Module C6

Describing change – an introduction to differential calculus

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Introduction

Have you ever heard the statement: 'the only thing that is constant in life is change'? A seemingly contradictory statement. But when we look around us everything does change. We grow, the planets move, bridges vibrate, medications are absorbed into the blood stream, profits go up and down etc., etc. etc. But just because we see things constantly changing does not mean that they are changing in a constant way. In fact, the majority of events that we observe do not change in a constant way. The field of mathematics that was developed to quantify these changes is called differential calculus and is just one part of a wider discipline called The Calculus that we will investigate further in a later module. The invention of differential calculus in the 17th century is thought to be one of the greatest inventions of modern mathematics and has revolutionized all of the science and mathematically based disciplines. At the time it created an enormous scandal as its two inventors (Isaac Newton and Gottfried Leibniz) battled for the right to be called first inventor (see boxed history later in the module). But of course by now the battles have died down and your job is to understand and use this great discovery which will bring together all of the skills and knowledge you have developed in the previous modules. More formally, at the end of this module you should be able to:

- use graphs and algebra to describe the rate of change of a function
- determine the instantaneous rate of change of a function
- apply the power, sum and difference rules to find the derivative of certain polynomial functions
- apply calculus to velocity and acceleration and other real life problems
- use gradient functions to determine the derivatives of trigonometric, exponential and logarithmic functions
- locate local stationary points of a function
- solve optimization problems.

6.1 Rate of change – the problem of the curve

Let's think about what we already know about rates of change. We know that if we want to compare values of a single variable we might use ratios or percentages, but if we want to compare two variables we would use a rate of change or more briefly a **rate**. You will have come across many different rates during your life, in either your work or study. Here are some you might be familiar with. Recall that rates always have units.

Speed	metres per second
	kilometres per hour
	revolutions per second
	degrees centigrade per minute
Flow rate	kilograms per second
Energy content	kilojoules per gram
Acceleration	kilometres per hour per hour
Density	grams per cubic centimetre
Pressure	dynes per square centimetre (pascals)
Power	Joules per second
Diffusion	square metres per second
Concentration	grams per cubic centimetre
Latent heat	kilojoules per kilogram

If a relationship between two variables is a straight line it is easy to find the rate of change. It is the gradient or slope of the straight line. Recall from module 3 that the gradient puts a value on the steepness of a straight line by comparing the change in height with the change in horizontal distance.

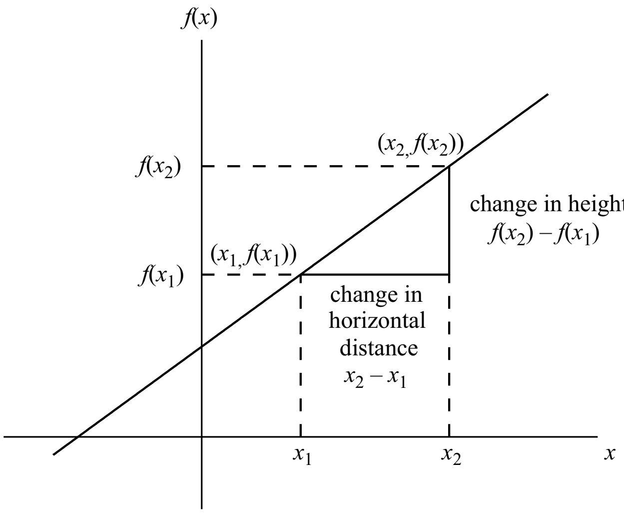
You may also recall this from *Tertiary Preparation Mathematics Level B* (11082) or equivalent as well.

In function notation, if we have the linear function $y = f(x)$, which has two points with horizontal values of x_1 and x_2, the rate of change m is

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}} = \frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

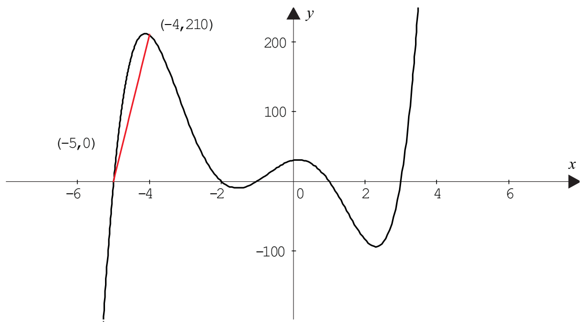
(Note: Δ is the Greek symbol delta used to mean a change, so that Δx means a change in x.)

Figure 6.1



It's easy to determine the gradient or rate of change of a function if it is a linear function, because linear functions always have a constant gradient or rate of change. Curved functions are not so easy, because the rate of change of one variable with respect to the other is always changing. However, we can approximate the process by finding the gradient between two points of interest on the graph. We call this the **average rate of change** of the curve. Using figure 6.2 from module 3, we have approximated the average rate of change of the curve between the points $x = –5$ and $x = –4$, by finding the gradient of the straight line connecting these two points. This type of line segment which intersects the curve in two places is called a **secant**.

Figure 6.2: Average rate of change of a curve



The average rate of change for the curve shown between the points (–5, 0) and (–4, 210) will be

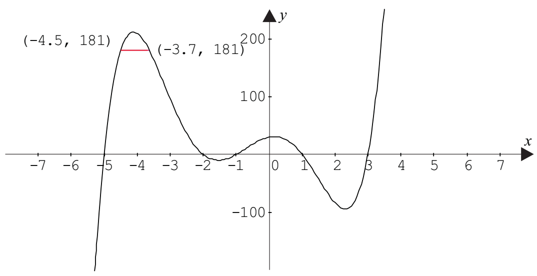
$$m = \frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
$$= \frac{210 - 0}{-4 - -5}$$
$$= 210$$

Between –5 and – 4 the curves changes at a rate of 210 units of y for each unit of x.

This means that if we know any two points on a curve, we can use them to calculate the average rate of change of the function, by calculating the gradient of the straight line connecting the points (gradient of the secant).

But what if we chose x-coordinates of –4.5 and –3.7.

Figure 6.3: Average rate of change of a curve



Here we would calculate that the average rate of change is zero. But as you can see from the graph there is actually quite a lot happening to the function and its rate of change between –4.5 and –3.7. In this situation and many others, it is not enough to find the average rate of change, we might want to find what is happening exactly at a particular instant (at a point on the graph). For example:

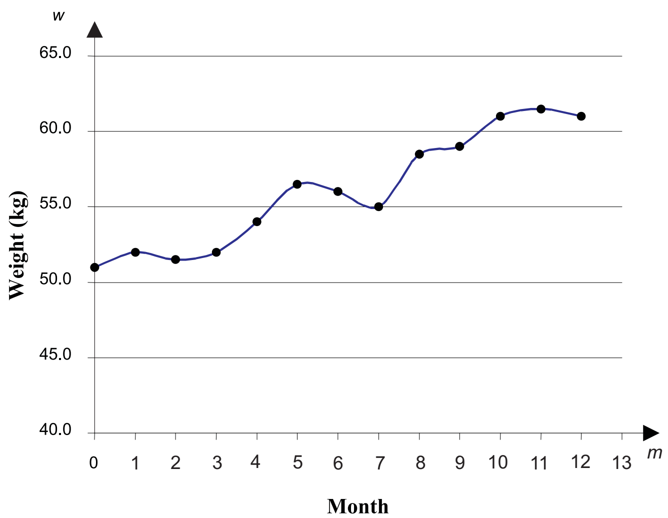
- If you were gathering data on the fuel consumed by a rocket, the instant the rate of fuel consumption changed would tell you that the rocket was boosting its speed.
- A manufacturer might monitor production costs over time so that they would know the instant there is a variation in the rate of change of cost with time. They could relate this instant to machinery or human error.
- Human population biologists would want to monitor the rate of change of population size so that predictions could be made for town planning for a particular year.
- Pattern makers know that there are a range of rectangles with different dimensions that can be made with the same perimeter. The rectangular dimensions that produce the maximum area can be found by determining at what specific width (or length) the rate of change of area is zero (at a turning point).

We will return to more of these examples later in the module. It appears that we need a more accurate idea of the rate of change of a function. We often need to know the rate of change at a particular instant (point on the function).

There must be a way to do this. Let's do some investigating.

Teenagers grow in leaps and bounds. The change in weight of a particular teenager could be described by the following graph.

Figure 6.4: Graph of the weight changes of a teenager over a 12 month period



In most instances we wouldn't monitor a teenager's weight in such a way. But it could be important if the teenager was an athlete training heavily, was ill or on medication. In such cases it would be important to know the highs and lows of their weight, when the teenager's rate of weight change increased or decreased, or what the rate of change of weight at a particular time was so that it could be related to a change in training program etc. Let’s look at such a relationship which can be described in words or by a 7th degree polynomial equation (we won’t attempt that now). Try to describe how the rate of change of the function is varying over time in words yourself. Remember that the rate we are describing is actually kg per month.

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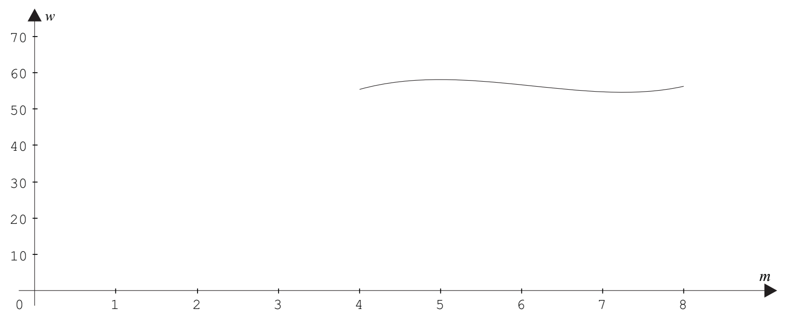
You will have noticed that the above function is complex with many increases and decreases in the rate of change. For example, the function appears to have local highs at $t = 1$, $t = 5$ and $t = 11$ where the rate of change of weight with time is zero and that the rate of change is stationary (does not change) between the 8th and 9th months.

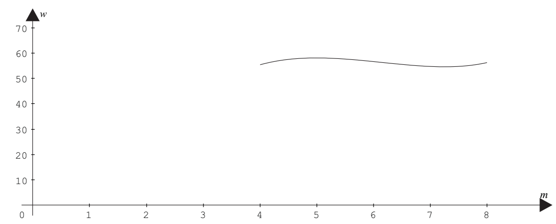
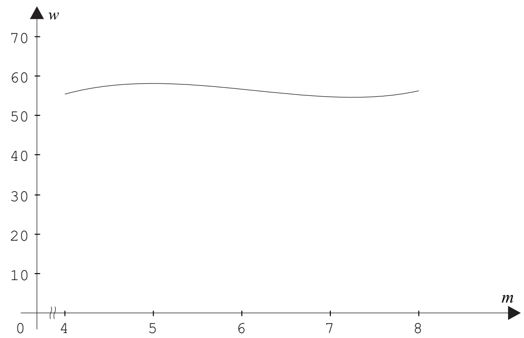
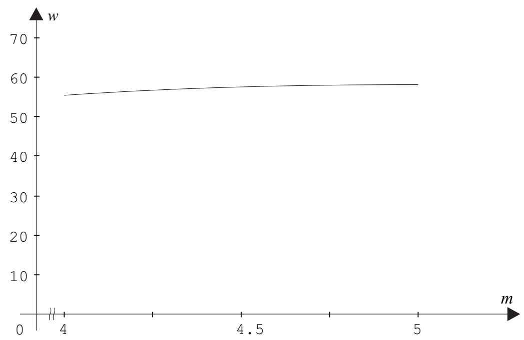
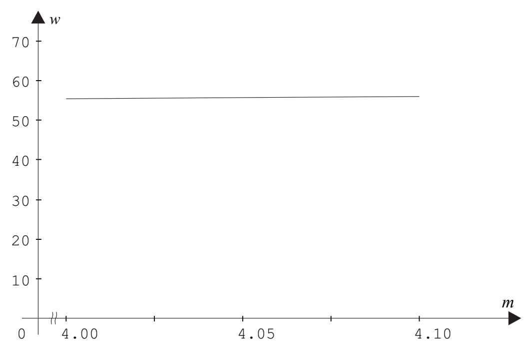
Let’s zoom in and look more closely at the part of the function in the domain $4 \le m \le 8$, between the fourth and eighth month of growth. In order to investigate this more easily, this part of the function could be represented by another function approximated by $W = 0.6m^3 - 11m^2 + 65m - 67$, where m is the month or part of the month and W is the weight in kg. Use Graphmatica to sketch this function.

Figure 6.5: $W = 0.6m^3 - 11m^2 + 65m - 67$



The curve at this stage looks to be moving between a maximum and a minimum value.

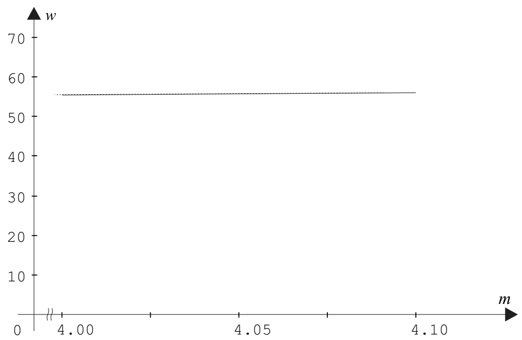
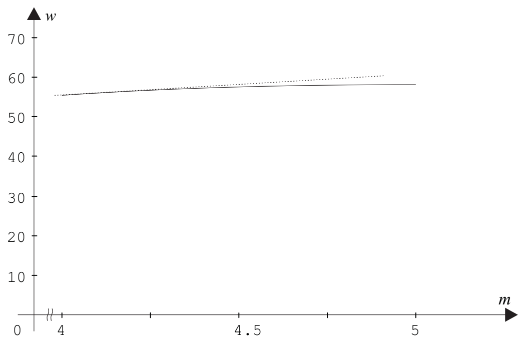
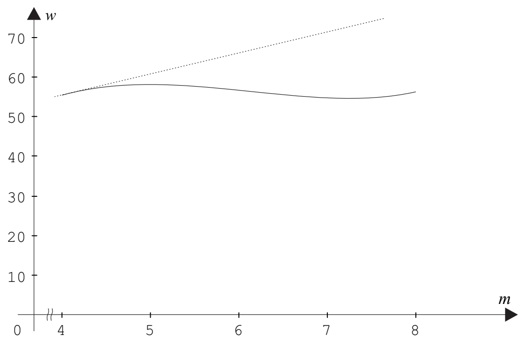
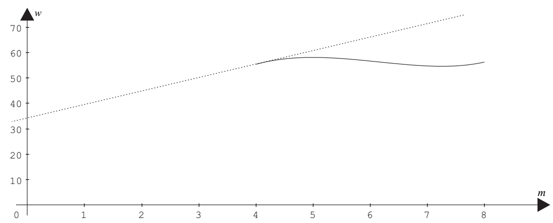
What does the curve look like as we zoom in closer and closer? You might get a series of pictures like these below. (See the introductory book for hints on the best way to zoom using Graphmatica.)

For *x* values 0 to 8 Zoom in	
For *x* values 4 to 8 Zoom in	
For *x* values 4 to 5 Zoom in	
For *x* values 4 to 4.1	

What did you notice? Well of course, the function has not changed, neither has its rate of change become different. But as we zoomed in closer and closer (looking at points on the curve closer and closer together) the curve appears to look more like a straight line.

(You can try this for yourself on Graphmatica using the magnifying glass key. Watch out that you don't lose the image, you might have to change the scale to accommodate it. Alternatively you could use the *Grid Range* selection to change the scale of the graph.)

If the curve does begin to look like a straight line segment when we get closer and closer, then we can use this to help us find the rate of change at a single point. A line that runs along the straight part of the curve will actually touch the curve in just one place when we zoom in very close. A straight line which behaves like this and touches the curve in just one place is called a **tangent to the curve**. At the single point on the curve, the curve and the tangent will fit snugly together. Let's look at this more closely using another Graphmatica tool. On the Graphmatica tool bar find the *Calculus* option and click on *Draw tangent*, then move the curser down onto a point of the curve, click the mouse and see what happens. The *Draw tangent* option draws a straight line which will touch the curve in just one place in the vicinity of your curser. We will use the curves (or parts of curves) we used before, this time we will zoom out instead of in. What do you notice?

For x values 4 to 4.1 Zoom in	
For x values 4 to 5 Zoom in	
For x values 4 to 8 Zoom in	
For x values 0 to 8	

When we were in the close up version of the curve, the tangent that Graphmatica drew fitted snugly onto the section of the curve. As we zoom out more we can see that the tangent to the curve is just another straight line. The difference being that it touches the curve at just one point in the area we are interested in.

Sketch the curve using Graphmatica now if you haven't done so already and include the tangent to the curve at a couple of points. Describe what you notice.

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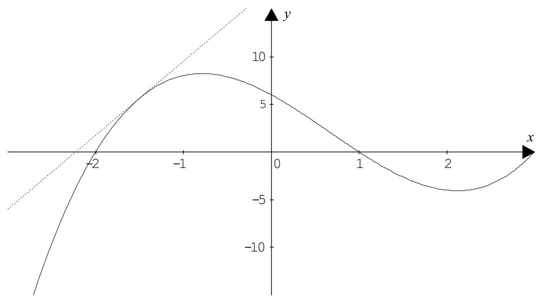
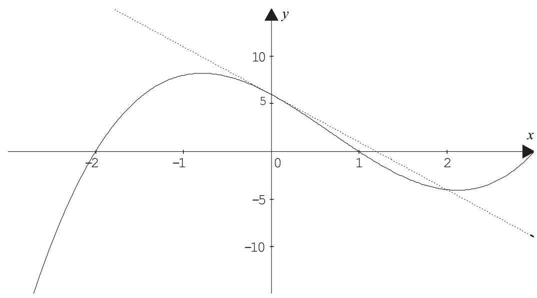
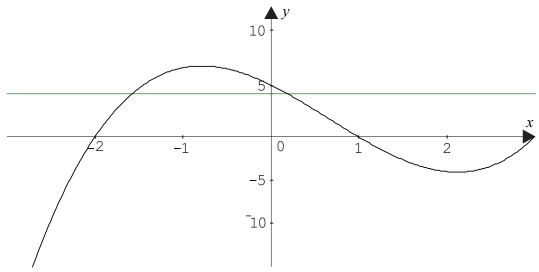
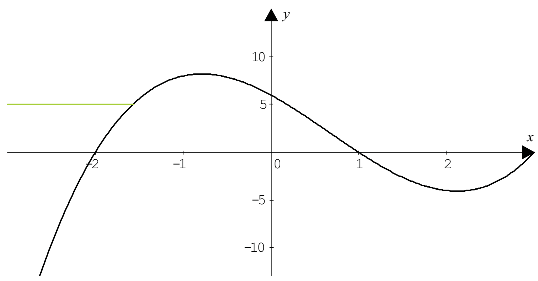
When is a straight line not a tangent	
This straight line is a tangent at $x = -1.5$. It touches the curve at one point only in the vicinity of $x = -1.5$. The straight line mimics the behaviour of the curve at $x = -1.5$.	
This straight line is a tangent at $x = 0$, but not at $x = 2$. The tangent does not just touch the curve at $x = 2$, but passes right through it.	
This straight line is not a tangent at any place on the curve. The line intersects the curve rather than just touching it. The straight line does not mimic the behaviour of the curve at any of the points where it intersects the curve.	
This line is not a tangent at the point $x = -1.5$. Even though the line segment touches the curve, if extended it would pass through the curve. The straight line segment does not mimic the behaviour of the curve at that point.	

