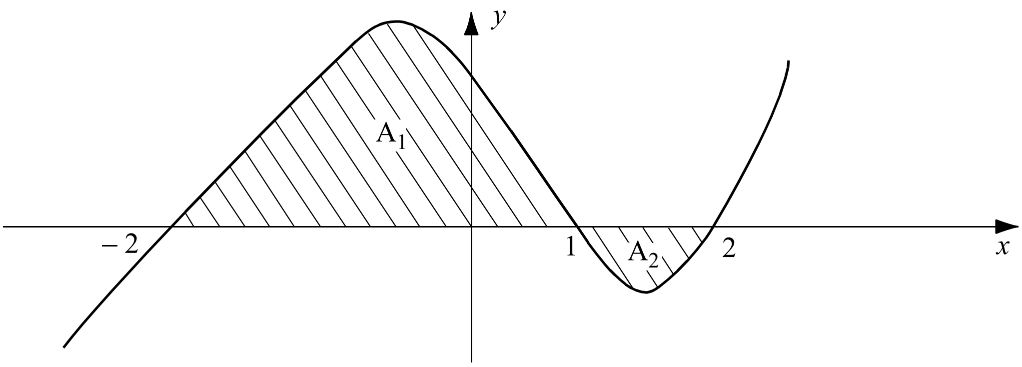
Solution

The curve cuts the x-axis when $y = 0$, i.e.

$(x-1)(x-2)(x+2) = 0$

$\therefore\ x-1 = 0,\ x-2 = 0$ or $x+2 = 0$

$\therefore\ x = 1,\ 2$ or -2



$$A_1 = \int_{-2}^{1} (x-1)(x-2)(x+2)\ dx = \int_{-2}^{1} (x-1)(x^2-4)\ dx$$

$$= \int_{-2}^{1} (x^3 - x^2 - 4x + 4)\ dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - 2x^2 + 4x\right]_{-2}^{1}$$

$$= \left(\frac{1}{4} - \frac{1}{3} - 2 + 4\right) - \left(\frac{16}{4} - \frac{-8}{3} - 2 \times 4 + 4 \times -2\right)$$

$$= \frac{23}{12} - \left(\frac{-28}{3}\right) = 11\frac{1}{4}$$

$$A_2 = \int_1^2 (x^3 - x^2 - 4x + 4) \ dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - 2x^2 + 4x\right]_1^2$$

$$= \left(\frac{16}{4} - \frac{8}{3} - 2 \times 4 + 4 \times 2\right) - \left(\frac{1}{4} - \frac{1}{3} - 2 + 4\right) = \frac{4}{3} - \frac{23}{12} = -\frac{7}{12}$$

Total area $= 11\frac{1}{4} + \frac{7}{12} = 11\frac{5}{6}$ units^2 (ignoring negative sign for A_2)

Note: Consider the single integral from – 2 to 2:

$$\int_{-2}^{2} (x^3 - x^2 - 4x + 4) \ dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - 2x^2 + 4x\right]_{-2}^{2}$$

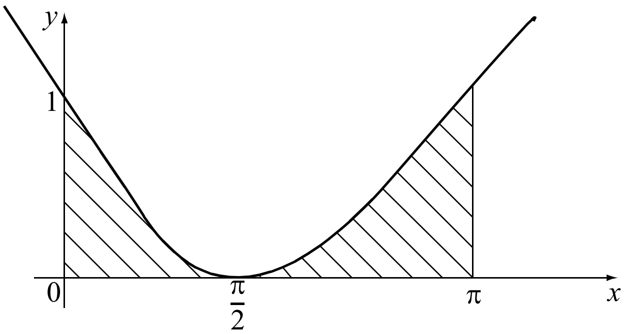
$$= \frac{4}{3} - \left(\frac{-28}{3}\right) = \frac{32}{3} = 10\frac{2}{3} \text{ which is incorrect.}$$

The area below the axis has been subtracted instead of added, i.e. the negative sign has not been ignored. Thus, it is essential to find positive and negative areas separately and to add their **absolute** values.