It appears that we could use these tangents to help us determine the rate of change of the curve at just one point, instead of averaging it between two points as we have done before. Let's see what we get. Try to estimate the gradient of the tangents you have drawn to the graph using Graphmatica. Recall you can do this easily by drawing a small triangle on the tangent and estimating the ratio of the vertical rise over the horizontal run or notice the figures along the bottom of the Graphmatica screen. When you draw a tangent these figures should show the point where the tangent touches the curve and the slope of the tangent.

Do it for points

$x \approx 4$, $y \approx 55.2$, slope ≈ 5.6	This means that weight is being gained at a rate of 5.6 kg per month at this point.
$x = 5$	
$x = 6$	
$x = 7$	
$x = 8$	

Did you get what you expected?

Now let's put this all together. You might have noticed two things as you zoomed in and out:

- the curve now appears straight in places; and
- the line drawn by Graphmatica (the tangent to the curve) now appears to lie along this straight part of the curve, i.e. touches the curve at one point.

So what does all this mean when it comes to determining more accurately the rate of change of a curve.

If we zoom in on a curve we can see that small parts of the curve become straighter as we zoom closer and that the tangent drawn to the curve fits snugly along those straight parts of the curve. So perhaps we **could** use the gradient of the tangent to the curve at a point to determine the gradient of the curve at that point. This would be the rate of change at that instant or the **instantaneous rate of change**.

The instantaneous rate of change at a point on a curve is determined by the gradient of the tangent to the curve at that point.

Something to talk about…

Write your own definition of a tangent and how you think it helps interpret rate of change at any instant and share it with your fellow students on the discussion group. You might use the teenager's weight fluctuations as an example.

So we have concluded that the gradient of the tangent to the curve would give us the instantaneous rate of change for a curve. But how could we find out exactly where the tangent is and what is its gradient?

Well, we could do as we did above, sketch the graph, get Graphmatica to draw the tangent, then estimate its gradient using the triangle method. But this has its shortcomings. It:

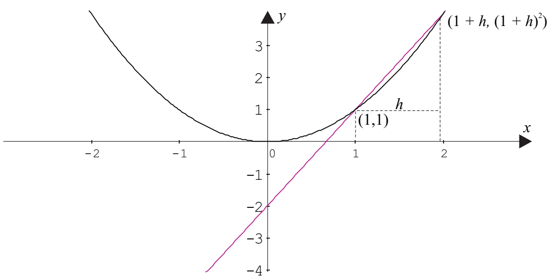
- is time consuming and inaccurate;
- always requires that we draw a graph; and
- requires us to use Graphmatica to insert the tangent. (Try doing this by hand and you will see how hard and inaccurate this can be.)

So let's look for a more robust method that is accurate and works every time, with or without the use of a graphing package or calculator. To do this we have to look back to two tools from module 3 that we have already used: the average rate of change and the concept of a limit. We'll do this through an example of a simple parabola $y = f(x) = x^2$, rather than the teenage weight change curve in the first instance.

So for $y = f(x) = x^2$ we want to find the instantaneous rate of change of the curve at the point $x = 1$. This means that we want to find the gradient of the tangent to the curve at $x = 1$. But at this stage all we can do algebraically is find the average rate of change between two points.

So let's pick the first point to be at $x = 1$, i.e. the point (1,1) and the second point to be h units along from $x = 1$. The second point will have an x-coordinate of $x = 1 + h$ and a y-coordinate of $y = f(1+h) = (1+h)^2$.

Figure 6.6: $y = f(x) = x^2$ and intersecting straight line



The gradient of this line segment will be $m = \dfrac{f(1+h) - f(1)}{1+h-1} = \dfrac{f(1+h) - f(1)}{h}$ where h is the distance along the x-axis between the first and second points.

What happens to the two points as h gets smaller and smaller (approaches zero)?

First point	Value of h	Second point
$x = 1$	$h = 1$	$x = 2$
$x = 1$	$h = 0.5$	$x = 1.5$
$x = 1$	$h = 0.1$	$x = 1.1$
$x = 1$	$h = 0.01$	$x = 1.01$
$x = 1$	$h = 0.001$	$x = 1.001$

As you probably guessed, as h gets smaller and approaches zero (i.e. as $h \to 0$) the two points get closer and closer together. If this happens then the line segment (**the secant**) becomes the tangent.

What happens to the gradient of the secant as h gets smaller and smaller (approaches zero)?

Value of h	$f(1+h) = (1+h)^2$	$m = \dfrac{f(1+h) - f(1)}{h}$
$h = 1$	4	3
$h = 0.5$	2.25	2.5
$h = 0.1$	1.21	2.1
$h = 0.01$	1.0201	2.01
$h = 0.001$	1.002001	2.001
$h = 0.0001$	1.00020001	2.0001

Yes, as you can see the gradient approaches 2 as h approaches zero.

So this means that as h approaches zero, the gradient of the tangent at $x = 1$ approaches 2. We can summarise this by the expression, $m = \lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{h} = 2$.

(Note that the second point actually never reaches the first point and h never actually becomes zero. If it did we would not be able to find the gradient of the line because we cannot divide by zero).

We could also do this algebraically.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$ Substitute values in for the point required, i.e. $x = 1$

$$= \lim_{h \to 0} \frac{(1+h)^2 - (1)^2}{h}$$ Evaluate the function at points $x = 1 + h$ and $x = 1$

$$= \lim_{h \to 0} \frac{(1+2h+h^2) - 1}{h}$$ Expand the brackets, recall that $(1+h)^2 = (1+h)(1+h)$

$$= \lim_{h \to 0} \frac{1+2h+h^2 - 1}{h}$$ Simplify the numerator.

$$= \lim_{h \to 0} \frac{2h+h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2+h)}{h}$$ Take h out as a common factor in the numerator.

$$= \lim_{h \to 0} (2+h)$$ Cancel out h from numerator and denominator. Recall that as h gets closer to zero then the term h will also approach zero and disappear when we take the limit as $h \to 0$

$$= 2$$

So in this example we have seen how we can use our knowledge of average gradient and limits to calculate the gradient of a tangent to a curve and hence determine the instantaneous rate of change of the curve.

Example

Imagine you are Newton dropping his famous apple. (He actually didn't drop it himself, just observed it dropping.) Of course you will have developed a function relating the distance dropped (s, in metres) since release (t, in seconds) so that: $s = 4.9t^2$.

Sketch a graph of the function and determine the following:

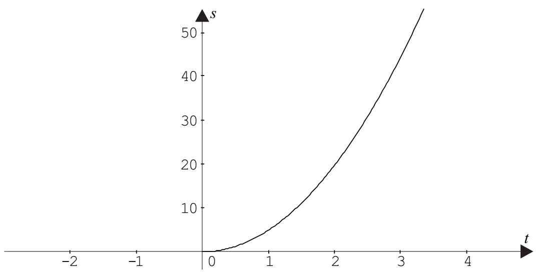
(i) If you were standing at the top of a tower 44.1 m high, how long will it take the apple to reach the ground?

(ii) What is the average speed of the apple between one second before it hits and the time it hits the ground?

(iii) By taking smaller and smaller time intervals, approximate the instantaneous speed of the apple as it hits the ground and what would be its approximate speed (rate of change of distance with time) the second before it hits the ground.

(iv) Use $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, to find the exact value of the instantaneous speed of the apple the second before it hits the ground.

The graph would be as below.



Note that the domain of the function will be $t \geq 0$, because we cannot have negative time.

(i) To determine when the apple will reach the ground, substitute $s = 44.1$ into $s = 4.9t^2$,

$$\begin{aligned} s &= 4.9t^2 \\ 44.1 &= 4.9t^2 \\ 9 &= t^2 \\ t &= \pm 3 \end{aligned}$$

Hence it will take 3 seconds for the apple to reach the ground.

Note in actuality, the domain will be $0 \leq t \leq 3$, because once the apple reaches the ground after 3 seconds it can fall no further.

(ii) To determine the average rate of change between 2 and 3 seconds we calculate the gradient of the straight line between the points of the function (2, 19.6) and (3, 44.1).

Average speed $= \dfrac{f(x+h)-f(x)}{h} = \dfrac{44.1-19.6}{1} = 24.5$ m/s

(iii) When finding the average speed of 24.5 m/s the value of h was 1. To determine smaller and smaller time intervals we have to reduce the value of h.

h	$f(2+h)=4.9\times(2+h)^2$	$\dfrac{f(2+h)-f(2)}{h}$
1	44.1	24.5000
0.1	21.609	20.0900
0.01	19.796	19.6490
0.001	19.6196	19.6049
0.0001	19.60196	19.60049

As h gets smaller and smaller the speed of the apple approaches 19.6 m/s, so 19.6 m/s would be a good approximation for the instantaneous speed of the apple 2 seconds before hitting the ground.

(iv) To find this exactly we need to evaluate the instantaneous speed,

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

Substitute values in for the point required, i.e. $x = 2$

$$=\lim_{h\to 0}\frac{4.9\times(2+h)^2-4.9\times(2)^2}{h}$$

Evaluate the function at points $x = 2+h$ and $x = 2$

$$=\lim_{h\to 0}\frac{4.9\times(4+4h+h^2)-4.9\times 4}{h}$$

Expand the brackets, recall that $(2+h)^2 = (2+h)(2+h)$

$$=\lim_{h\to 0}\frac{4.9\times 4+4.9\times 4h+4.9\times h^2-4.9\times 4}{h}$$

Simplify the numerator.

$$=\lim_{h\to 0}\frac{4.9\times 4h+4.9\times h^2}{h}$$

$$=\lim_{h\to 0}\frac{h(4.9\times 4+4.9h)}{h}$$

Take h out as a common factor in the numerator.

$$=\lim_{h\to 0}4.9\times 4+4.9h$$

Cancel out h from numerator and denominator.
Recall that as h gets closer to zero then the term $4.9h$ will also approach zero and disappear when we take the limit as $h\to 0$

$$=4.9\times 4$$

$$=19.6$$

We can see that the exact instantaneous speed 2 seconds before hitting is 19.6 m/s.

Activity 6.1

1. During a recent flood, scientists noted that the height of a river above its normal level could be modelled by the following equation:

   $$H = 5.5t^2$$

   where H was the height in centimetres above the river's normal height and t the time in hours after 6 am.

   (a) Sketch the curve showing the height of the river between 6 am and 12 noon, when it reached its peak.

   (b) What was the average rate of change of water level between 8 am and 9 am?

   (c) By taking smaller and smaller time intervals, approximate the instantaneous rate of change of water level at 8 am.

   (d) If $f(t) = H = 5.5t^2$ use $\lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$, to find the exact value of the instantaneous rate of change of water level at 8 am.

2. The temperature on a given day between 10 am and 2 pm, was known to be modelled by the equation:

   $$T = 0.25t^2 + 3.5t + 20$$

   where T was the temperature in degrees Centigrade and t the time in hours after 10 am.

   (a) Sketch the curve showing the temperature between 10 am and 2 pm.

   (b) What was the average rate of change of temperature between 1 pm and 2 pm?

   (c) By taking smaller and smaller time intervals, approximate the instantaneous rate of change of temperature at 1 pm.

   (d) If $f(t) = T = 0.25t^2 + 3.5t + 20$ use $\lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$, to find the exact value of the instantaneous rate of change of temperature at 1 pm.

6.2 Instantaneous rates of change and the derivative function

You can see that we are able to determine the instantaneous rate of change for any point, but is there any way we could do this more generally for the whole function instead of point by point. Well the answer is yes. Consider this for $y = f(x) = x^2$ again.

In this case instead of taking a particular point, $x = 1$, let's take the more general point, $P(x, y)$ or in function notation $P(x, f(x))$ and find the instantaneous rate of change at this point using:

$$\text{Instantaneous rate of change} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ — Substitute values in for the point required, i.e. $x = x$

$= \lim_{h\to 0}\frac{(x+h)^2-(x)^2}{h}$ — Evaluate the function at points $x = x + h$ and $x = x$

$= \lim_{h\to 0}\frac{(x+2xh+h^2)-x^2}{h}$ — Expand the brackets, recall that $(x+h)^2 = (x+h)(x+h)$

$= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$ — Simplify the numerator.

$= \lim_{h\to 0}\frac{2xh+h^2}{h}$

$= \lim_{h\to 0}\frac{h(2x+h)}{h}$ — Take h out as a common factor in the numerator.

$= \lim_{h\to 0}(2x+h)$ — Cancel out h from numerator and denominator.

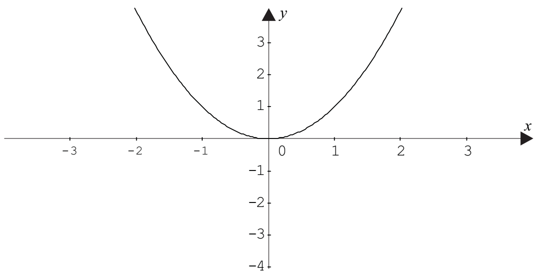
$= 2x$ — Recall that as h gets closer to zero then the term h will also approach zero and disappear when we take the limit as $h \to 0$

So if the instantaneous rate of change at the point x is $2x$ what does this mean? Well the $2x$ is another function which will allow us to calculate the instantaneous rate of change for any value of x. So that

Value of x in $f(x) = x^2$	**Instantaneous rate of change using $2x$**
3	6
2	4
1	2
0	0
–1	–2
–2	–4
–3	–6

Check that this fits with your intuitive idea of what is happening to the rate of change of the function, $f(x) = x^2$ as we move from $x = –3$ through $x = 0$ to $x = 3$. You can also check it on Graphmatica using the *Draw tangent* option if you want to.

Figure 6.7: Graph of $f(x) = x^2$



This process of finding a general formula for the instantaneous rate of change of a function is so important and useful that it has been given a name of its own and associated vocabulary.

The formula for determining the instantaneous rate of change of a function at any point is called the **derivative** of the function. The process of finding the derivative is called **differentiation**.

The processes of finding and using the derivative of a function are called **differential calculus**.

A bit of history… **For interest only**

Who invented calculus?

In the mid 1660's, about one thousand years after the Greeks first thought of the concepts, Isaac Newton (an Englishman) is said to have developed his method of fluxions, which today we call calculus. But Newton is described as a suspicious, neurotic and tortured personality and only distributed his discovery to a few colleagues. He did not publish it. Ten years later Gottfried Leibniz (a German) made virtually the same discoveries. Letters passed between Newton and Leibniz, but still Newton did not publish. Leibnitz was the first to publish on differential calculus in1684. He did not acknowledge Newton's work.

Sounds like something out of the Sunday papers doesn't it? But it doesn't end there. Charge and countercharge followed. A commission held by the Royal Society in 1713 supported Newton's claims. Not surprising perhaps as by this time Newton was the President of the Royal Society. But still accusations continued.

Newton is often thought to be the greatest genius of all time making discoveries in mathematics physics, astronomy and numerous other sciences. Leibniz was also extremely talented and made numerous discoveries in mathematics from his early years. Dunham (1994) has said that 'The mutual denunciations of two of the greatest mathematicians of all time make a sad chapter in European intellectual history. That individuals of such genius descended to petty and outrageous mudslinging does not bode well for those of us with more modest intellects'. Personally I think it makes them appear just more human.

In conclusion, you might be interested to know that today many people incorrectly associate only Newton's name with calculus, but it was actually Leibniz who first coined the word *calculus* and whose mathematical notation we use today.

(Source: Dunham, W 1993, *The mathematical universe*, Wiley, New York.)

There are a number of different notations for the derivative of a function, the most common are below.

The derivative of $y = f(x)$ is denoted by $f'(x)$ or $\frac{dy}{dx}$.

The two notations mean exactly the same thing and are pronounced

$f'(x)$ is 'f dash x'

$\frac{dy}{dx}$ is 'dee y dee x'

Summary to date:

- The instantaneous rate of change of function is determined from the gradient of a tangent at a point on the function which is calculated from,

  $$\text{Instantaneous rate of change} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

- The instantaneous rate of change of a function is termed the derivative, so that

  $$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Example

When an apple drops from a tree the distance fallen in metres is related to time (seconds) by the function $y = f(x)$, what are the practical interpretations of $f(1.5) = 11.025$ and $f'(1.5) = 14.7$.

$f(1.5) = 11.025$ means that after 1.5 seconds the apple has fallen 11.025 metres, while $f'(1.5) = 14.7$ means that at the instant 1.5 seconds after leaving the tree the apple is travelling at a speed of 14.7 m/s.

(Note when you evaluate a derivative it will have units just like other rates of change discussed earlier in the module.)

Example

If the function $y = x^3 + x^2 - 4x - 4$ has a derivative function, $\frac{dy}{dx} = 3x^2 + 2x - 4$, find the values of $\frac{dy}{dx}$ when $x = 2,\ x = -2,$ and $x = -1$. Sketch the graph of the original function using Graphmatica and interpret the values you obtained for $\frac{dy}{dx}$ in terms of the behaviour of the graph.

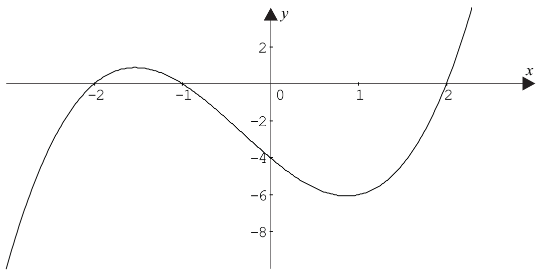
Using $\dfrac{dy}{dx} = 3x^2 + 2x - 4$ when,

$$x = 2, \quad \frac{dy}{dx} = 12$$

$$x = -2, \quad \frac{dy}{dx} = 4$$

$$x = -1, \quad \frac{dy}{dx} = -3$$

The graph of the original function $y = x^3 + x^2 - 4x - 4$ is:



At $x = 2$, the function is increasing and you would expect the gradient of the tangent at this point to be positive. It is changing at that instant at a rate of 12 units of y for each unit of x.

At $x = -2$, the function is increasing and you would expect the gradient of the tangent at this point to be positive, but not as large as at $x = 2$. It is changing at that instant at a rate of 4 units of y for each unit of x, slower than at $x = 2$.

At $x = -1$, the function is decreasing and you would expect the gradient of the tangent at this point to be negative. It is changing at that instant at a rate of –3 units of y for each unit of x.

Activity 6.2

1. When a cannon ball is fired from a cannon, its height above the ground is given by the function:

   $$h = f(t)$$

   where h is the height in metres and t the time in seconds since leaving the cannon.

   What are the practical interpretations of the following statements:

   (a) $f(1.5) = 20$

   (b) $f'(1) = 12$

2. A company finds that the cost of producing x widgets is given by some function:

   $$C = g(x)$$

   where C is the total cost in dollars of producing x widgets. What are the practical interpretations of the following statements?

   (a) $g(20) = 170$

   (b) $g'(20) = 2.70$

3. The function $y = \dfrac{x^4 - 2x^2}{4}$ has the derivative function $\dfrac{dy}{dx} = x^3 - x$.

   (a) Find the value of the derivative function when $x = 0$, $x = -1$, and $x = 1$.