Exercise 6.10

Find the area beneath the curve $y = 1 - \sin x$ and the x-axis from $x = 0$ to $x = \pi$.

Solution



Since both areas are positive, a single integral may be used. (Note that radians must be used.)

$$\text{Area} = \int_0^{\pi} (1 - \sin x)\ dx$$

$$= [x + \cos x]_0^{\pi} = (\pi + \cos\pi) - (0 + \cos 0)$$

$$= (\pi - 1) - (0 + 1) = \pi - 2 = 3.14159 - 2 = 1.14159 \text{ units}^2$$

Additional Exercises

1. Evaluate the following definite integrals:

(a) $\int_1^2 dx$ (b) $\int_1^2 x^2\ dx$ (c) $\int_2^4 (x^3 + x)\ dx$

(d) $\int_{-1}^1 x^3\ dx$ (e) $\int_0^1 (1 - x)^2\ dx$ (f) $\int_1^2 (x - 1)(2 - x)\ dx$

(g) $\int_{-1}^1 e^x\ dx$ (h) $\int_0^2 e^{\frac{x}{2}}\ dx$ (i) $\int_{-2}^{-1} \frac{dx}{(x - 2)^3}$

2. Find the areas under the following curves and above the x-axis between the x-values indicated.

   (a) $y = x^2 + 3,\ x = 0$ and $x = 1$ (b) $y = \dfrac{1}{x},\ x = 2$ and $x = 3$

   (c) $y = 4x^3 + 3x^2 + 1,\ x = 1$ and $x = 2$

3. Find the areas between the following curves and the x-axis.

   (a) $y = (x-2)(x+3)$ (b) $y = (x-1)(x+2)(x-3)$

Answers to Additional Exercises

1. (a) $[x]_1^2 = 1$ (b) $\left[\dfrac{x^3}{3}\right]_1^2 = \dfrac{7}{3}$ (c) $\left[\dfrac{x^4}{4} + \dfrac{x^2}{2}\right]_2^4 = 66$

   (d) $\left[\dfrac{x^4}{4}\right]_{-1}^1 = 0$ (e) $\left[-\dfrac{(1-x)^3}{3}\right]_0^1 = \dfrac{1}{3}$ (f) $\left[-2x + \dfrac{3x^2}{2} - \dfrac{x^3}{3}\right]_1^2 = \dfrac{1}{6}$

   (g) $[e^x]_{-1}^1 = e - \dfrac{1}{e} = 2.3504$ (h) $\left[2e^{\frac{x}{2}}\right]_0^2 = 2e - 2 = 3.4366$

   (i) $\left[\dfrac{-1}{2(x-2)^2}\right]_{-2}^{-1} = -\dfrac{1}{2\times 9} + \dfrac{1}{2\times 16} = -0.0243$

2. (a) $\displaystyle\int_0^1 (x^2+3)\ dx = \left[\frac{x^3}{3} + 3x\right]_0^1 = \frac{10}{3}$

   (b) $\displaystyle\int_2^3 \frac{dx}{x} = [\,ln x]_2^3 = 0.4055$

   (c) $\displaystyle\int_1^2 (4x^3 + 3x^2 + 1)\ dx = [x^4 + x^3 + x]_1^2 = 23$

3. (a) Intersects x-axis at $x = -3$ and $x = 2$

   $$\int_{-3}^{2} (x^2 + x - 6)\ dx = \left[\frac{x^3}{3} + \frac{x^2}{2} - 6x\right]_{-3}^2 = -\frac{125}{6} \quad \therefore \quad \text{Area} = \frac{125}{6}$$

   (b) Intersects x-axis at $x = -2,\ x = 1$ and $x = 3$

   $$A_1 = \int_{-2}^{1} (x^3 - 2x^2 - 5x + 6)\ dx = \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x\right]_{-2}^1 = \frac{63}{4}$$

   $$A_2 = \int_{-2}^{3} (x^3 - 2x^2 - 5x + 6)\ dx = -\frac{16}{3} \quad \therefore \quad \text{Area} = \frac{63}{4} + \frac{16}{3} = \frac{253}{12}$$