   (b) What implications do these results have on the shape of the graph of $y = \dfrac{x^4 - 2x^2}{4}$?

   (c) Use Graphmatica to draw the function $y = \dfrac{x^4 - 2x^2}{4}$, what do you notice at the points $x = 0$, $x = -1$, and $x = 1$?

4. The population of a city is increasing according to the function $P = h(t)$, where P is the population t years after 1940. Interpret the meaning of the mathematical statement:

   $$h'(50) = 1200$$

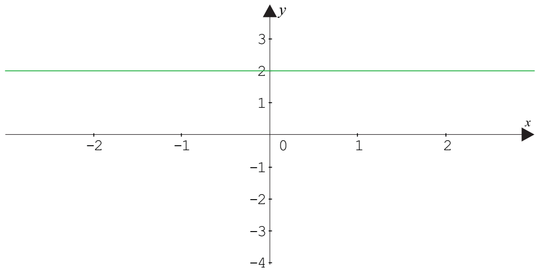
6.3 Shortcuts for differentiation

You will have noticed that so far we have not spent much time actually determining the derivatives for a range of functions using $f'(x) = \dfrac{dy}{dx} = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$. The reason for this is that we need extensive algebra skills and a lot of free time to calculate the derivative for many functions using this formula. For these reasons mathematicians have developed shortcut methods to help quickly calculate the derivatives of functions. Let's investigate some of these shortcut methods now for a range of functions.

6.3.1 Polynomial and other power functions

Let's think about some common polynomial functions and draw them on Graphmatica. For example we know that the constant function, $y = 2$ (say) is a straight line parallel to the horizontal axis. The gradient of this line is zero.

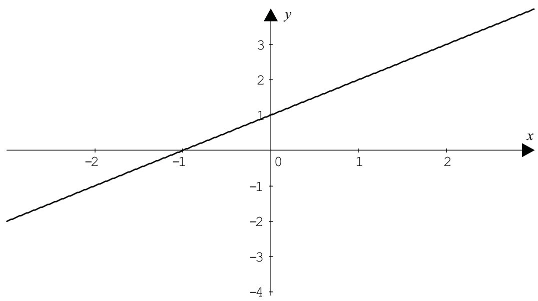
Figure 6.8: Straight line parallel to *x*-axis



$$y = 2$$
$$\frac{dy}{dx} = 0$$

The straight line function, $y = x + 1$, (say) will have a gradient of 1.

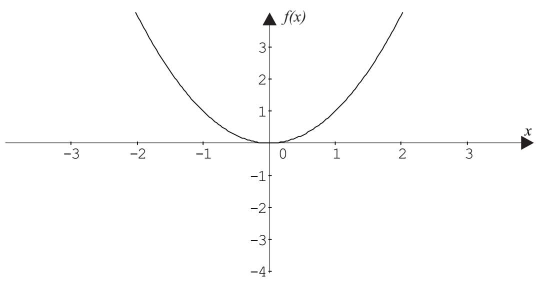
Figure 6.9: Straight line



$$y = x + 1$$
$$\frac{dy}{dx} = 1$$

From our previous work we know that the gradient function for $f(x) = x^2$ is $f'(x) = 2x$.

Figure 6.10: Quadratic function



$$f(x) = x^2$$
$$f'(x) = 2x$$

Let's use $f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to find the derivative of $f(x) = x^3$.

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ **Do not learn this, for explanation only**

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$
$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$
$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$
$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$
$$= \lim_{h \to 0} 3x^2 + 3xh + h^2$$
$$= 3x^2$$

We can now put what we know together, remembering the index rules e.g.

$$3 = 3 \times 1 = 3 \times x^0$$
$$3x = 3 \times x^1$$

Original function, $f(x)$	**Derivative function, $f'(x)$**
$1 = 1 \times x^0$	0
$x = x^1$	1
x^2	$2x$
x^3	$3x^2$

Try to finish the rest of the table

x^4	
x^5	
x^6	
x^7	
x^8	

Have you noticed the pattern?

$$\boxed{\text{If } f(x) = x^n \text{ then } f'(x) = nx^{n-1}}$$

This shortcut works whether the index is positive, negative or a fraction.

Example

If $y = x^{27}$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 27 \times x^{27-1} = 27x^{26}$$

Example

Find the derivative of $g(x)=\dfrac{1}{x^2}$.

First step is to write $g(x)=\dfrac{1}{x^2}$ in the form x^n.

$$g(x)=\frac{1}{x^2}=x^{-2}$$
$$g'(x)=-2\times x^{-2-1}$$
$$=-2x^{-3} \quad \text{or} \quad \frac{-2}{x^3}$$

Watch out this is tricky. Make sure you write the function as x to some power before you differentiate. If you have forgotten your index rules now is a good time to revise them.

Example

Find $f'(x)$, when $f(x)=\sqrt{x}$.

First step is to write $f(x)=\sqrt{x}$ in the form x^n.

$$f(x)=\sqrt{x}=x^{\frac{1}{2}}$$
$$f'(x)=\frac{1}{2}\times x^{\frac{1}{2}-1}$$
$$=\frac{1}{2}\times x^{-\frac{1}{2}} \quad \text{or} \quad \frac{1}{2\sqrt{x}}$$

Watch out this is tricky. Make sure you write the function as x to some power before you differentiate. If you have forgotten your index rules now is a good time to revise them.

Activity 6.3

1. Given the functions $f(x)$ below, find the derivatives, $f'(x)$.

   (a) $f(x)=x^{10}$

   (b) $f(x)=x^{-3}$

   (c) $f(x)=x$

   (d) $f(x)=2$

   (e) $f(x)=x^{\frac{2}{3}}$

   (f) $f(x)=\dfrac{1}{x}$

   (g) $f(x)=\dfrac{1}{x^3}$

   (h) $f(x)=\sqrt[4]{x^2}$

2. If the displacement (in metres) of an object after t seconds is given by $s = t^5$ find the values of its derivative, $\frac{ds}{dt}$, when $t = 3$ seconds.
3. Differentiate the function $p(x) = 2\pi$, that is find $\frac{dp(x)}{dx}$.
4. If $h(t) = \sqrt[3]{t}$, find the value of $h'(8)$, that is, find the value of its derivative at $t = 8$.
5. Angela was asked to differentiate the function $y = \frac{1}{x^4}$, her answer is shown below:

   $$\frac{dy}{dx} = \frac{1}{4x^3}$$

   Can you explain why this answer is incorrect?
6. Barry differentiated the function $y = 2e$ and obtained the answer $\frac{dy}{dx} = 2$. Why is this the incorrect answer?

These shortcuts are very useful, but polynomial functions are not always that simple, so we need to know how we can find the derivative of functions such as:

$$f(x) = 2x^7 + x$$

$$q(t) = \frac{1}{t} + 3t - 1$$

$$p = 7r^{-3} - \frac{r}{3} + 2r^3$$

Let's think firstly about functions such as $y = 2x$ or $y = 7x^2$ where we have the basic function multiplied by a constant.

Think about a family of straight lines and their associated gradients (or derivative functions). Draw these on Graphmatica if you want to see what is happening.

Line	Gradient	Derivative function
$y = x$	1	$\frac{dy}{dx} = 1$
$y = 2x$	2	$\frac{dy}{dx} = 2$
$y = 3x$	3	$\frac{dy}{dx} = 3$
$y = 4x$	4	$\frac{dy}{dx} = 4$

It appears that to find the derivative we just multiply the derivative of the basic function by a constant. So for the function $y = 25x$, the derivative will be 25 times the derivative of $y = x$. So $\frac{dy}{dx} = 25$.

Does the same thing occur with polynomials of higher powers?

Draw the functions $y = x^2$, $y = 2x^2$, and $y = 3x^2$ on Graphmatica. Then use the *Draw tangent* tool to draw a tangent at $x = 1$.

Describe what you notice about the gradient of the tangent to the three curves at $x = 1$.

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Did you notice that the gradient of the tangent increased for $y = x^2$ to $y = 3x^2$. In fact if you had been able to estimate the gradient you would have found that the gradient of the tangent at for $y = 3x^2$ was three times that of $y = x^2$ and that $y = 2x^2$ was twice that of $y = x^2$.

So it appears that the rule for finding the derivative of straight lines applies to polynomials to the power two. In fact, it applies to all functions, including polynomials, so that we have the general result.

$$\boxed{\text{If } y = a \times f(x) \quad \text{then} \quad \frac{dy}{dx} = a \times f'(x)}$$

Example

Find the derivative of $y = 2x^7$

$$\frac{dy}{dx} = 2 \times 7 \times x^{7-1} = 14x^6$$

Example

If $q(t) = \frac{3}{t^3}$, find $q'(t)$

The first step is to write $q(t) = \frac{3}{t^3}$ in the form t^n.

$$q(t) = \frac{3}{t^3} = 3 \times t^{-3}$$

$$q'(t) = 3 \times -3 \times t^{-3-1}$$

$$= -9t^{-4} \quad or \quad \frac{-9}{t^4}$$

Activity 6.4

1. Find the derivative of the following functions.

   (a) $3x^9$

   (b) $7x$

   (c) $-\frac{1}{2}x^{-2}$

   (d) $\frac{2}{\sqrt{x}}$

   (e) $-\frac{3}{x^3}$

2. If $y = 3x^9$ find $\frac{dy}{dx}$

3. If $a(r) = \pi r^2$ find its derivative $a'(r)$.

4. Differentiate the function $y = 2ex^3$.

5. If $V(r) = \frac{4}{3}\pi r^3$ find the value of $V'(2)$.

6. Jason was asked to differentiate the function $s = 2\pi t \times t^3$ and his answer was

   $$\begin{aligned}\frac{ds}{dt} &= 2\pi \times 3t^2 \\ &= 6\pi t^2\end{aligned}$$

   Why is this incorrect?

7. Find the derivative of the function $p(x) = 2ex\sqrt{x}$. Try and simplify it fully before you differentiate.

When we found the derivative of the function, $f(x) = 3x^2$, we found that it was just three times the derivative of $f(x) = x^2$. Now since $f(x) = 3x^2 = x^2 + x^2 + x^2$, this must mean that if we have a function made up of different functions added together, we must be able to add the derivatives of the individual functions. So

If $f(x) = 3x^2 = x^2 + x^2 + x^2, \quad f'(x) = 6x = 2x + 2x + 2x$

If $f(x) = x^3 + x^2 + 1$ then $f'(x) = 3x^2 + 2x + 0$.

This is exactly what happens and is true for polynomial and all other functions so that

$$\text{If } y = f(x) \pm g(x) \text{ then } \frac{dy}{dx} = f'(x) \pm g'(x)$$

Example

If $f(x) = 2x^7 + x$, then determine its derivative.

$$\begin{aligned} f'(x) &= 2 \times 7 \times x^{7-1} + 1 \times x^{1-1} \\ &= 14x^6 + 1 \end{aligned}$$

Example

If $q(t) = \dfrac{1}{t} + 3t - 1$ find $q'(t)$.

The first step is to write the original function is the correct form.

$$\begin{aligned} q(t) &= \frac{1}{t} + 3t - 1 = t^{-1} + 3t - 1 \\ q'(t) &= -1 \times t^{-1-1} + 3 - 0 \\ &= -t^{-2} + 3 \quad \text{or} \quad -\frac{1}{t^2} + 3 \end{aligned}$$

Example

Differentiate $p = 7r^{-3} - \dfrac{r}{3} + 2r^3$ with respect to *r*.

$$\begin{aligned} p &= 7r^{-3} - \frac{r}{3} + 2r^3 = 7r^{-3} - \frac{1}{3} \times r + 2r^3 \\ \frac{dp}{dr} &= 7 \times -3 \times r^{-3-1} - \frac{1}{3} \times 1 \times r^{1-1} + 2 \times 3 \times r^{3-1} \\ &= -21r^{-4} - \frac{1}{3} + 6r^2 \quad \text{or} \quad 6r^2 - \frac{21}{r^4} - \frac{1}{3} \quad . \end{aligned}$$

Activity 6.5

1. Find the derivative of the polynomial $y = 3x^4 - 2x^3 + 5x^2 - x - 1$.
2. If $h(t) = \dfrac{3}{t^2} + t - \pi$, find the value of $h'(2)$.
3. Find $\dfrac{ds}{dp}$ if $s = 3p^3 - 12p + 35.7$.
4. Differentiate $V = 2\pi r^2 + 8\pi r$.
5. Differentiate the function $s = 2t^2 + \sqrt{t} - 3e$.
6. Find the derivative of the function $P = 2t^2(3t - 1)$, remember to simplify it first.

7. Find the value of $v'(4)$ given that $v(t) = 2\sqrt{t} + 12t^2 + 1$.

8. Given that $T = \dfrac{3x^2}{x^5}$ find the derivative $\dfrac{dT}{dx}$.

9. Differentiate $y = \dfrac{x^3 + 2x^2 - 5x}{x}$.

10. If $H = 3t^{-1} + 2t^3 - \sqrt{t}$ find the value of $\dfrac{dH}{dt}$ when $t = 4$.

11. Find the derivative of $E = \dfrac{2\pi}{r} - r^3$.

12. Calculate the instantaneous rate of change of the function $P = 3h^4 - 2h + 8$ when $h = -3$.

13. Determine the gradient of the tangent to the function $y = 2x^2 + 6x - 5$ at $x = 4$.

14. For temperatures over 200°C the length of a certain metal bar begins to increase due to expansion. This length can be found from the formula:

    $$L = 200 + \sqrt[4]{T}$$

    where L is the length of the bar in mm and T the temperature in °C.

    (a) Find the length of the bar when the temperature has reached 800°C.

    (b) Calculate the value of the derivative at 800°C, what does this value tell us?

15. The height of a launched missile above the ground is given by the formula where t is the time in seconds after it is launched and $h(t) = 300t - 5t^2$ its height in metres.

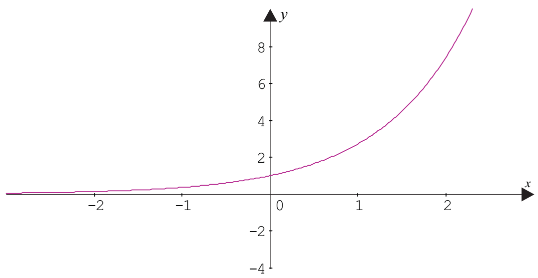
    (a) Calculate the value of $h(2)$. What does this value tell us about the missile?

    (b) Calculate the value of $h'(30)$. What does this value tell us about the missile?

6.3.2 Exponential functions

The exponential is an important function we have come across in modules 3 and 5. Let's revise what we know about its rate of change by examining the graph of $y = e^x$

Figure 6.11: $y = e^x$

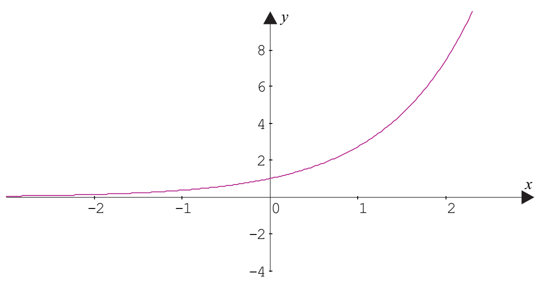


We know that it is an increasing function, so that the rate of change must always be positive. So the function that describes the derivative must be a function for which all of its values are positive. You can check that the gradient of the tangent to the function is always positive either by hand or by using the *Draw tangent* tool to draw some tangents to the curve.

But what is the actual function that describes the derivative? To help us understand what the function might be let's use $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$. The algebra is too hard for us to evaluate $f'(x) = \lim_{h \to 0} \dfrac{e^{x+h} - e^x}{h}$, but we can get a feel for what is happening by approximating $f'(x)$ when h is very small say, $h = 0.001$ and sketching this function on Graphmatica.

Use Graphmatica now to sketch the function $y = \dfrac{e^{x+0.001} - e^x}{0.001}$ (if you have trouble with this refer to the instructions in the introductory book for this unit). The function should look like the graph below.

Figure 6.12: $y = \dfrac{e^{x+0.001} - e^x}{0.001}$



What type of function best describes the function you have drawn?

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Yes, the function looks remarkably similar to the original exponential function. In fact, the rate of change of the function at any point is equal to the value of the function at that point. This is a specific property of all exponential functions and is unique in mathematics making the function important and powerful. We have already seen some of this power in module 3 (see examples on radioactive decay, rate of cooling or compound interest) but you will undoubtedly investigate it more in your further mathematical studies. So to summarize we can say that,

$$\boxed{\text{If } f(x) = e^x \text{ then } f'(x) = e^x}$$

Example

If $f(x) = x^2 + 2e^x$, find $f'(x)$.

$f'(x) = 2x + 2e^x$

Example

Two antique dealers are competing and one believes that the value of her stock is increasing more rapidly than her competitor's. To solve the argument, graphs of the value (dollars) of a particular antique over time (years) were drawn up. The functions representing these functions were

$$f(x) = 0.5e^x$$
$$g(x) = 0.8e^x$$

Compare the instantaneous rate of change at the end of one year for each dealer and comment on which function is increasing at the greater rate at that time.

To find the instantaneous rate of change for each, we must first find the derivative for each function and then determine its value at $x = 1$.

$f(x) = 0.5e^x$

$f'(x) = 0.5 \times e^x$

When $x = 1$, $f'(x) = 0.5 \times e^1 \approx 1.359$

The instantaneous rate of change is approximately \$1.36 per year at the first year.

$g(x) = 0.8e^x$

$g'(x) = 0.8 \times e^x$

When $x = 1$, $g'(x) = 0.8 \times e^1 \approx 2.1746$

The instantaneous rate of change is approximately \$2.17 per year.

At year 1, the second dealers' value of a particular antique is changing at nearly 1.5 times the rate of the first dealer. Because they are exponential functions the first dealer will never catch up.

Activity 6.6

1. Find the derivative of the following functions:

   (a) $y = 3e^x + x$

   (b) $v(t) = 3t^4 - 12e^t$

   (c) $P = \dfrac{e^{2t}}{e^t}$ (simplify the expression first)

2. Find the instantaneous rate of change of the function $P = 12h + 3e^h$ when $h = 2$.

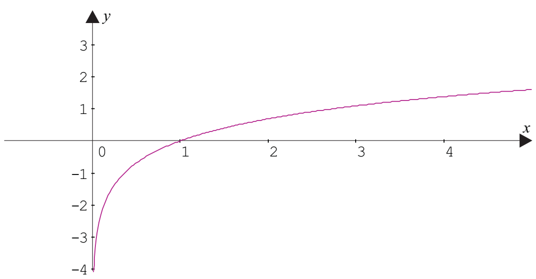
3. Use $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ and Graphmatica to find the equation of the derivative for the function $y = e^{-x}$.

4. Suppose a student was asked to find the derivative of the function $y = e^x$ and gave the answer $\dfrac{dy}{dx} = xe^{x-1}$. Can you explain clearly to the student why this is wrong?

6.3.3 Logarithmic functions

What do you recall about the logarithmic function and its rate of change? Sketch the function on Graphmatica.

Figure 6.13: $y = \ln x$



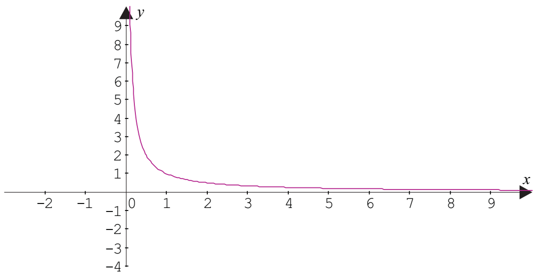
Now using the *Draw tangent* tool on Graphmatica find the gradient of the tangent at the following points and sketch a graph of these values by hand (put the values of the gradient of the tangent on the vertical axis).

***x* value**	**Gradient of tangent**
$x = 0.1$	
$x = 0.5$	
$x = 1$	
$x = 2$	
$x = 3$	
$x = 4$	

Describe in your own words how its rate of change is changing.

Do you know any functions that might behave this way? Using the same technique we used for the exponential function try sketching the function $y = \dfrac{\ln(x + 0.001) - \ln x}{0.001}$ to get an approximate derivative function.

Figure 6.14: $y = \dfrac{\ln(x + 0.001) - \ln x}{0.001}$



What equation could describe the function? If you said $y = \dfrac{1}{x},\ x > 0$ you would be correct.

This means that the derivative of $y = \ln x$ is $\dfrac{dy}{dx} = \dfrac{1}{x}$.

Example

Find the derivative of $p=\frac{1}{t}-\ln t$.

First stage is to write the original function in a form that is easy to differentiate.

$$\begin{aligned}
p&=\frac{1}{t}-\ln t=t^{-1}-\ln t\\
\frac{dp}{dt}&=-1\times t^{-1-1}-\frac{1}{t}\\
&=-t^{-2}-\frac{1}{t}\\
&=-\frac{1}{t^2}-\frac{1}{t}\quad or\quad \frac{-1-t}{t^2}
\end{aligned}$$

Example

The two functions $y=2\ln x$ and $y=\sqrt{x}$ behave very similarly. By differentiating each function, find the value of x where the instantaneous rates of change are equal. (You might like to sketch the two function on Graphmatica first so you can get an estimate of your answer before commencing)

If $y=2\ln x$ then $\frac{dy}{dx}=\frac{2}{x}$

If $y=\sqrt{x}=x^{\frac{1}{2}}$ then $\frac{dy}{dx}=\frac{1}{2}x^{\frac{1}{2}-1}=\frac{1}{2\sqrt{x}}$

To find out where the instantaneous rates of change are equal we must find the value of x which satisfies the equation,

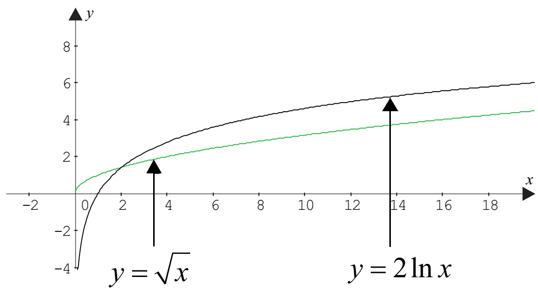
$$\begin{aligned}
\frac{2}{x}&=\frac{1}{2\sqrt{x}}\\
\frac{4\sqrt{x}}{x}&=1\\
4\sqrt{x}&=x\\
16x&=x^2\\
x^2-16x&=0\\
x(x-16)&=0\\
x&=0\quad\text{or}\quad x=16
\end{aligned}$$

Since $y=2\ln x$, is not defined for $x = 0$, $x = 16$ must be the solution.

This means that at $x = 16$ both functions must be changing at the same rate. To check draw the graphs on Graphmatica and substitute into each derivative.

$$x = 16, \quad \frac{dy}{dx} = \frac{2}{x} = \frac{2}{16} = \frac{1}{8}$$

$$x = 16, \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{16}} = \frac{1}{2 \times 4} = \frac{1}{8}$$



The graph confirms that the two rates of change are the same at $x = 16$.

Activity 6.7

1. Differentiate the following functions:

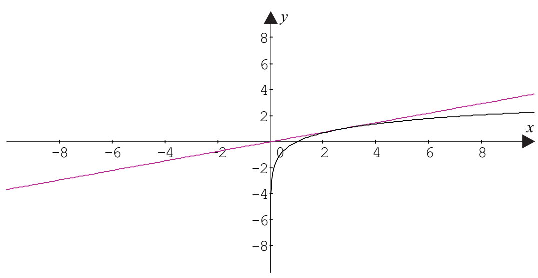
   (a) $y = \ln x - 3x^5$

   (b) $P(t) = 3e^t - 2\log_e t + 2\pi$

   (c) $T = 5x^2 + 3\ln x + 2e$

2. Find the gradient of the tangent to the function $y = 2\ln x$ when $x = 2$.

3. At what point on the curve $y = \ln x$ does the tangent at that point pass through the origin? (See the diagram below)



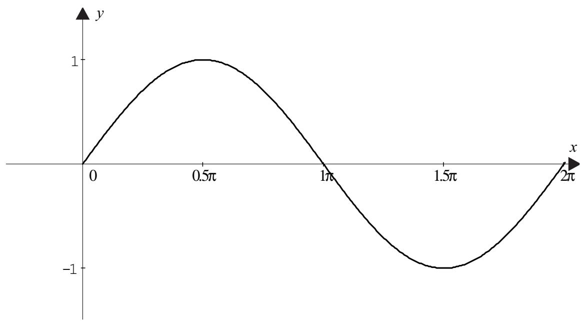
6.3.4 Trigonometric functions

The last functions we have to consider are the trigonometric functions,

$$y = f(x) = \sin x$$
$$y = g(x) = \cos x$$

Examine the sine function now after sketching it on Graphmatica.

Figure 6.15: $y = f(x) = \sin x$



Describe the behaviour of its rate of change in your own words.

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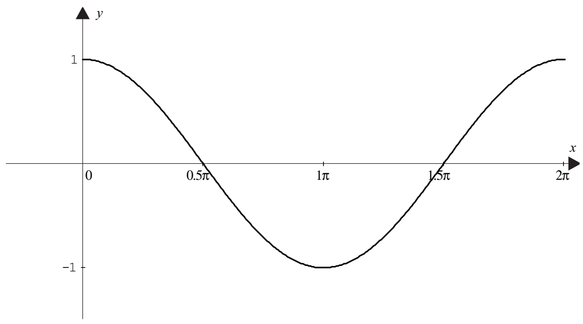
You might have noticed that it changed rapidly at first from $x = 0$ slowing gradually until it reached zero (i.e. no change) at $x = \frac{\pi}{2}$. After this, the function decreased so that its rate of change was negative but changed slowly at first then sped up to a maximum when $x = \pi$ only to change slowly again as it approached $x = \frac{3\pi}{2}$. What function do you know behaves this way? To get an idea of what this could be sketch an approximation of the gradient function for $y = \sin x$, by sketching $y = \frac{\sin(x + 0.001) - \sin x}{0.001}$ on Graphmatica.

Figure 6.16: $y = \dfrac{\sin(x+0.001)-\sin x}{0.001}$



If you thought that this function looked very similar to the cosine function you would be correct.

You could do the same exercise for cos *x*, so in general we could say that:

$$\text{If } f(x)=\sin x \text{ then } f'(x)=\cos x$$
$$\text{If } f(x)=\cos x \text{ then } f'(x)=-\sin x$$

Important note: Because of the way in which the derivative of the sine function is derived, any calculations involving the calculus of trigonometric function should be in **radians** rather than degrees.

Something to talk about…

We have given you the derivative function for $f(x)=\cos x$, think about why it is the shape it is. Share your reasoning with your fellow students on the discussion group.

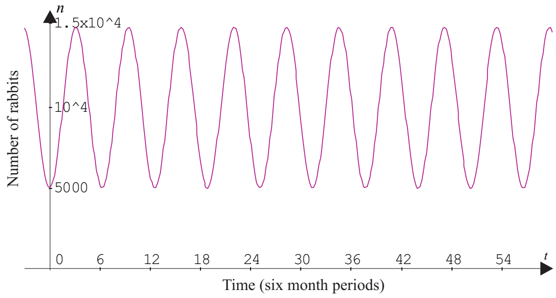
Example

Differentiate the function $y = 2\sin x - 3\cos x + 2x - 1$

$$\begin{aligned}\frac{dy}{dx} &= 2\cos x - 3\times -\sin x + 2 - 0 \\ &= 2\cos x + 3\sin x + 2\end{aligned}$$

Example

The figure below represents the changes in a rabbit population over 30 years. Calculation of the total number of rabbits was performed every 6 months, so 6 represents 3 years or 6 lots of 6 month periods. The graph is approximated by the function $n = 10\,000 - 5\,000\cos t$. (Note when evaluating the function and its derivative *t* should be calculated using radian measure.)



What is the rate of increase of the rabbit population at the end of the first 3 months?

If the function is $n = 10\,000 - 5\,000\cos t$ then the derivative function will be

$$\frac{dn}{dt} = 0 - 5\,000 \times -\sin t = 5\,000\ \sin t$$

at the end of 3 months, t = 0.5 ($\frac{1}{2}$ of 6 months).

When $t = 0.5$, then $\frac{dn}{dt} = 5000 \times \sin(0.5) \approx 2397.13$ (calculated in radians).

This means that at the point at the end of the first 3 months the rabbit population is increasing at approximately 2397 rabbits per 6 month period.

Activity 6.8

1. Differentiate the following functions:

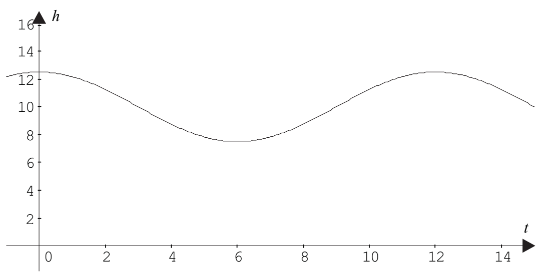
   (a) $y = 2\cos x + 3$

   (b) $y = \sin x - \cos x + 2e^x$

   (c) $y = x^2 - 12\sin x + \cos x$

2. Find the instantaneous rate of change of the function $y = 3\cos x$ when $x = 1.1$.

3. The height of water against a beachside peer was measured commencing at 12 noon. A graph is shown below:



The equation of the curve was found to closely resemble the function $h = 2.5\cos(\frac{\pi}{6}t) + 10$, where h is the height in metres of the water and t the time after 12 noon.

(a) On the graph, draw the tangent to the curve at 4 pm. Estimate the gradient of this tangent. What does this figure mean?

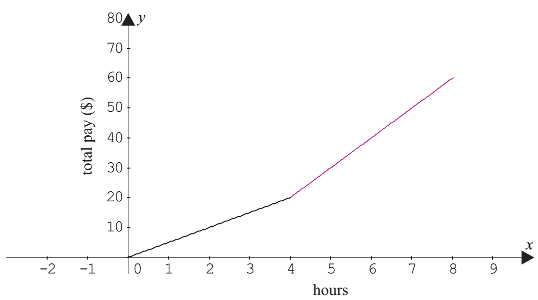
(b) At this level of calculus, you haven't learnt how to differentiate more complicated functions, but using your knowledge of calculus and your answer to (a), what do you think the $\frac{dh}{dt}$ will be?

(c) When will the rate of change of tide height be zero? How long will it stay at zero?

6.3.5 Where can't you find a derivative of a function?

In all the functions we have looked at so far we have been able to easily find the derivative of the original function at any point and we have also been able to confirm that it was correct by graphing the function and interpreting what is happening to the rate of change. But there are some functions in which this is not that easy. Consider this pay scale function. For the first 4 hours of work you are paid at $5 per hour and the next 4 hours you are paid at $10 per hour. What will be the rate of pay at exactly at the 4th hour. Draw a graph to represent the function.

Figure 6.17: Pay rate function



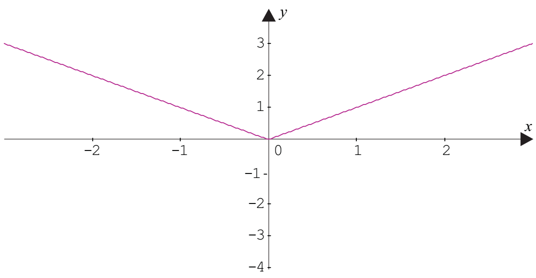
The graph has a corner at $x = 4$, and it is hard to place a single tangent at this point and so determine the rate of change. The graph actually consists of two line segments which meet at $x = 4$ and no matter how much we zoom in we cannot get it to look like a straight line. If we can't do this then we cannot find a tangent and its slope. We say that graphs that have corners are **not differentiable**.

Another example of a graph with a corner is the absolute value function. You will not have seen this function before, so let's discuss it before we graph it. Refer to module 2 to get a definition of absolute value. The absolute value function is $y = |x|$. Recall that absolute value means that the value of the function will always be positive. We could write the function in two parts:

$$y = x, \quad \text{for} \quad x \geq 0$$
$$y = -x, \quad \text{for} \quad x < 0$$

The function looks like the figure below.

Figure 6.18: $y = |x|$

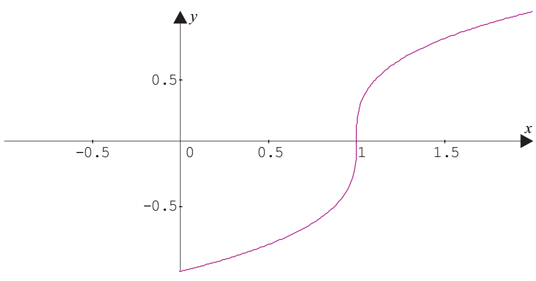


What is happening to the rate of change around $x = 0$? To the left of $x = 0$ it is negative, while to right it is positive. It is difficult to know where to put the tangent (there are an infinite number of alternatives to find the rate of change). If you zoom in at the point $x = 0$, the function never starts to looks like a straight line, it will always appear as a corner. This is another case where there is a sharp corner and we say that the function is not differentiable at $x = 0$ i.e. we cannot find the derivative at this point.

Other functions that pose difficulties are those with discontinuities. For example $y = \frac{1}{x}$ does not have a derivative at $x = 0$ as there is an asymptote at this point and the function does not exist there. There is a discontinuity at $x = 0$. If you have forgotten about the shape of these types of functions look back at module 3.

The final type of function for which we cannot find a derivative is the function which has a vertical tangent. Think about what the gradient of vertical line would be. Yes it actually has infinite value. So, a function with vertical tangent at a point will not be able to be differentiated. The function $y = \sqrt[3]{(x-1)}$ is an example of such a function. It is not differentiable at $x = 1$.

Figure 6.19: $y = \sqrt[3]{(x-1)}$



Note: If you do more calculus you will find that $\frac{dy}{dx} = \frac{y}{3\sqrt[3]{\left(x-1^2\right)}}$ which doesn't exist at $x = 1$ since division by zero is not possible.

In summary the derivative is not defined at:

- a sharp corner;
- a point of discontinuity; or
- where the tangent line is vertical.

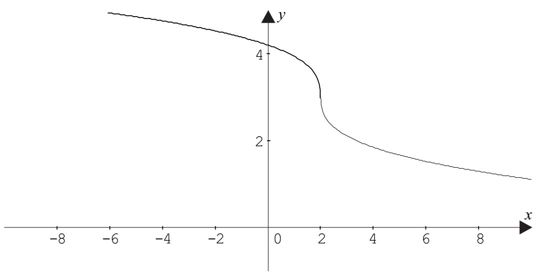
Now that you have completed all the shortcuts for a range of functions, complete this table to summarize the rules for differentiation.

$f(x)$	$f'(x)$
x^n	
$\sin x$	
$\cos x$	
e^x	
$\ln x$	
$a \times f(x)$	
$f(x) \pm g(x)$	

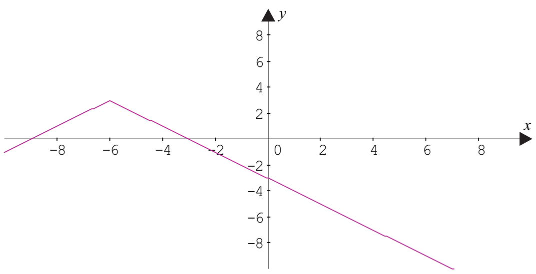
Activity 6.9

1. Examine the graphs shown below and determine whether the function shown is differentiable. In each case state why you think it is or is not differentiable.

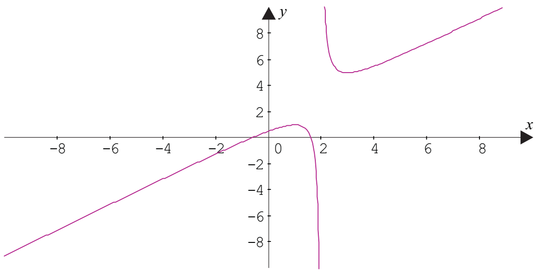
(a)



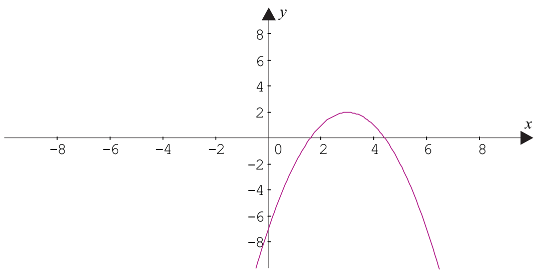
(b)



(c)



(d)



2. Find the derivative of the following functions:

   (a) $y = 3\ln x - 2\cos x + 2x^3$

   (b) $P = 3\sqrt{t} + 2\sin t$

   (c) $h(t) = 12e^t - \dfrac{1}{t^2}$

   (d) $y = 2x^5 - 4\sin x + 3\ln x$

   (e) $P = 12\cos a - 3\sqrt{a}$

6.4 Some applications of differential calculus

In each century of its existence, mathematicians and others have demonstrated the power of differential calculus to inform studies in the sciences, engineering, surveying, economics, business and psychology. The rate measurements described at the beginning of this module all can be derived theoretically from derivative functions.

One of the strengths of differential calculus lies in the ability to reduce complicated questions into simple procedures and rules. Yet, we must be careful that we do not treat differential calculus as just a set of rules. It is very important that as students of science based disciplines you have a good understanding of how best to apply these rules to real world situations. In the next section we will extend your understanding of the calculus concepts through a series of applied case studies.

6.4.1 Displacement-velocity-acceleration: when derivatives are meaningful in their own right

In our society we all drive or ride in cars. In the instance described below we might be driving such a car along a straight road. The displacement in metres from an original departure point is given by the equation, $s = t^2$, where s is in metres and t is in seconds.

If we differentiate this equation with respect to t, we would get $\frac{ds}{dt} = 2t$. This is the rate of change of displacement with respect to time and we know that it is termed **velocity**. Recall that we use this term because velocity is a measured quantity that has direction as well as magnitude (a vector), compared with speed which only has magnitude (a scalar). In the example the velocity might be 2 m/s at 1 second of travelling. It is positive because we are travelling away from the starting point in a positive direction and it is 2 because at the first second we a travelling the equivalent of 2 metres per second. See module 5 for more discussion on quantities with direction and magnitude (vectors).