6.5 Trapezoidal Rule

Sometimes when we want the definite integral of a function, we are faced with one of two situations:

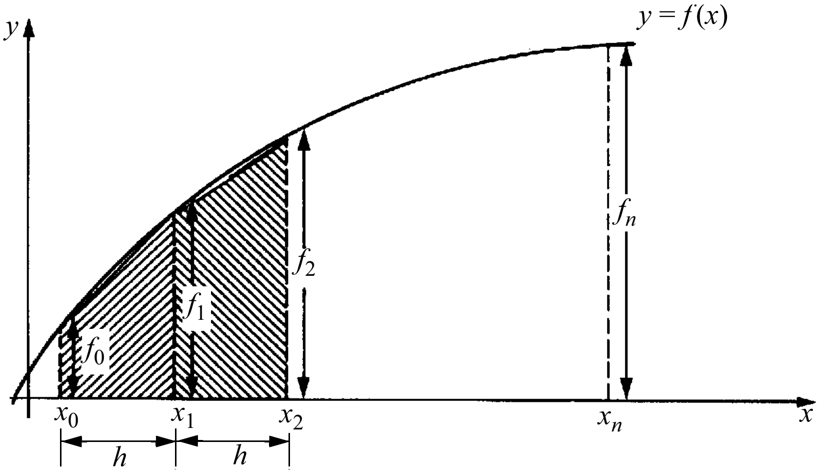
(a) the integration is difficult or impossible to perform,

e.g. $\int e^{-x^2}\,dx$ cannot be determined analytically as no antiderivative for e^{-x^2} is known.

(b) the function is unknown and only its values at various points are known, e.g. the depth measurements of a lake.

In these situations we use numerical methods to approximate the area under the curve.

Consider the area beneath the curve $y = f(x)$ between $x = x_0$ and $x = x_n$.



This can be divided into n equal strips of width h. If we join the top of these strips, we form a series of trapeziums whose areas approximate the area under the curve. The area of the first trapezium is given by:

$$\frac{h}{2}(f_0 + f_1)$$

where f_0, f_1 are the values of $f(x)$ at x_0, x_1 respectively. Thus, the total area under the curve is approximately:

$$\int_{x_0}^{x_n} f(x)\,dx \approx \frac{h}{2}\left\{(f_0 + f_1) + (f_1 + f_2) + \ldots + (f_{n-1} + f_n)\right\}$$

$$\therefore \quad \boxed{\text{Area} \approx \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \ldots + 2f_{n-1} + f_n\right)} \qquad (12.1)$$

where $h = \dfrac{x_n - x_0}{n}$

This is known as the **Trapezoidal Rule**. Its accuracy is improved by increasing the number of strips, i.e. reducing *h*. If the curve is concave down, as in the diagram above, the approximation is less than the exact area and if it is concave up, the approximation is more than the exact area. These errors tend to cancel in an undulating curve.

Exercise 6.11

Evaluate $\displaystyle\int_1^4 \frac{dx}{x}$ using 6 strips

Solution

$h = \dfrac{\text{total width}}{\text{no. of strips}} = \dfrac{4-1}{6} = 0.5, \; f(x) = \dfrac{1}{x}$

x	1	1.5	2	2.5	3	3.5	4
$f(x) = \dfrac{1}{x}$	1	0.667	0.5	0.4	0.333	0.286	0.25

[Note that *x* covers the range of the integral, i.e. from $x = 1$ to $x = 4$, in steps of size $h = 0.5$. The values of *f* are the values when each value of *x* is substituted into $f(x)$. These *f* values are then substituted into (12.1).]

$$\text{Area} \approx \frac{0.5}{2}\,(1 + 2 \times 0.667 + 2 \times 0.5 + 2 \times 0.4 + 2 \times 0.333 + 2 \times 0.286 + 0.25)$$