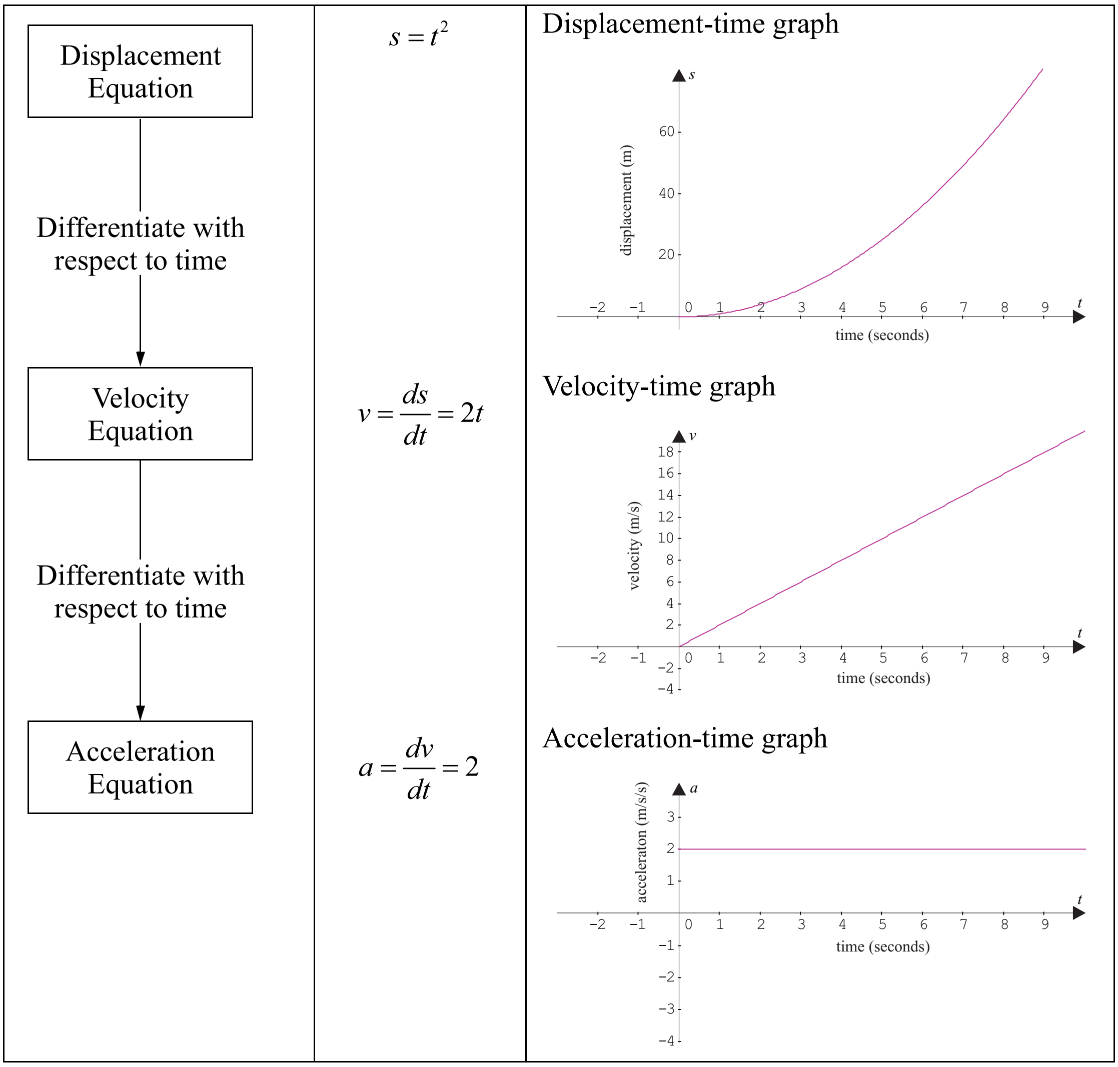
So this means that $v = \frac{ds}{dt} = 2t$, where velocity is measured in metres per second.

If we now differentiate v with respect to t, we would get $\frac{dv}{dt} = 2$. The rate of change of velocity is called **acceleration**, and it also is a quantity which has direction and magnitude (a vector). In this case the acceleration is positive because the car is moving in the positive direction and 2, because the velocity is changing at a rate of 2 metres per second per second. This means that if we started with the velocity of 2 m/s after 1 second we would be travelling at 4 m/s. We would now have the equation $a = \frac{dv}{dt} = 2$, where acceleration is measured in metres per second per second.

(**For your interest only** we could also differentiate the displacement equation twice to get the acceleration equation producing a second derivative of s with respect to t, i.e. $a = \frac{d^2s}{dt^2} = 2$.)

Let's put all this together in a graphical way and make some comparisons between the three equations and their graphs.

Figure 6.20: Motion graphs



From the displacement-time equation we can conclude:

- the car is moving away from the departure point in a positive direction
- the rate of change of displacement with time (velocity) is positive hence the car is moving away at an **increasing** rate in a positive direction.

From the velocity-time equation we can conclude:

- the velocity is positive so the car is moving away from the departure point in a positive direction
- the rate of change of velocity with time (acceleration) is positive and going away from the departure point, hence the car is accelerating.

From the acceleration-time equation we can conclude:

- the acceleration and velocity are positive so the car is moving away from its departure point in a positive direction
- the car's acceleration is not changing over time, it is a constant 2 m/s/s
- the car's acceleration is positive so the car must be increasing its velocity.

Notice also that as we differentiate we reduce the degree of the polynomial by 1 so that we have moved from a parabola to a straight line to a constant function.

It is easy to develop acceleration from displacement and in module 7 we will investigate how to do the reverse process and go from the acceleration equation to the displacement equation.

For interest only

Newton's Laws of motion are very common in many physics courses. Here is how they are inter-linked.

Consider a moving object for which we have the relationship between displacement, *s*, and *t* in SI units (i.e. in metres and seconds). Then

$s = ut + \frac{1}{2}at^2$, where u and a are constants.

If we differentiate the equation with respect to *t*, we get

$$\frac{ds}{dt} = v = u + at$$

Note that if we put $t = 0$ in the equation $v = u + at$, $v = u$, so u must be the initial velocity.

If we differentiate velocity with respect to *t* we get

$$\frac{dv}{dt} = a$$

So a must be the acceleration of the object.

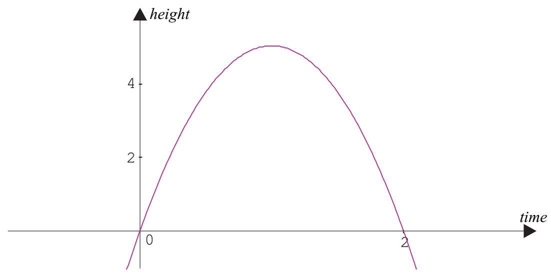
Do not learn Newton's Laws only notice the application

Something to talk about…

The relationships between displacement, velocity and acceleration are not always clear. Why not talk about your confusions on the discussion group. Talking through a problem is often a good way to clarify the concepts. Look in the introductory book for a some internet sites where animations of the above relationship are available.

Activity 6.10

1. The graph shows the displacement of an object as it is thrown into the air.



   (a) When will the velocity of the object be zero?

   (b) How far from its starting point will the object be when it has fallen to the ground?

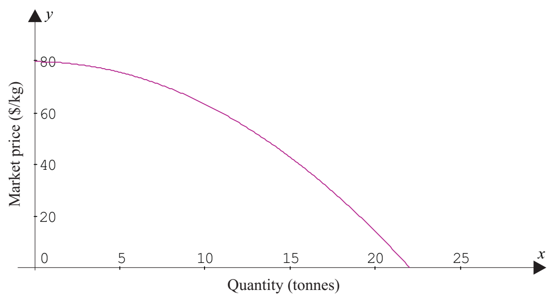
2. If the equation of the above graph was $s = 10t - 5t^2$, determine the following:

   (a) The velocity of the object after 0.8 seconds.

   (b) The acceleration of the object at its highest point.

3. The following graph shows the demand curve for potatoes supplied to the Toowoomba market.

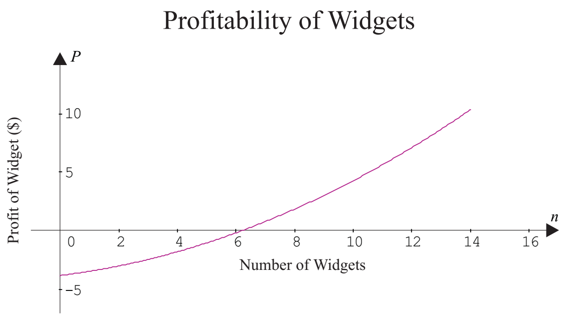
Toowoomba Potato Demand Curve



Notice that as the quantity flooding the market increases, the price available to the farmers decreases to such an extent that they have to give them away (that's in theory anyway).

If the equation of the above demand curve is $P = -0.165q^2 + 80$ (where P is the price and q the quantity produced) find the equation of its derivative and the value of this at $q = 5$. What does this answer represent in the above situation?

4. A company manufactures Widgets. The profit they make on each Widget depends on the number they are able to sell. The graph of profit against number sold is shown below:



If the equation of the above curve is $P = 0.05n^2 + 0.3n - 3.8$ (where P is the profit (in $) for each Widget and n the number of Widgets produced), find the equation of its derivative. This new function is sometimes called the marginal profit, and gives the increase in profit if one extra unit (in this case Widget) is produced. Determine the marginal profit when the company is producing 10 Widgets.

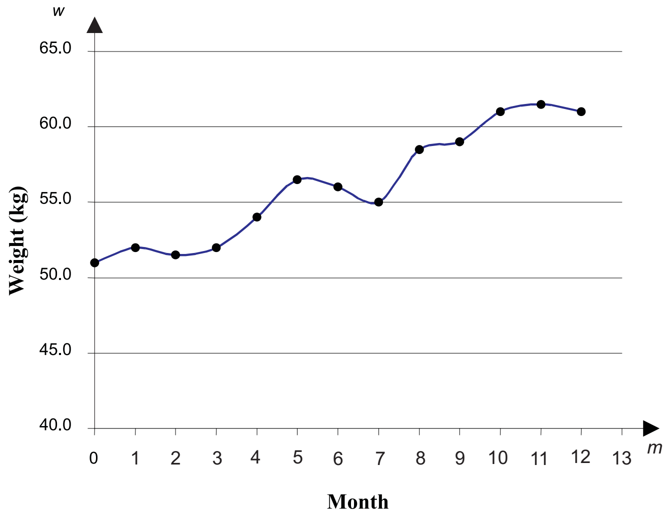
6.4.2 Twists and turns

So far we have investigated and compared a number of graphs as they have twisted and turned through a relationship between two variables. We have compared rates of change and approximated where a major event such as a turning point might have occurred. For example recall the teenage weight change curve at the beginning of this module. This function had a number of highs and lows. In this example we previously described the behaviour of the function in words, approximating that it:

- started at 51 kg and finished at 61 kg
- had peaks in weight at 1, 5 and 11 months
- had troughs in weight at 2 and 7 months
- a stable episode at around 8.5 months.

The lowest point occurred at $m = 0$, when weight was approximately 51 kg, while the highest point occurred when $m = 11$ at approximately 61.5 kg. But between these values we noticed that there are a number of other highs and lows. The overall maximum and minimum values are called **global maxima** or **minima**, while the other highs and lows occur at turning points and are called **local maxima** or **minima**.

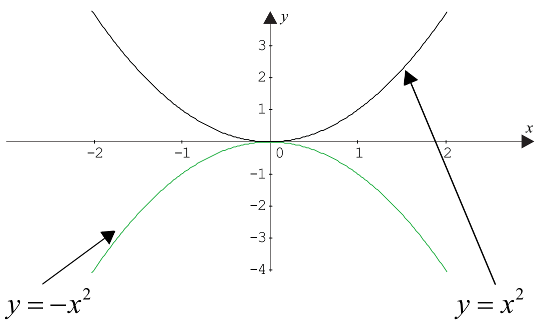
Figure 6.21: Teenage weight change curve



The global maximum and minimum values are often easy to determine, but how could we find the local turning points exactly, and so precisely describe the behaviour of this function. We will come back to the weight change function later.

Let's now look at two functions we are more familiar with: $y = x^2$ and $y = -x^2$.

Figure 6.22: $y = x^2$ and $y = -x^2$



From our previous knowledge of parabolas, we know that $y = x^2$ has a minimum value at $x = 0$, while $y = -x^2$ has a maximum at $x = 0$.

If we consider $y = x^2$, describe in your own words what is happening to the gradient of the tangent on the left of the minimum, what happens on the right and what happens at the minimum? If you find this difficult quickly sketch a picture of the function and roughly draw in some tangent lines on either side of $x = 0$.

We know that we can find the gradient of the tangent at different values of x by substituting into the derivative of the function, which is $\frac{dy}{dx} = 2x$. Let's do it.

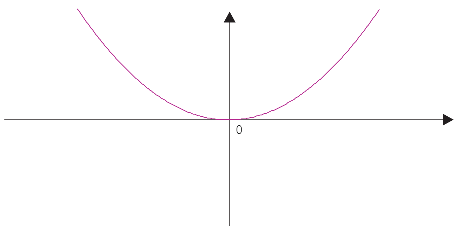
	$x = -1$	$x = -0.5$	$x = 0$	$x = 0.5$	$x = 1$
$\frac{dy}{dx} = 2x$	-2	-1	0	1	2
Direction of tangent					

We notice that at the minimum value, the gradient of the tangent is zero as expected. Is this also true for the gradient of the tangent at $x = 0$, for $y = -x^2$?

These results are common to all maximum and minimum points of all curves, not just polynomial functions. In summary we would say this:

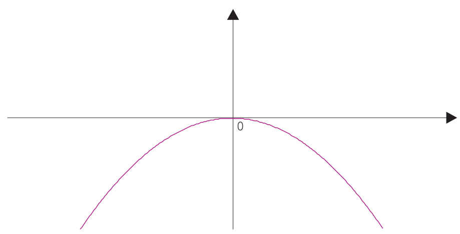
Figure 6.23: Turning points

Minimum turning point



For minimum points the gradient of the tangent is zero, changing from negative to positive as we moved from left to right of the minimum.

Maximum turning point



For maximum points the gradient of the tangent is zero, changing from positive to negative as we moved from left to right of the maximum.

Check this for yourself using Graphmatica.

Example

Consider the function $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 6x$, confirm that there are a local maximum and a minimum turning point at $x = -2$ and $x = 3$ respectively.

The gradient of the tangent is zero at maximum and minimum turning points. To confirm this, first determine the derivative of the function.

$$f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 6x$$

$$\begin{aligned} f'(x) &= \frac{1}{3} \times 3 \times x^2 - \frac{1}{2} \times 2 \times x - 6 \\ &= x^2 - x - 6 \end{aligned}$$

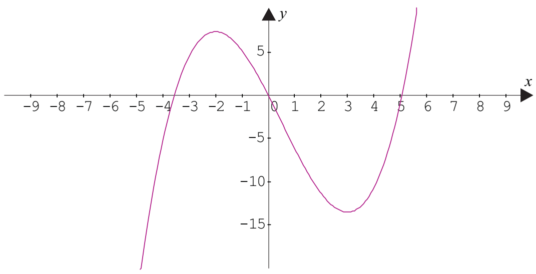
Substituting the value into the derivative function we find that $f'(3)=0$ and $f'(-2)=0$.

To decide which of the points is a minimum and which is a maximum, we need to assess the direction of the gradient of the tangent either side of the turning points.

	$x=-3$	$x=-2$	$x=0$	$x=3$	$x=4$
$\frac{dy}{dx}=x^2-x-6$	6	0	–6	0	6
Direction of tangent	/	—	\	—	/

So we can confirm the $x=-2$ is the maximum turning point and $x=3$ is the minimum turning point.

(Note that you could also reassure yourself that this is correct by examining the behaviour of the graph using strategies from module 3). Using Graphmatica, the function is represented by the graph below.



Example

Does the function $f(x)=x^3-3x^2-24x+5$ have any turning points? If so, find the position of all local maximum and minimum turning points. What is the maximum value of the function in the domain $-3 \le x \le 2$.

(This is an important example as it helps you find where turning points will occur.)

The power of the polynomial indicates that the function should have at most two turning points.

The turning points will have gradients of the tangent equal to zero, so $f'(x)=0$. We can solve this equation to find which points produce a tangent gradient of zero.

$$f(x) = x^3 - 3x^2 - 24x + 5$$
$$f'(x) = 3x^2 - 6x - 24$$

If $f'(x) = 0$ then $3x^2 - 6x - 24 = 0$

$$3x^2 - 6x - 24 = 0$$
$$3(x^2 - 2x - 8) = 0$$
$$x^2 - 2x - 8 = 0$$
$$(x-4)(x+2) = 0$$
$$x = 4, \quad x = -2$$

So the function will have turning points at $x = 4$ and $x = –2$. To determine whether they are maximum or minimum turning points consider the gradient of the tangents either side of these points.

	$x = -3$	$x = -2$	$x = 0$	$x = 4$	$x = 5$
$\frac{dy}{dx} = 3(x^2 - 2x - 8)$	21	0	–24	0	21
Direction of tangent	/	—	\	—	/

Hence, the maximum turning point is at $x = –2$ and the minimum is at $x = 4$. To determine the actual positions of these points evaluate the value of the function at each point.

$$f(4) = x^3 - 3x^2 - 24x + 5 = 4^3 - 3 \times 4^2 - 24 \times 4 + 5 = -75$$

$$f(-2) = x^3 - 3x^2 - 24x + 5 = (-2)^3 - 3 \times (-2)^2 - 24 \times (-2) + 5 = 33$$

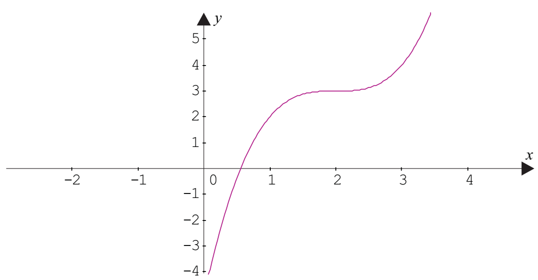
The local maximum is (–2, 33). The local minimum is (4, –75).

The maximum value of the function in the domain $-3 \le x \le 2$ will be 33.

However, we know from our work in module 3 that even though quadratic equations will always have a maximum or a minimum, some cubic functions do not have maxima or minima but still twist. Let's now examine one such function to ascertain what other types of twists/ turns exist.

Consider the function $y = x^3 - 6x^2 + 12x - 5$. Sketch graph of the function on Graphmatica.

Figure 6.24: $y = x^3 - 6x^2 + 12x - 5$



There is not a maximum or minimum turning point on the graph, but the function does appear to have a twist at approximately $x = 2$. Use the *Draw tangent* tool on Graphmatica to see what is happening to the gradient of the tangent at around $x = 2$.

Let's now see what is happening to the gradient of the tangent at $x = 2$ by calculating the value of the derivative at this point.

$$y = x^3 - 6x^2 + 12x - 5$$

$$\frac{dy}{dx} = 3x^2 - 12x + 12$$

When $x = 2$, $\dfrac{dy}{dx} = 3x^2 - 12x + 12 = 3\times 2^2 - 12\times 2 + 12 = 0$

Note, we could have found the value where the function turns by solving the equation

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 - 12x + 12 = 0 \\ 3(x^2 - 4x + 4) &= 0 \\ 3(x-2)^2 &= 0 \\ (x-2)^2 &= 0 \\ x &= 2 \end{aligned}$$

Let's investigate the behaviour of the tangent around $x = 2$.

	$x = 0$	$x = 1$	$x = 2$	$x = 3$	$x = 4$
$\frac{dy}{dx} = 3x^2 - 12x + 12$	12	3	0	3	12
Direction of tangent					

This type of point is not termed a turning point because the gradient of the tangent does not change from positive to negative or vice versa. This type of point is called a **horizontal point of inflection**.

Maximum and minimum turning points are grouped with the horizontal points of inflection to be called **stationary points**. They are called this because all have derivative functions of zero at these points, hence causing the rate of change of the function to be stationary at that point.

In summary,

- Places where the gradient of the tangent is zero, $f'(x) = 0$, are called stationary points.
- There are three types of stationary points: maximum and minimum turning points and horizontal points of inflection.

Before we do an example, let's think about how we approached analyzing the behaviour of a function in module 3. At the end of that module we have developed a series of questions to help with these investigations. Now that we know about instantaneous rates of change and stationary points we can add some more detail to the questions. Here are the revised questions:

- What is the shape of the graph?
- Where will it cut the vertical axis?
- Where will it cut the horizontal axis?
- Is it a continuous function?
- What happens when the variables get very large or small?
- What is the rate of change of the dependent variable with respect to the independent variable?
- Will the function have any stationary points? What are their characteristics?
- Will it have an inverse function?

Example

Examine the behaviour of the function, $f(x) = x^4 - 2x^3$, include in your answer the exact values of the vertical and horizontal intercepts, stationary points, any asymptotes and a hand drawn sketch of the graph.

The function is a polynomial of degree 4. We would expect that as $x \to \pm\infty$, the function would approach $+\infty$, because the degree is even.

It will have a vertical intercept when $x = 0$ at $y = 0$

It will have horizontal intercepts when $y = 0$,

$$\begin{aligned} 0 &= x^4 - 2x^3 \\ x^3(x-2) &= 0 \\ x &= 0, x = 2 \end{aligned}$$

Polynomials are continuous function with no breaks, corners or asymptotes so we would anticipate that it is differentiable throughout the domain of all real values of x.

The function is of degree 4 so we would expect it to have at most 3 stationary points.

To find the location of stationary points of a function we have to determine where $f'(x) = 0$

$$f(x) = x^4 - 2x^3$$
$$f'(x) = 4x^3 - 6x^2$$

If $\quad f'(x) = 0, \quad$ then $\quad 4x^3 - 6x^2 = 0$

$$\begin{aligned} 4x^3 - 6x^2 &= 0 \\ 2x^2(2x-3) &= 0 \\ x &= 0, x = \frac{3}{2} \end{aligned}$$

To find the coordinates of these two points evaluate $f(0)$ and $f(\frac{3}{2})$

$$f(0) = 0 \text{ and } f(\frac{3}{2}) = (\frac{3}{2})^4 - 2\times(\frac{3}{2})^3 = -\frac{27}{16}$$

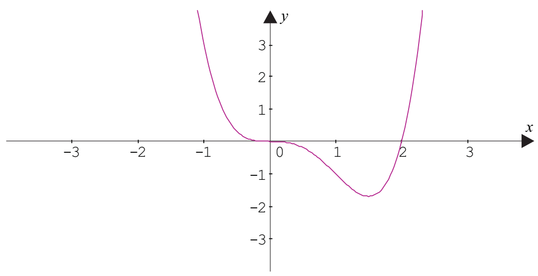
So the stationary points are (0, 0) and $(\frac{3}{2}, -\frac{27}{16})$.

To determine what type of stationary points they are we have to consider the behaviour of the gradient of the tangent in their vicinity.

	$x=-1$	$x=0$	$x=1$	$x=\frac{3}{2}$	$x=2$
$f'(x)=4x^3-6x^2$	-10	0	-2	0	8
Direction of tangent					

So there is a horizontal point of inflection at (0, 0) and a minimum turning point at $(\frac{3}{2}, -\frac{27}{16})$.

If we put all this information together we will get a graph as below.



Example

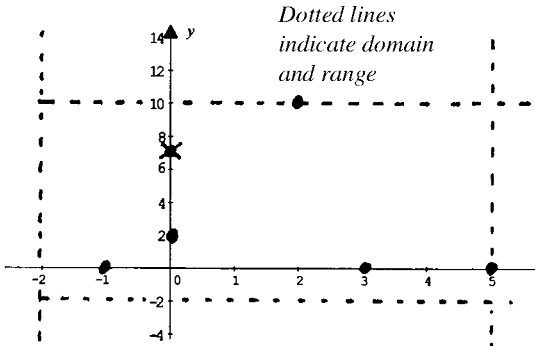
Given the following set of conditions, sketch the graph of the function $y = f(x)$ by hand.

- The domain is $-2 \le x \le 5$ and the range is $-2 \le f(x) \le 10$
- The function differentiable at all points except at $x = -1$ and $x = 0$
- $f(0) = 2$
- When $x = 2$, the function has a maximum value of 10
- When $-2 \le x < -1$, $f'(x) = -2$ and when $-1 < x < 0$, $f'(x) = 2$
- When $x = 4$ the derivative is zero
- The function is continuous throughout its domain
- Intercepts on the x-axis occur at -1, 3 and 5

This type of question is like doing a jigsaw puzzle. The trick is to do the points on the graph first,

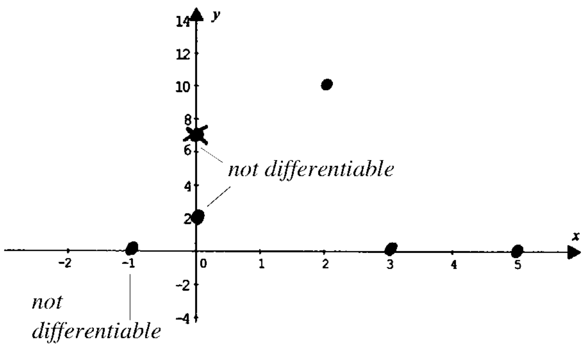
Do the points on the graph first,

- The domain is $-2 \le x \le 5$ and the range is $-2 \le f(x) \le 10$
- $f(0) = 2$
- When $x = 2$, the function has a maximum value of 10
- Intercepts on the x-axis occur at -1, 3 and 5



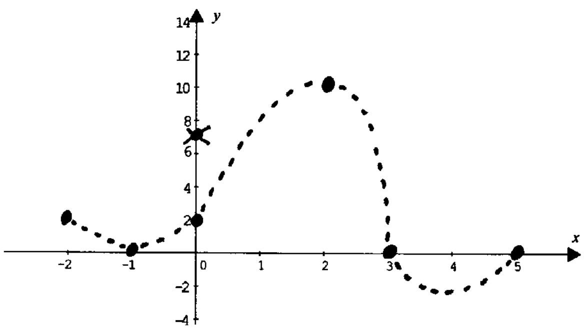
Check where the function is differentiable

- The function differentiable at all points except at $x = -1$ and $x = 0$



Interpret all available derivatives

- When $-2 \le x < -1, f'(x) = -2$ and when $-1 < x < 0, f'(x) = 2$
- When $x = 4$ the derivative is zero



Note this is one possible solution.

Activity 6.11

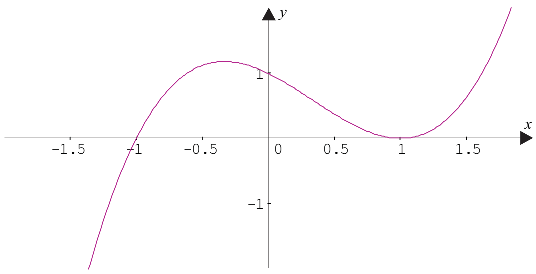
1. Consider the function $y = x^2 + x - 2$ over the domain $-1 \le x \le 4$.

   (a) Use the derivative to find the turning point in this domain.

   (b) What is the global maximum for this function in this domain?

   (c) What is the global minimum for this function in this domain?

2. The graph of the function $y = f(x)$ is shown below:



   From the graph determine the answers to the following questions:

   (a) For what values of x is $f(x) = 0$?

   (b) Find the value of $f(0)$.

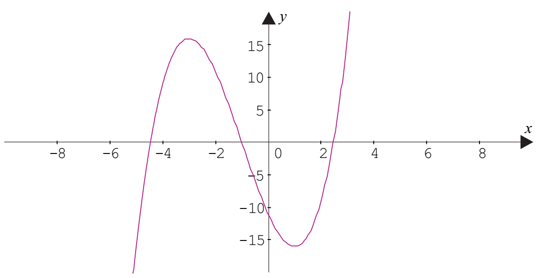
   (c) For what values of x is $f'(x) = 0$?

   (d) Estimate the value of $f'(0.5)$.

3. Find the location of the maximum and minimum turning points for the function $y = 2\sin x$ in the domain $0 \le x \le 2\pi$.

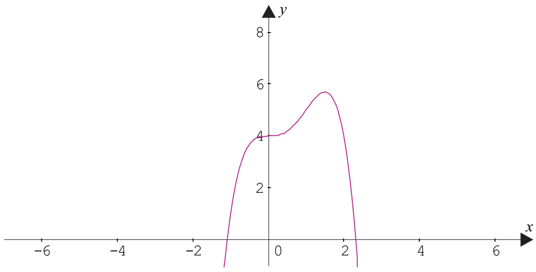
4. Determine the turning points for the function $y = 2x^3 - 6x^2 - 18x + 1$, find the nature of these turning points (that is whether they are maximum or minimum) and then draw a sketch of the curve.

5. The graph of the function $y = x^3 + 3x^2 - 9x - 11$ is shown below:



   Describe the behaviour of the derivative in the domain $-4 \le x \le 4$, that is where the derivative is negative, zero and positive.

6. The graph of the function $y = 4 + 2x^3 - x^4$ is shown below. Close inspection of this graph shows that the function has a maximum turning point at $x = 1.5$ and a horizontal point of inflection at $x = 0$. Use algebraic methods to confirm that these are the stationary points.



7. Draw a sketch of the function $y = f(x)$ given the following clues:

   (a) $f(x) = 0$ when $x = 0$ and $x = 4$.

   (b) $f(0) = 0$

   (c) $f'(x) = 0$ when $x = 0$ and $x = 3$.

   (d) $f'(x) \geq 0$ in the domain $-\infty < x < 3$.

   (e) $f'(x) < 0$ in the domain $3 < x < \infty$.

6.4.3 Optimization

Believe it or not the ability to determine maximum and minimum values of a function lies at the heart of many decisions that are made daily in our world.

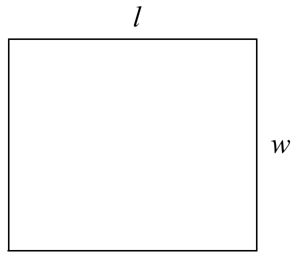
- How can we maximise profit?
- How can we minimise petrol consumption?
- How can we build this with the minimum amount of materials?
- What is the optimum time for running this experiment?

Let's now link the techniques of determining stationary points with some real world decision making situations. We'll do this through a number of examples.

Example

A farmer has a roll of fencing wire 300 m long and wants to know how long and wide to make a rectangular paddock so that the area is maximised.

Step 1: Draw an image of the problem.



Step 2: Name the variables involved. In this case call the length l, width w and the area, A.

Step 3: Identify which variable will be the dependent variable and which the independent. In this case we want to maximise area, so this must be the dependent variable and length and width would be the independent variables.

Step 4: Use information in the question to find a relationship between the two independent variables, length and width. In this case the perimeter must be 300 i.e.

$$\begin{aligned} 2(l+w) &= 300 \\ l+w &= 150 \\ l &= 150-w \end{aligned}$$

Step 5: Write a formula for the function that we have to maximise and try to write it with only one independent variable. In this case, the function is area, let's call it A.

$$\begin{aligned} A &= l \times w \\ A &= (150-w)\times w \\ A &= 150w - w^2 \end{aligned}$$

Step 6: The maximum value of area will occur when $\frac{dA}{dw}=0$, so to find the paddock width that will make the area maximum we first find the derivative then find the value of w which makes it equal to zero.

$$\begin{aligned} A &= 150w - w^2 \\ \frac{dA}{dw} &= 150 - 2w \end{aligned}$$

When

$$\begin{aligned} \frac{dA}{dw} &= 0 \\ 150 - 2w &= 0 \\ w &= 75 \end{aligned}$$

Step 7: Confirm that this is a maximum by using the first derivative rule or noticing that the original function is a parabola in which the coefficient of the *x* squared term is negative.

	$w = 70$	$w = 75$	$w = 80$
$\frac{dA}{dw} = 150 - 2w$	10	0	–10
Direction of tangent	/	—	\

Step 8: Determine that if the width is 75 m then the length must also be 75 m ($l = 150 - w$). Hence the shape which maximises the area is a square with dimensions 75×75 m.

Example

A fish biologist is asked to stock a farm dam with fish. The biologist and the farmer know that the more fish that are in the dam the more competition for food and so the fish will gain weight more slowly. However, from previous work the biologist knows that when there are *n* fish per unit area of water, the average amount of weight (in grams) that each fish gains during the season is given by $W = 600 - 30n$ What is the value of *n* that leads to the maximum total production of weight gains of fish (*P*)?

(Arya, J. & Lardner, R. 1979, *Mathematics for the Biological Sciences*, Prentice-Hall, Englewood Cliffs, New Jersey.)

If the gain in weight of the fish is given by the function, *W*, for each fish, then the total production (*P*) of the dam must be $n \times W$ per unit area.

$$P = n \times W$$
$$P = n(600 - 30n)$$
$$P = 600n - 30n^2$$

To find the amount of fish that would maximise the production we need to know the maximum value of the production function i.e. find the maximum turning point.

To determine this, find the value of *n* for which $\frac{dP}{dn} = 0$

$$P = 600n - 30n^2$$
$$\frac{dP}{dn} = 600 - 60n$$
$$\text{When } \frac{dP}{dn} = 0,$$
$$600 - 60n = 0$$
$$n = 10$$

This means that there is a stationary point for this function at $n = 10$. We know that this is a maximum value because the function is a parabola with a negative coefficient of the x^2 term. We could however, check the behaviour of the gradient of the tangent around the value of $n = 10$ to confirm that it is the maximum value.

This means that it would require 10 fish per unit area of dam to produce maximum production.

Example

The production cost C per day (in dollars) for steel tube depends on the length L, in metres, produced per day and is given by the formula $C = 30L + \dfrac{300\,000}{L}$.

Report on the length of tube produced each day that would optimise costs.

Optimisation of costs really means finding the minimum costs. So we would have to minimise the cost equation with respect to length i.e. to find the minimum cost, find the value of L which satisfies $\dfrac{dC}{dL} = 0$.

$$C = 30L + \frac{300\,000}{L}$$

$$\frac{dC}{dL} = 30 - \frac{300\,000}{L^2}$$

$$\text{When } \frac{dC}{dL} = 0,$$

$$\begin{aligned} 30 - \frac{300\,000}{L^2} &= 0 \\ 30L^2 - 300\,000 &= 0 \\ L^2 &= 10\,000 \\ L &= \pm 100 \end{aligned}$$

This means that there is a stationary point for this function at $L = 100$. We do not need to consider the negative value, because we cannot produce a negative length of steel tube. We know that this is a minimum value but we could look at the behaviour of the gradient of the tangent around the value of $L = 100$ to confirm that it is the minimum value.

	$L = 90$	$L = 100$	$L = 110$
$\dfrac{dC}{dL} = 30 - \dfrac{300\,000}{L^2}$	–7.04	0	5.2
Direction of tangent	$\diagdown$	—	$\diagup$

To find the cost each day for this length of tube we evaluate

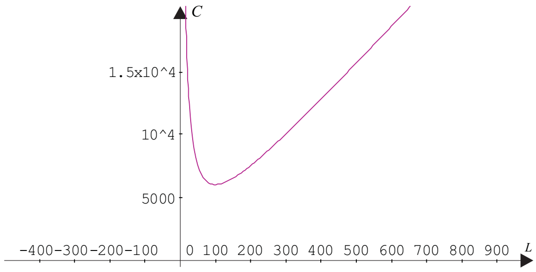
$$C = 30L + \frac{300\,000}{L} \text{ when } L = 100$$

$$C = 30 \times 100 + \frac{300\,000}{100}$$

$$C = 6\,000$$

The cost is minimised to \$6 000 when the length of tube produced is 100 m/day. The cost to produce this length of tube is \$6 000 per day.

If you consider the graph of the function we can confirm that the cost of producing a tube decreases until the length is 100 m and then increases.



Example

A cylindrical can without a top is to be made from a piece of sheet metal 942.5 cm^2. If the volume of the can must be a maximum, find the radius of the base and the height of the can. (Hint: calculations are easier if you use $942.5 \approx 300\pi$ and if the radius is chosen as the independent variable).

If we let r cm be the radius of the base and h be the height of the can then the volume of the can will be $V = \pi r^2 h$. We need this formula because we are asked to maximise volume.

To express V in terms of one variable only (the hint suggests that we should write the equation in terms of r) we need to find a relationship between h and r. We use the surface area given in the question to help with this. Let surface area be S, so that $S = 300\pi$.

Surface area of the can will be the area of the base plus the area of the side of the can.

$$S = 2\pi rh + \pi r^2$$
$$300\pi = 2\pi rh + \pi r^2$$
$$2\pi rh = 300\pi - \pi r^2$$
$$h = \frac{300\pi - \pi r^2}{2r\pi}$$
$$h = \frac{300 - r^2}{2r}$$
$$h = \frac{150}{r} - \frac{r}{2}$$

Substituting the value of h into the volume equation we get,

$$V = \pi r^2 h$$
$$V = \pi r^2 \left(\frac{150}{r} - \frac{r}{2}\right)$$
$$V = 150\pi r - \frac{\pi r^3}{2}$$

We now have a volume equation with only one variable.

To find the maximum we need to find the value of r which satisfies the equation, $\frac{dV}{dr}=0$

$$V = 150\pi r - \frac{\pi r^3}{2}$$

$$\frac{dV}{dr} = 150\pi - \frac{3}{2}\pi r^2$$

$$150\pi - \frac{3}{2}\pi r^2 = 0$$

$$r^2 = -150\pi \div -\frac{3}{2}\pi$$

$$r^2 = -150\pi \times -\frac{2}{3\pi}$$

$$r^2 = 100$$

$$r = \pm 10$$

The negative value of r is not possible, so the radius is 10 cm. However we must confirm first that this is in fact a maximum value by investigating the gradient of the tangent around this point.

	$x = 9$	$x = 10$	$x = 11$
$\frac{dV}{dr} = 150\pi - \frac{3}{2}\pi r^2$	≈ 90	0	≈ -99
Direction of tangent	/	—	\

A radius of 10 cm gives the maximum can volume. The height of the can is determined from

$$h = \frac{150}{r} - \frac{r}{2}$$

$$h = \frac{150}{10} - \frac{10}{2}$$

$$h = 15 - 5 = 10$$

The height of the can would be 10 cm.

Note: You could have attempted to find V in terms of h but this would have lead to a function $V = \pi h\left(-h + \sqrt{h^2 + 300}\right)^2$ which is difficult to differentiate at this stage. However, if you graphed this and the function with r as the independent variable, both functions would have a maximum at the equivalent of $r = 10$.

Tips for completing optimization problems

1. Make sure you know the quantity or function to be optimised (maximised or minimised).
2. Obtain a formula for the function to be optimised. Using information given in the problem try to eliminate variables so that the function has only one independent variable.
3. Use the derivative of the function to find any stationary points and determine if they are maximum or minimum values of the function.