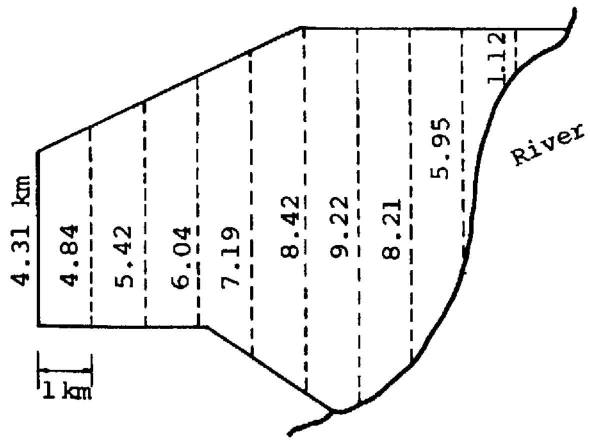
$$= 0.25 \times 5.622 = 1.405$$

[Exact answer is $[\ln x]_1^4 = \ln 4 - \ln 1 = 1.386$]

Exercise 6.12

The Forestry Commission has a pine plantation with the dimensions below taken from a plan drawn to scale. Estimate the number of trees if a planting pattern of 224 trees per hectare is used.

[**Note:** $1\text{ha} = 10^4 \text{ m}^2$]



(Note that only the width and length of the strips are of any importance – the shape of the baseline does not matter.)

$h = 1$

Area $= \dfrac{1}{2}\ \{4.31 + 2\,(4.84 + \ldots + 1.12) + 0\}$ [Note that the final value of *f* is 0 rather than 1.12.]

$= 0.5 \times 117.13 = 58.565 \text{ km}^2$

$= 58.565 \times 10^6 \text{ m}^2 = 5856.5 \text{ ha}$

Number of trees $= 5856.5 \times 224 = 1312000$

Additional Exercises

1. The work done by a horizontal force $F(x)$ in moving an object along the x-axis from $x = a$ to $x = b$ is defined to be $W = \int_b^a F(x)\ dx$. The horizontal force (in Newtons) is measured at 2m intervals as it moves an object from $x = 1$ to $x = 11$ and it is recorded below. Find approximately the work done.

x	1	3	5	7	9	11
$F(x)$	14	13	10	8	5	1

2. The distance, s, covered along a straight road in h hours by a car with velocity $v(t)$ is given by:

$$s = \int_0^h v(t)\ dt.$$

The following table records the velocity (in km h^{-1}) at 0.1 hour intervals for one hour. Find the approximate distance covered in that hour.

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$v(t)$	0	75	88	80	77	90	110	80	84	85	100

3. Find the approximate values of the integrals below using the given number of strips (n).

(a) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{\sin x\ dx}{x}\ ;\ n = 4.$ (b) $\displaystyle\int_0^1 e^{-x^2}\ dx\ ;\ n = 5$

Answers to Additional Exercises

1. $W = \dfrac{2}{2}\ \{14 + 2\,(13 + 10 + 8 + 5) + 1\} = 87$ Newton metres

2. $s = \dfrac{0.1}{2}\ \{0 + 2\,(75 + \ldots + 85) + 100\} = 0.05 \times 1638 = 81.9$ km.

3. (a) $h = \dfrac{\left(\frac{5\pi}{6} - \frac{\pi}{6}\right)}{4} = \dfrac{\pi}{6}$ (Note that radians must be used,) $f(x) = \dfrac{\sin x}{x}$

x	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$
f	0.9549	0.8270	0.6366	0.4135	0.1910

Area $= \dfrac{\frac{\pi}{6}}{2}\ \{0.9549 + 2\,(0.8270 + 0.6366 + 0.4135) + 0.1910\} = 1.2828$

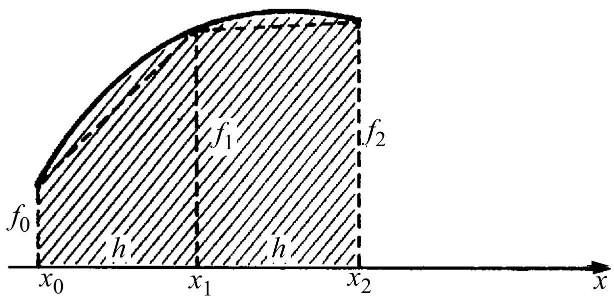
(b) $h = \dfrac{1-0}{5} = 0.2\ ,\ f(x) = e^{-x^2}$

x	0	0.2	0.4	0.6	0.8	1
f	1	0.9608	0.8521	0.6977	0.5273	0.3679

Area $= \dfrac{0.2}{2}\ \{1 + 2(0.9608 + \ldots + 0.5273) + 0.3679\} = 0.7444$

6.6 Simpson's Rule

The trapezoidal rule can result in fairly large errors, particularly if the function is sharply curved. A better fit is obtained by using a parabola instead of a straight line as the 'top' of the trapeziums. Consider the pair of strips below:



The area beneath the parabola covering **two** strips is

$$\frac{h}{3}(f_0 + 4f_1 + f_2)$$

$\therefore$ The total area beneath the curve is given by:

$$\int_{x_0}^{x_n} f(x)\ dx \approx \frac{h}{3}\left\{(f_0 + 4f_1 + f_2) + (f_2 + 4f_3 + f_4) \ldots + (f_{n-2} + 4f_{n-1} + f_n)\right\}$$

$$\boxed{\text{Area} \approx \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + \ldots + 2f_{n-2} + 4f_{n-1} + f_n)} \qquad (12.2)$$

This is known as **Simpson's 1/3 Rule** and it is much more accurate than the Trapezoidal Rule. Note that it requires an **even** number of strips.

Exercise 6.13

Evaluate $\int_1^4 \frac{dx}{x}$ using Simpson's Rule with 6 strips and compare your answer with Exercise 12.1.

Solution

$$h = \frac{3}{6} = 0.5$$

x	1	1.5	2	2.5	3	3.5	4
f	1	0.6667	0.5	0.4	0.3333	0.2857	0.25

$$\text{Area} = \frac{0.5}{3} \quad (1 + 4 \times 0.6667 + 2 \times 0.5 + 4 \times 0.4 + 2 \times 0.3333 + 4 \times 0.28576 + 0.25)$$

$$= \frac{1}{6} \times 8.3262 = 1.3877$$

[Exact answer using the Fundamental Theorem of Calculus is 1.3863: Trapezoidal Rule gives 1.405]

Exercise 6.14

Evaluate $\int_0^3 e^{-x^2}\,dx$ using Simpson's Rule with 6 strips.

Solution

This is a good example of when the Fundamental Theorem of Calculus cannot be used as there is no known antiderivative of e^{-x^2}.

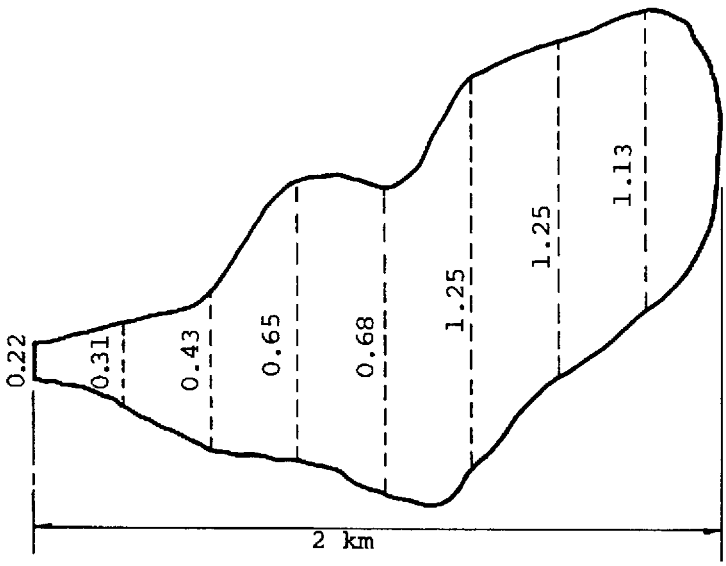
x	0	0.5	1	1.5	2	2.5	3
f	1	0.77880	0.36788	0.10540	0.01832	0.00193	0.00012

$$\int_0^3 e^{-x^2}\,dx = \frac{0.5}{3}\ \{1 + 4\,(0.77880 + 0.1054 + 0.00193) + 2\,(0.36788 + 0.01832) + 0.00012\}$$

$$= \frac{0.5}{3} \times 5.31703 = 0.88617$$

Exercise 6.15

The dimensions (in km) of a dammed lake were measured from an aerial photograph with the results below. Estimate its area using both rules and compare the results.