Activity 6.12

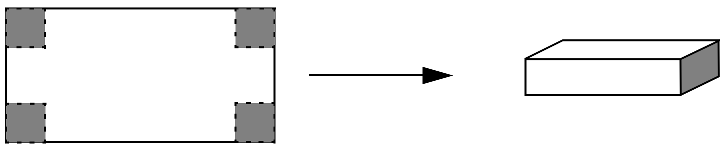
1. The cost of running a truck on a journey of 100 km depends on both the time the trip takes and the average speed of the truck during the trip. A firm has estimated that for one of its older trucks the cost of a journey in dollars is given by the function:

   $C = 0.5x + \dfrac{2500}{x}$,where x is the average speed of the truck in km/h.

   Determine the average speed that will produce a journey of minimum cost. What will that cost be?

2. A farmer wishes to construct a new rectangular paddock adjacent to an existing straight section of fence. If he has 350 m of fencing available and he intends to use the existing fence as one of the sides of his new paddock, find the dimensions of the new paddock such that its area will be a maximum.

3. A metal worker has a rectangular piece of sheet metal of length 500 mm and of width 350 mm. She intends to cut four small squares of equal size from each corner of the sheet and then fold it together to form an open topped tray. Find the size of the small squares such that the volume of the tray is a maximum. See the diagram below:



4. The current flowing in a section of a conductor is found to vary according to the function:

   $i = 150\sin t + 150\cos t$

   where i is the current in amperes and t the time in ms after the current is switched on.

   Find the time when the current first reaches a minimum and also calculate the value of the current at this time.

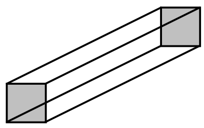
5. The power delivered by a belt driven pulley system varies according to both the tension in the belt and the rotational speed of the pulley. The relationship is shown below:

   $P = Tx - 0.0002x^3$, where T is the tension (in Newtons) in the belt, x the rotational speed in revolutions per minute (rpm) and P the power in kW.

   A certain belt driven system has a belt tension of 500 N, calculate the rotational speed x necessary to obtain maximum power.

6. In a certain country, a Post Office regulation states that the dimensions of parcels must be governed by the following restriction: ‘The length plus the girth must not exceed 190 cm’. A company in that country wishes to package their goods in boxes which are square based prisms (see diagram below) and which comply with Post Office restrictions. What is the size of the square on the base in order that it has a maximum volume?

   Note: The girth of a parcel is the perimeter of the smallest face.



7. A cylindrical can with no lid, is made from 150 cm² of sheet metal, find the dimensions of the can so that its volume is a maximum.

8. The power developed by an internal combustion engine in kW depends on the internal pressure measured in kilopascals. For a particular engine, experimental analysis revealed that the power developed and the internal pressure were related according to the following function:

   $W = 115 + 15.7p - 2.81p^2$, where W is the power in kW and p the internal pressure in kPa.

   Find the internal pressure necessary to deliver maximum power for this engine.

Something to talk about…

Differential calculus is one of the most powerful tools in our mathematical tool kit. Have you ever come across its use before, in your work or other experiences. Use these experiences to develop a question that uses differential calculus. Discuss it with your fellow students in the discussion group to make sure it works.

That’s the end of this module. By completing this module you have completed a difficult topic involving many new concepts. You should have a feeling of achievement, even if you did find it tough.

But before you are really finished you should do a number of things:

1. Have a close look at your action plan for study. Are you on schedule? Or do you need to restructure you action plan or contact your tutor to discuss any delays or concerns?

2. Make a summary of the important points in this module noting your strengths and weaknesses. Add any new words to your personal glossary. This will help with future revision.

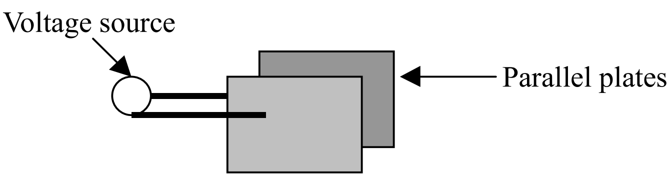
3. Practice some real world problems by having a go at ‘A Taste of Things to Come’.

4. Check your skill level by attempting the post-test.

6.5 A taste of things to come

Capacitance in an Alternating Current (AC) circuit

A capacitor is essentially a pair of parallel plates, separated by an insulator. When a potential difference (i.e. a voltage) is applied to either side of the capacitor, charge builds up on the plates. When this voltage is removed, the plates discharge. See the diagram below:



The amount of charge that collects on the plates depends on the voltage that has been applied and the capacitance of the plates (C).

That is:

$$q = Cv$$

Where q is the charge, C the capacitance, measured in farads (F) and v the voltage in volts.

When the capacitor is charging or discharging the rate at which this happens is given by:

$$\frac{dq}{dt} = C\frac{dv}{dt}$$ (differentiate both sides of the above equation with respect to t)

In electronics, the rate at which the charge varies is defined as the current i in amperes. That is:

$$i = \frac{dq}{dt} = C\frac{dv}{dt}$$

Here is one for you to try:

The voltage applied to a circuit is known to be given by $v = 12\sin t$ where t is the time after switching on the voltage supply, in ms. If this voltage is applied to a capacitor of 0.5 mF, describe the resulting current. What will be the value of this current 1ms after the voltage is applied?

Multi-variable functions and partial differentiation

The functions we have met so far in this module have consisted of one independent and one dependent variable. Real life, however, is rarely as simple as this, with many variables being dependent on more than one other variable. For example, the pressure of an ideal gas depends on both the volume of gas present (V) and the temperature of the gas (T), according to the relationship:

$$P = \frac{kT}{V}$$

Where k is a constant.

A chemist may be interested in studying the rate at which the pressure changes when the volume is changed, in other words in the derivative $\frac{dP}{dV}$. Because this function has two variables, we assume that one variable is momentarily fixed and we differentiate the function as if this were a constant. The resulting derivative is called a partial derivative and is represented by the symbol $\frac{\partial P}{\partial V}$.

In the above case:

$$\begin{aligned} P &= kTV^{-1} \\ \frac{\partial P}{\partial V} &= kT(-1V^{-2}) \\ &= \frac{-kT}{V^2} \end{aligned}$$

If we wished to find how temperature changes affected the pressure, we may wish to determine the partial derivative $\frac{\partial P}{\partial V}$. In this case it would be determined as if the volume were kept constant.

$$\begin{aligned} P &= \frac{k}{V}T \\ \frac{\partial P}{\partial T} &= \frac{k}{V} \times 1 = \frac{k}{V} \end{aligned}$$

Here is one for you to try:

If $h = x\ln y + x^2 y$ find the partial derivatives $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$.

6.6 Post-test

1. Find the derivative of the following functions:

   (a) $y = 3x^5 - 2x^3 + 2$

   (b) $y = \dfrac{2}{x^2}$

   (c) $y = (x+7)^2$

   (d) $y = 3\sqrt{x} - 2x + 1$

   (e) $y = 12\cos x$

2. If $g(x) = 2\sin x + 2\ln x$ find the value of $g'(2.5)$.

3. Determine $\dfrac{dV}{dx}$ when $V = 2 + 3e^x - 2\cos x$

4. Use Graphmatica to sketch the function $y = |3x+2|$. At what point(s) on the function is it not differentiable?

5. Determine the location and nature of the turning points of the function $y = x^4 - 32x + 48$. What will be the y-intercept of the function?

   What will happen to the function at its extremities? That is, what happens for large and small values of x?

   Use the above information to sketch the function.

6. Given the following clues, sketch the function:

   (a) The function is differentiable for all values of x.

   (b) $f(0) = 0.5$

   (c) $f'(0) = 0$

   (d) $\lim_{x\to\infty} f(x) = 0^+$

   (e) $\lim_{x\to-\infty} f(x) = 0^+$

7. The cost of running an ocean liner on a cruise depends on its speed according to the following function:

   $$C = v\left(\frac{1500}{v^2} + 5\right)$$

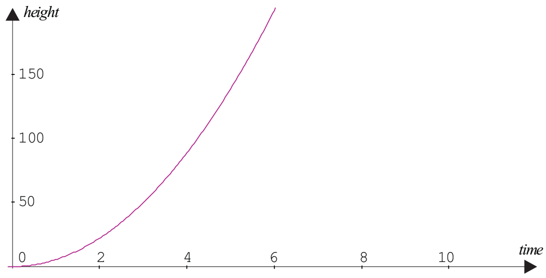
   Where C is the cost in thousands of dollars and v the average speed in knots. Find the speed at which the cost is a minimum and find this cost.

6.7 Solutions

Solutions to activities

Activity 6.1

1. (a) The graph is shown below:



(b) The average rate of change of water level between 8 am and 9 am, represents the gradient of the line segment joining the two points: $(2,\ 5.5\times 2^2)$ and $(3,\ 5.5\times 3^2)$, that is between (2, 22) and (3, 49.5).

$$\text{Average rate of change} = \frac{f(t+h)-f(t)}{h} = \frac{49.5-22}{1} = 27.5 \text{ cm/h}$$

(c) The following table shows the value of this average rate of change, when h becomes small.

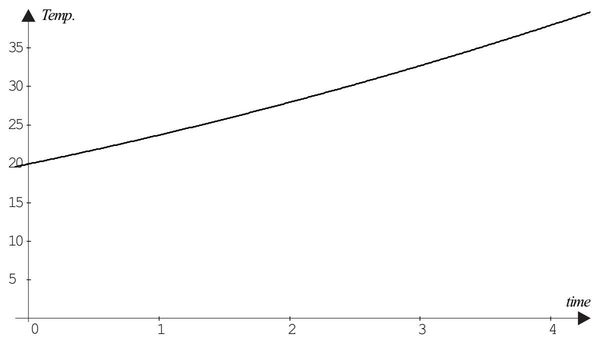
h	$f(2+h)=5.5\times(2+h)^2$	$\dfrac{f(2+h)-f(2)}{h}$
1	49.5	27.5
0.1	$5.5\times 2.1^2 = 24.255$	$\dfrac{24.255-22}{0.1} = 22.55$
0.0001	$5.5\times 2.0001^2 \approx 22.0022$	$\dfrac{22.0022-22}{0.0001} = 22.00055$

It would seem that the instantaneous rate of change of water level at 8 am is approaching 22 cm/h.

(d) The exact rate of change can be evaluated using limits.

$$\begin{aligned}\text{rate of change} &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}\\ &= \lim_{h\to 0}\frac{5.5\times(2+h)^2-5.5\times 2^2}{h}\\ &= \lim_{h\to 0}\frac{5.5\times(4+4h+h^2)-22}{h}\\ &= \lim_{h\to 0}\frac{22+22h+5.5h^2-22}{h}\\ &= \lim_{h\to 0}\frac{22h+5.5h^2}{h}\\ &= \lim_{h\to 0}\frac{h(22+5.5h)}{h}\\ &= \lim_{h\to 0}(22+5.5h)\\ &= 22\text{ cm/h}\end{aligned}$$

2. (a) The graph is shown below:



(b) The average rate of change of temperature between 1 pm and 2 pm is found by finding the gradient of the line passing through the points (3, 32.75) and (4, 38).

$$\text{Average rate of change} = \frac{f(t+h)-f(t)}{h} = \frac{38-32.75}{1} = 5.25\text{ degrees/hour}$$

(c) The following table shows the value of this average rate of change when h becomes small.

h	$f(3+h)=0.25\times(3+h)^2+3.5\times(3+h)+20$	$\frac{f(3+h)-f(3)}{h}$
1	38	5.25
0.1	$0.25\times(3.1)^2+3.5\times 3.1+20=33.2525$	$\frac{33.2525-32.75}{0.1}=5.025$
0.0001	$0.25\times 3.0001^2+3.5\times 3.0001+20=32.7505$	$\frac{32.7505-32.75}{0.0001}$ $=5.000025$

It would seem that the instantaneous rate of change of temperature at 1 pm is 5 degrees/hour.

(d) To confirm this we use limits.

$$\begin{aligned}
\text{rate of change} &= \lim_{h\to 0}\frac{f(3+h)-f(3)}{h}\\
&= \lim_{h\to 0}\frac{0.25\times(3+h)^2+3.5\times(3+h)+20-32.75}{h}\\
&= \lim_{h\to 0}\frac{0.25\times(9+6h+h^2)+10.5+3.5h+20-32.75}{h}\\
&= \lim_{h\to 0}\frac{2.25+1.5h+0.25h^2+10.5+3.5h+20-32.75}{h}\\
&= \lim_{h\to 0}\frac{5h+0.25h^2}{h}\\
&= \lim_{h\to 0}\frac{h(5+0.25h)}{h}\\
&= \lim_{h\to 0}(5+0.25h)\\
&= 5 \text{ degrees/hour}
\end{aligned}$$

Activity 6.2

1. (a) This means that after 1.5 seconds have elapsed, the cannon ball has reached a height of 20 m.

   (b) The statement means that after 1 second the rate at which the cannon ball's height is increasing is 12 m every second. That is, its vertical velocity at that moment is 12 m/s.

2. (a) The total cost of producing 20 widgets will be \$170.

   (b) When the Company is producing 20 widgets, the rate of change of cost compared to the number of widgets produced is \$2.70. That is the cost of producing an extra widget will be \$2.70.

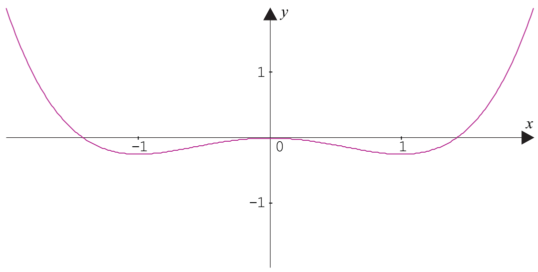
3. (a) When $x = 0$, $\frac{dy}{dx} = x^3 - x = 0^3 - 0 = 0$

   When $x = -1$, $\frac{dy}{dx} = x^3 - x = (-1)^3 - (-1) = 0$

   When $x = 1$, $\frac{dy}{dx} = x^3 - x = 1^3 - 1 = 0$

   (b) As $\frac{dy}{dx}$ is a measure of the gradient of a tangent to the curve at a particular point, we would expect that at each of the points above, the gradient of the tangent would be zero.

(c) The graph is shown below:



Notice that at each of the points, corresponding to $x = 0$, $x = -1$ and $x = 1$ the curve is turning.

4. The meaning of the statement $h'(50) = 1200$ is that 50 years after 1940, that is in 1990, the rate of change of the population was 1200. In other words in 1990 the city's population was increasing at a rate of 1200 people per year.

Activity 6.3

1. (a) $f(x) = x^{10}$

$$f'(x) = 10 \times x^{10-1}$$
$$= 10x^9$$

(b) $f(x) = x^{-3}$

$$f'(x) = -3 \times x^{-3-1}$$
$$= -3x^{-4}$$

(c) $f(x) = x \;\; = x^1$

$$f'(x) = 1 \times x^{1-1}$$
$$= x^0$$
$$= 1$$

(d) $f(x) = 2 = 2x^0$

$$f'(x) = 2 \times 0 \times x^{0-1}$$
$$= 0$$

(e) $f(x) = x^{\frac{2}{3}}$

$$f'(x) = \frac{2}{3} \times x^{\frac{2}{3}-1}$$
$$= \frac{2}{3} \times x^{-\frac{1}{3}}$$
$$= \frac{2}{3} x^{-\frac{1}{3}} \quad \text{or} \quad \frac{2}{3x^{\frac{1}{3}}} \quad \text{or} \quad \frac{2}{3\sqrt[3]{x}}$$

(f) $f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -1 \times x^{-1-1} = -1 \times x^{-2}$

$= -x^{-2}$ or $\frac{-1}{x^2}$ or $-\frac{1}{x^2}$

(g) $f(x) = \frac{1}{x^3} = x^{-3}$

$f'(x) = -3 \times x^{-3-1} = -3 \times x^{-4}$

$= -3x^{-4}$ or $\frac{-3}{x^4}$ or $-\frac{3}{x^4}$

(h) $f(x) = \sqrt[4]{x^2} = (x^2)^{\frac{1}{4}} = x^{\frac{2}{4}} = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2} \times x^{\frac{1}{2}-1} = \frac{1}{2} \times x^{-\frac{1}{2}}$

$= \frac{1}{2}x^{-\frac{1}{2}}$ or $\frac{1}{2x^{\frac{1}{2}}}$ or $\frac{1}{2\sqrt{x}}$

2. $s = t^5$

$\frac{ds}{dt} = 5t^4$

when $t = 3$,

$\frac{ds}{dt} = 5 \times 3^4$

$= 405$ m/s

3. If $p(x) = 2\pi$, then its derivative $\frac{dp}{dx} = 0$. This is because the original function is a constant with a value approximately equal to 6.28. Such a function has a zero gradient.

4. The function $h(t) = \sqrt[3]{t}$ can be written $h(t) = t^{\frac{1}{3}}$ therefore its derivative is:

$h'(t) = \frac{1}{3} \times t^{\frac{1}{3}-1}$

$= \frac{1}{3} \times t^{-\frac{2}{3}}$ or $\frac{1}{3t^{\frac{2}{3}}}$ or $\frac{1}{3\sqrt[3]{t^2}}$ or $\frac{1}{3(t^{\frac{1}{3}})^2}$

when $t = 8$,

$h'(8) = \frac{1}{3(8^{\frac{1}{3}})^2}$

$= \frac{1}{12}$

5. In order to differentiate the function $y = \frac{1}{x^4}$, it should be firstly written as $y = x^{-4}$ and consequently its derivative will be:

$$\frac{dy}{dx} = -4x^{-5}$$
$$= -\frac{4}{x^5}$$

Angela has treated the function as if it is just $y = x^4$, found the derivative of this function which is $4x^3$, then put it all over one to get $\frac{1}{4x^3}$. In actual fact the function we have to differentiate is not $y = x^4$ but $y = \frac{1}{x^4}$ which should be written in the form x^n before it is differentiated.

6. The function $y = 2e$ is actually a constant since $2e \approx 5.437$, therefore its derivative will be zero. Recall that all constant functions (lines which run parallel to the horizontal axis) will have a gradient of zero. Barry has mistakenly treated the constant 'e' as a variable.

Activity 6.4

1. (a) $y = 3x^9$

$$\frac{dy}{dx} = 3 \times 9 \times x^{9-1} = 27x^8$$

(b) $y = 7x$

$$\frac{dy}{dx} = 7 \times x^{1-1} = 7x^0 = 7$$

(c) $y = -\frac{1}{2}x^{-2}$

$$\frac{dy}{dx} = -\frac{1}{2} \times -2 \times x^{-2-1} = 1 \times x^{-3}$$
$$= x^{-3} \quad \text{or} \quad \frac{1}{x^3}$$

(d) $y = \frac{2}{\sqrt{x}} = 2 \times x^{-\frac{1}{2}}$

$$\frac{dy}{dx} = 2 \times -\frac{1}{2} \times x^{-\frac{1}{2}-1} = -1 \times x^{-\frac{3}{2}}$$
$$= -x^{-\frac{3}{2}} \quad \text{or} \quad -\frac{1}{x^{\frac{3}{2}}} \quad \text{or} \quad -\frac{1}{\sqrt{x^3}}$$

(e) $y = -\frac{3}{x^3} = -3 \times x^{-3}$

$$\frac{dy}{dx} = -3 \times -3 \times x^{-3-1} = 9x^{-4} \quad \text{or} \quad \frac{9}{x^4}$$

2. If $y = 3x^9$ then $\frac{dy}{dx} = 3 \times 9 \times x^{9-1} = 27x^8$.

3. If $a(r) = \pi r^2$ then $a'(r) = \pi \times 2 \times r^{2-1} = \pi \times 2 \times r = 2\pi r$.

4. If $y = 2ex^3$ then:

$$\begin{aligned}\frac{dy}{dx} &= 2e \times 3 \times x^{3-1} \\ &= 2e \times 3 \times x^2 \\ &= 6ex^2\end{aligned}$$

   Remember that $2e$ should be regarded as a constant as it approximately equals 5.437.