Solution

$h = \frac{2}{8} = 0.25$

By Simpson's Rule,

$$\text{Area} = \frac{0.25}{3} \; \{0.22 + 4\,(0.31 + 0.65 + 1.25 + 1.13) + 2\,(0.43 + 0.68 + 1.25) + 0\}$$

$\uparrow$

Note: include 0 as last $f(x)$ value

$$= \frac{0.25}{3} \times 18.3 = 1.525 \text{ km}^2$$

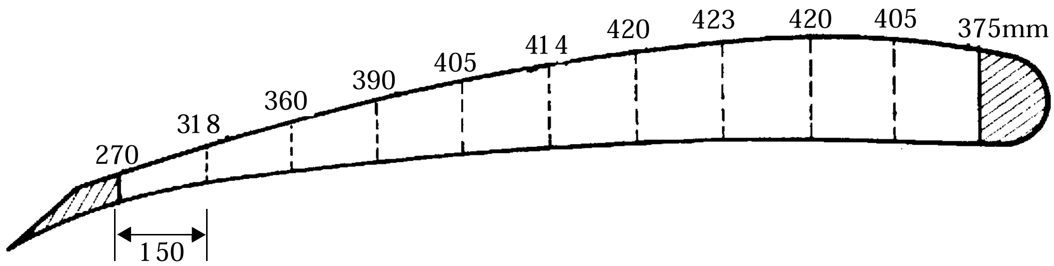
By the Trapezoidal Rule,

$$\text{Area} = \frac{0.25}{2} \; \{0.22 + 2\,(0.31 + 0.43 + \ldots + 1.13) + 0\} = 0.125 \times 11.62 = 1.453 \text{ km}^2$$

Note that there is quite a large difference between these answers. This is due to the rapid change in the values at the right-hand side which creates errors in the Trapezoidal Rule.

Exercise 6.16

The design of a new plane has the fuel tank in the wing. From scale drawings of its cross-section, the depth of the tank at 150 mm intervals is determined. The tank maintains this cross-section for its entire length of 3 metres. Find its volume in litres, using both rules and compare your answers.



Solution

Simpson's Rule:

$$\text{Area} \approx \frac{150}{3}\{270 + 4(318 + 390 + 414 + 423 + 405) + 2(360 + 405 + 420 + 420) + 375\}$$

$$= 50 \times 11655 = 582750 \text{ mm}^2 = 0.58275 \text{ m}^2$$

$$\text{Volume} = 3 \times 0.58275 = 1.74825 \text{ m}^3 = 1748\ l$$

Trapezoidal Rule:

$$\text{Area} \approx \frac{150}{3}\{270 + 2(318 + \ldots + 405) + 375\}$$

$$= 75 \times 7755 = 581625 \text{ mm}^2 = 0.581625 \text{ m}^2$$

$$\text{Volume} = 3 \times 0.581625 = 1.744875 \text{ m}^3 = 1745\ l$$

The two answers are very close since the values do not change rapidly.

Additional Exercises

1. Repeat Additional Exercise 2 from the previous section using Simpson's Rule.

2. Use Simpson's Rule with the given number, n, of strips indicated to find the approximate values of the integrals below.

   (a) $\displaystyle\int_0^1 \sqrt{1-x^2}\ dx \quad ; \ n = 4$

(b) $\displaystyle\int_{0.8}^{1.4} \frac{e^x}{x}\, dx \qquad ; \; n = 6$

(c) $\displaystyle\int_{0}^{2} \sqrt{1 + x^4}\; dx \qquad ; \; n = 8$

Answers to Additional Exercises

1. $s \approx \dfrac{0.1}{3} \; \{0 + 4\,(75 + 80 + 90 + 80 + 85) + 2\,(88 + 77 + 110 + 84) + 100\}$

   $= \dfrac{0.1}{3} \times 2458 = 81.93 \text{ km}$

2. (a) $h = \dfrac{1 - 0}{4} = 0.25\,,\; f(x) = \sqrt{1 - x^2}$

x	0	0.25	0.5	0.75	1
f	1	0.9682	0.8660	0.6614	0

Area $\approx \dfrac{0.25}{3} \; \{1 + 4 \times 0.9682 + 2 \times 0.8660 + 4 \times 0.6614 + 0\} = 0.7709$

(b) $h = \dfrac{1.4 - 0.8}{6} = 0.1\,,\; f(x) = \dfrac{e^x}{x}$

x	0.8	0.9	1	1.1	1.2	1.3	1.4
f	2.7819	2.7329	2.7183	2.7311	2.7668	2.8225	2.8966

Area $\approx \dfrac{0.1}{3} \; \{2.7819 + 4(2.7329 + 2.7311 + 2.8225) + 2(2.7183 + 2.7668) + 2.8966\}$

$= 1.6598$

(c) $h = \dfrac{2 - 0}{8} = 0.25\,,\; f(x) = \sqrt{1 + x^4}$

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
f	1	1.0020	1.0308	1.1473	1.4142	1.8551	2.4622	3.2216	4.1231

Area $= \dfrac{0.25}{3} \; \{1 + 4 \times 1.0020 + 2 \times 1.0308 + \ldots + 4.1231\} = 3.6535$

Practice Test

This test is for practice purposes only. You should attempt the questions without referring to your notes. The answers are given overleaf.

1. Differentiate the following functions with respect to x.

   (a) $4 + 3x - 2x^2 + \frac{1}{x}$

   (b) $\cos x^2$

   (c) $(1 + e^x) \sin x$

   (d) $\frac{x-1}{x+1}$

   (e) $x^3 e^{2x}$

2. Find the minimum value of the function, $y = 3x^2 + 4x - 3$.

3. The rate of population growth of pigeons in a large city is given by:

   $$R = 400P^2 - \frac{1}{3}P^3$$

   where P is the pigeon population. For what population is the growth rate a maximum?

4. Use Newton's Method to find the positive solution of $x^2 = \cos x$ correct to 3 decimal places.

5. Integrate the following functions with respect to x:

   (a) $\frac{2}{x} + 4\cos x$

   (b) $3e^{2x} + 4\sqrt{x}$

   (c) $4x^3 e^{x^4}$

   (d) $\frac{6x^2 + 4}{2x^3 + 4x - 2}$

6. Evaluate:

   (a) $\int_2^4 (3x + x^2)\ dx$

   (b) $\int_0^{\frac{\pi}{2}} (\cos 2\theta + \sin\theta)\ d\theta$

7. Find the area beneath the curve $y = 3x^2 + 2x + 1$ and the x = axis between $x = 0$ and $x = 1$.

8. Find the area enclosed by the curve $y = x(x-1)(x-3)$ and the x-axis.

9. A curve passes through the points given below. Use (a) the Trapezoidal Rule and (b) Simpson's Rule to find the area beneath this curve between $x = 0$ and $x = 2$.

x	0	0.5	1	1.5	2
f	0	0.874	2.621	5.220	8.052

Answers

1. (a) $3 - 4x - \dfrac{1}{x^2}$ (b) $-2x \sin x^2$ (c) $e^x \sin x + (1 + e^x) \cos x$

   (d) $\dfrac{2}{(x+1)^2}$ (e) $3x^2 e^{2x} + 2x^3 e^{2x}$

2. $y = -\dfrac{13}{3}$ when $x = -\dfrac{2}{3}$

3. P = 800

4. 0.824

5. (a) $2 \ln x + 4 \sin x + C$ (b) $\dfrac{3e^{2x}}{2} + \dfrac{8x^{\frac{3}{2}}}{3} + C$

   (c) $e^{x^4} + C$ (d) $\ln(2x^3 + 4x - 2) + C$

6. (a) $\dfrac{110}{3}$ (b) 1

7. 3

8. $\dfrac{37}{12}$

9. (a) 6.3705 (b) 6.2783