5. Since $V(r) = \frac{4}{3}\pi r^3$ then:

$$\begin{aligned}V'(r) &= \frac{4}{3} \times \pi \times 3 \times r^{3-1} \\ &= 4 \times \pi \times r^2 \\ &= 4\pi r^2\end{aligned}$$

   Therefore when $r = 2$,

$$\begin{aligned}V'(2) &= 4 \times \pi \times 2^2 \\ &= 16\pi\end{aligned}$$

6. Jason has simply found the derivative of the function t and the function t^3 and then multiplied these together. Derivatives of function cannot be easily multiplied like this. The best way to complete this differentiation is to simplify the function first. In this case:

$$\begin{aligned}s &= 2\pi t \times t^3 \\ &= 2\pi t^4\end{aligned}$$

   Consequently its derivative will be:

$$\begin{aligned}\frac{ds}{dt} &= 2 \times \pi \times 4 \times t^{4-1} \\ &= 8\pi t^3\end{aligned}$$

7. The function $p(x) = 2ex\sqrt{x}$ is best completed by simplifying first as in the previous question, so that:

$$\begin{aligned}p(x) &= 2ex\sqrt{x} \\ &= 2 \times e \times x \times x^{\frac{1}{2}} \\ &= 2ex^{\frac{3}{2}}\end{aligned}$$

Differentiating this function gives:

$$\begin{aligned} p'(x) &= 2 \times e \times \frac{3}{2} \times x^{\frac{3}{2}-1} \\ &= e \times 3 \times x^{\frac{1}{2}} \\ &= 3e\sqrt{x} \end{aligned}$$

Activity 6.5

1. $y = 3x^4 - 2x^3 + 5x^2 - x - 1$

$$\begin{aligned} \frac{dy}{dx} &= 3 \times 4 \times x^{4-1} - 2 \times 3 \times x^{3-1} + 5 \times 2 \times x^{2-1} - 1 - 0 \\ &= 12x^3 - 6x^2 + 10x - 1 \end{aligned}$$

2. $h(t) = \dfrac{3}{t^2} + t - \pi = 3t^{-2} + t - \pi$

$$\begin{aligned} h'(t) &= 3 \times -2 \times t^{-2-1} + 1 - 0 \\ &= -6t^{-3} + 1 \\ &= -\frac{6}{t^3} + 1 \end{aligned}$$

when $t = 2$,

$$h'(2) = -\frac{6}{2^3} + 1 = -\frac{6}{8} + 1 = -\frac{3}{4} + 1 = \frac{1}{4}$$

3. $\dfrac{ds}{dp} = 3 \times 3p^2 - 12 + 0$

$$= 9p^2 - 12$$

4. $V = 2\pi r^2 + 8\pi r$

$$\begin{aligned} \frac{dV}{dr} &= 2\pi \times 2r^1 + 8\pi \times 1 \\ &= 4\pi r + 8\pi \end{aligned}$$

5. $s = 2t^2 + \sqrt{t} - 3e = 2t^2 + t^{\frac{1}{2}} - 3e$

$$\begin{aligned} \frac{ds}{dt} &= 2 \times 2t^1 + \frac{1}{2} \times t^{\frac{1}{2}-1} - 0 \\ &= 4t + \frac{1}{2}t^{-\frac{1}{2}} \\ &= 4t + \frac{1}{2\sqrt{t}} \end{aligned}$$

6. $P = 2t^2(3t-1) = 6t^3 - 2t^2$

$$\frac{dP}{dt} = 6 \times 3t^2 - 2 \times 2t$$
$$= 18t^2 - 4t$$

7. $v(t) = 2\sqrt{t} + 12t^2 + 1 = 2t^{\frac{1}{2}} + 12t^2 + 1$

$$v'(t) = 2 \times \frac{1}{2} \times t^{-\frac{1}{2}} + 12 \times 2t + 0$$
$$= t^{-\frac{1}{2}} + 24t$$
$$= \frac{1}{\sqrt{t}} + 24t$$

when $t = 4$,

$$v'(4) = \frac{1}{\sqrt{4}} + 24 \times 4$$
$$= 96.5$$

8. $T = \dfrac{3x^2}{x^5}$

$= 3x^{-3}$ (after simplifying)

$$\frac{dT}{dx} = 3 \times -3x^{-3-1}$$
$$= -9x^{-4}$$
$$= -\frac{9}{x^4}$$

9. $y = \dfrac{x^3 + 2x^2 - 5x}{x}$

$$= \frac{x(x^2 + 2x - 5)}{x}$$
$$= x^2 + 2x - 5$$

$$\frac{dy}{dx} = 2x + 2$$

10. $H = 3t^{-1} + 2t^3 - \sqrt{t}$

$$= 3t^{-1} + 2t^3 - t^{\frac{1}{2}}$$

$$\frac{dH}{dt} = 3 \times -1 \times t^{-2} + 2 \times 3t^2 - \frac{1}{2} \times t^{-\frac{1}{2}}$$

$$= -3t^{-2} + 6t^2 - \frac{1}{2} \times \frac{1}{\sqrt{t}}$$

$$= -\frac{3}{t^2} + 6t^2 - \frac{1}{2\sqrt{t}}$$

when $t = 4$,

$$\frac{dH}{dt} = \frac{-3}{4^2} + 6 \times 4^2 - \frac{1}{2\sqrt{4}}$$

$$= \frac{-3}{16} + 96 - \frac{1}{4}$$

$$= 95\frac{9}{16}$$

11. $E = \frac{2\pi}{r} - r^3$

$$= 2\pi r^{-1} - r^3$$

$$\frac{dE}{dr} = 2\pi \times -1r^{-2} - 3r^2$$

$$= -\frac{2\pi}{r^2} - 3r^2$$

12. The instantaneous rate of change is found by firstly differentiating the function and then substituting the value $h = -3$ into the derivative.

$$\frac{dP}{dh} = 3 \times 4h^3 - 2 \times 1 + 0$$

$$= 12h^3 - 2$$

when $h = 3$

$$\frac{dP}{dh} = 12 \times (-3)^3 - 2$$

$$= -324 - 2$$

$$= -326$$

13. To find the gradient of the tangent, we need to firstly differentiate the function $y = 2x^2 + 6x - 5$ and then substitute $x = 4$.

$$\begin{aligned}\frac{dy}{dx} &= 2\times 2x + 6\times 1 - 0\\ &= 4x + 6\end{aligned}$$

when $x = 4$, the gradient of the tangent will be:

$$\begin{aligned}m &= 4\times 4 + 6\\ &= 22\end{aligned}$$

14. (a) When $T = 800$,

$$\begin{aligned}L &= 200 + \sqrt[4]{T}\\ &= 200 + \sqrt[4]{800}\\ &\approx 205.3 \text{ mm}\end{aligned}$$

(b) We need to firstly differentiate the function and then substitute into the derivative the value $T = 800$

$$\begin{aligned}L &= 200 + \sqrt[4]{T}\\ &= 200 + T^{\frac{1}{4}}\\ \frac{dL}{dT} &= 0 + \frac{1}{4}\times T^{\frac{1}{4}-1}\\ &= \frac{1}{4}T^{-\frac{3}{4}}\\ &= \frac{1}{4\sqrt[4]{T^3}}\end{aligned}$$

When $T = 800$,

$$\begin{aligned}\frac{dL}{dT} &= \frac{1}{4}\times 800^{-\frac{3}{4}}\\ &= \frac{1}{4}\times 800^{-0.75}\\ &\approx 0.0017\end{aligned}$$

This tells us that at a temperature of 800°C, the length of the bar is changing at a rate of about 0.0017 mm/°C.

15. (a) $h(t) = 300t - 5t^2$

$$\begin{aligned} h(2) &= 300 \times 2 - 5 \times 2^2 \\ &= 600 - 20 \\ &= 580 \end{aligned}$$

This tells us that the missile has a height of 580 m two seconds after its launch.

(b) We need to firstly differentiate the function and then substitute in the value $t = 30$

$$\begin{aligned} h'(t) &= 300 - 5 \times 2t \\ &= 300 - 10t \\ h'(30) &= 300 - 300 \\ &= 0 \end{aligned}$$

This result tells us that after thirty seconds the missile's rate of change of height with respect to time is neither increasing nor decreasing…it is stationary. This means that the missile must be at the point where it will turn and fall to the ground. Notice that the original function is a parabola so we would expect it to have a maximum turning point. It must occur at the 30th second.

Activity 6.6

1. (a) $\dfrac{dy}{dx} = 3e^x + 1$

(b) $v'(t) = 3(4t^3) - 12(e^t) = 12t^3 - 12e^t$

(c)
$$\begin{aligned} P &= \frac{e^{2t}}{e^t} \\ &= e^{2t-t} \\ &= e^t \\ \frac{dP}{dt} &= e^t \end{aligned}$$

2. The instantaneous rate of change can be found by differentiating the function and then substituting the value $h = 2$ into this equation.

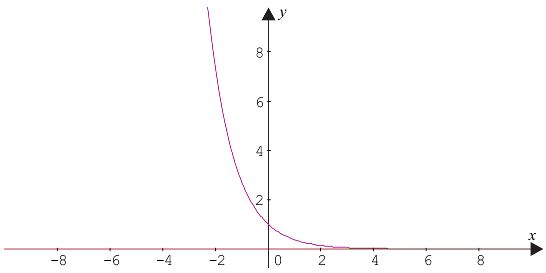
$$\begin{aligned} P &= 12h + 3e^h \\ \frac{dP}{dh} &= 12 + 3e^h \end{aligned}$$

when $h = 2$,

$$\begin{aligned} \frac{dP}{dh} &= 12 + 3e^2 \\ &\approx 34.17 \end{aligned}$$

Therefore the instantaneous rate of change of the function at $h = 2$ is approximately 34.17.

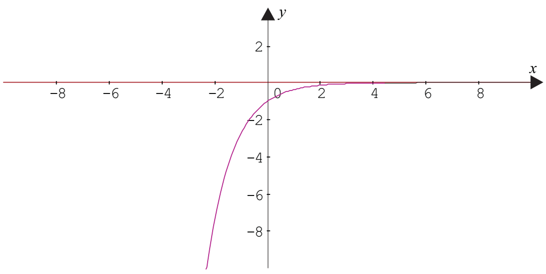
3. To begin with let's draw the graph of $y = e^{-x}$, see below:



Now use the formula $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ with a small value of h to try and predict the equation of the function's derivative. We will use h = 0.001 and then Graphmatica to draw the function:

$$y = \frac{e^{-(x+0.001)} - e^{-x}}{0.001}$$

This graph is shown below:
(Note: we will need to use the expression y=(e^-(x+0.001)-e^-x)/0.001 for Graphmatica)



This graph looks to be the reflection of the original graph over the x-axis, in other words it is the same shape however all its y-values are negative, we could conclude that the equation is: $y = -e^{-x}$. If you graph this function using Graphmatica, you will see that it is the function shown above. That is the derivative of $y = e^{-x}$ is $y = -e^{-x}$.

4. The student has treated e as a variable and x as the constant. The student must recall that when $y = e^x, \dfrac{dy}{dx} = e^x$

Activity 6.7

1. (a) $\dfrac{dy}{dx} = \dfrac{1}{x} - 3 \times 5x^4$

$= \dfrac{1}{x} - 15x^4$

(b) $P'(t) = 3 \times e^t - 2 \times \dfrac{1}{t} + 0$

$= 3e^t - \dfrac{2}{t}$

(c) $\dfrac{dT}{dx} = 5 \times 2x^1 + 3 \times \dfrac{1}{x} + 0$

$= 10x + \dfrac{3}{x}$

2. We need to differentiate the function $y = 2\ln x$ and then substitute into the derivative the value $x = 2$.

$$\frac{dy}{dx} = 2 \times \frac{1}{x} = \frac{2}{x}$$

when $x = 2$, the gradient of the tangent is:

$$m = \frac{2}{2} = 1$$

3. The equation of the tangent will be of the form $y = mx + b$ as it is a straight line. In this case the intercept is zero, therefore $b = 0$.

Let the point of contact be (x_1, y_1)

To determine the gradient of the tangent requires that we differentiate the function $y = \ln x$.

$$\frac{dy}{dx} = \frac{1}{x}$$

when $x = x_1$ the gradient of the tangent will be:

$$m = \frac{1}{x_1}$$

Consequently the equation of the tangent will be:

$$y = mx + b$$
$$y = \frac{1}{x_1}x$$

At the point of contact, where $x = x_1$, the value of y will be:

$$y = \frac{1}{x_1} \times x_1 = 1$$

Consequently the point of contact has coordinates $(x_1, 1)$. Since this point also lies on the curve $y = \ln x$ then it must satisfy that equation.

Therefore:

$$1 = \ln x_1$$
$$e^1 = x_1$$

The point of contact is $(e, 1)$

Activity 6.8

1. (a) $\dfrac{dy}{dx} = 2 \times -\sin x + 0 = -2\sin x$

   (b) $\dfrac{dy}{dx} = \cos x - (-\sin x) + 2e^x$

   $= \cos x + \sin x + 2e^x$

   (c) $\dfrac{dy}{dx} = 2x - 12 \times \cos x + -\sin x$

   $= 2x - 12\cos x - \sin x$

2. To find the instantaneous rate of change, we need to differentiate the function and substitute $x = 1.1$ into this function.

$$y = 3\cos x$$
$$\frac{dy}{dx} = 3 \times -\sin x$$
$$= -3\sin x$$

when $x = 1.1$,

$$\frac{dy}{dx} = -3\sin(1.1)$$
$$\approx -3 \times 0.89 \quad \text{(remember to put your calculator into radian mode)}$$
$$\approx -2.67$$

3. (a) The gradient of the tangent is about –1.2. This means that the water is falling at a rate of about 1.2 metres per hour at 4 pm. (Using Graphmatica, the gradient is –1.1336)

   (b) A time of 4 pm corresponds to the 4 hours after 12 noon. To find the rate at which the water level is rising at that time we need to differentiate the function

   $h = 2.5\cos(\frac{\pi}{6}t) + 10$ and substitute into this the value $t = 4$. Since we have only differentiated simple trigonometrical functions we could guess:

   $$\begin{aligned}\frac{dh}{dT} &= 2.5 \times -\sin(\frac{\pi}{6}t) + 0 \text{ (maybe)}\\ &= -2.5\sin(\frac{\pi}{6}t)?\end{aligned}$$

   When $t = 4$,

   $$\begin{aligned}\frac{dh}{dT} &= -2.5\sin(\frac{\pi}{6} \times 4)\\ &\approx -2.165 \text{ m/hour??}\end{aligned}$$

   We expected the derivative to be –1.1336. So if we multiplied –2.165 by 0.5236 then we get the right answer. So our derivative must be multiplied by 0.5236. Where does this figure come from? Well $\frac{\pi}{6}$ is 0.5236, so

   $$\begin{aligned}\frac{dh}{dt} &= -2.5\sin(\frac{\pi}{6} \times 4) \times \frac{\pi}{6}\\ &\approx -1.1336 \text{ m/hour}\end{aligned}$$

   Recall from module 4 that multiplying t by a number, stretches or compresses a graph, so this then changes the gradient by the same factor. This is in fact what happens with derivatives of these more complicated functions. You will learn about these as you study further mathematics.

   (c) The rate of change of tide height will be zero when $t = 6$ and when $t = 12$ (midnight). It will remain at zero only instantaneously.

Activity 6.9

1. (a) The function is NOT differentiable for all values of its domain (i.e. its x-values), as the tangent to the curve at $x = 2$ will be vertical.

   (b) This function is NOT differentiable for all values of its domain, as it contains a sharp corner at $x = –6$.

   (c) This function is NOT differentiable for all values of its domain, as it has a discontinuity at $x = 2$.

   (d) This function is differentiable for all values of its domain.

2. (a) $\dfrac{dy}{dx} = 3 \times \dfrac{1}{x} - 2 \times -\sin x + 2 \times 3x^2$

$= \dfrac{3}{x} + 2\sin x + 6x^2$

(b) $P = 3t^{\frac{1}{2}} + 2\sin t$

$\dfrac{dP}{dt} = 3 \times \dfrac{1}{2} \times t^{-\frac{1}{2}} + 2 \times \cos t$

$= \dfrac{3}{2} t^{-\frac{1}{2}} + 2\cos t$

$= \dfrac{3}{2\sqrt{t}} + 2\cos t$

(c) $h(t) = 12e^t - t^{-2}$

$h'(t) = 12 \times e^t - -2 \times t^{-3}$

$= 12e^t + 2t^{-3}$

$= 12e^t + \dfrac{2}{t^3}$

(d) $\dfrac{dy}{dx} = 2 \times 5x^4 - 4 \times \cos x + 3 \times \dfrac{1}{x}$

$= 10x^4 - 4\cos x + \dfrac{3}{x}$

(e) $P = 12\cos a - 3a^{\frac{1}{2}}$

$\dfrac{dP}{da} = 12 \times -\sin a - 3 \times \dfrac{1}{2} a^{-\frac{1}{2}}$

$= -12\sin a - \dfrac{3}{2} a^{-\frac{1}{2}}$

$= -12\sin a - \dfrac{3}{2\sqrt{a}}$

Activity 6.10

1. (a) Since the velocity is the rate of change of displacement, we can examine the graph for points where its gradient is zero, i.e. where a tangent drawn to the curve will be zero. This will occur at the highest point of the graph, when $t = 1$.

   (b) The object returns to its original position, as its final displacement is 0. The answer is NOT 2, as this is the time it takes to return to its original position.

2. (a) Since the displacement is given by $s = 10t - 5t^2$, the velocity will be given by its derivative,

$$v = \frac{ds}{dt} = 10 - 10t$$

So when $t = 0.8$,

$$\begin{aligned} v &= 10 - 10t \\ &= 10 - 10 \times 0.8 \\ &= 2\,\text{m/s} \end{aligned}$$

(b) From the graph we see that the highest point occurs at $t = 1$, to find the acceleration we need to differentiate the velocity equation,

$$a = \frac{dv}{dt} = -10$$

Consequently the acceleration is –10 m/s/s for all values of time.

3. We need to firstly differentiate the equation $P = -0.165q^2 + 80$

$$\begin{aligned} \frac{dP}{dq} &= -0.165 \times 2q \\ &= -0.33q \end{aligned}$$

When $q = 5$,

$$\begin{aligned} \frac{dP}{dq} &= -0.33 \times 5 \\ &= -1.65 \end{aligned}$$

In this context, this means that when the quantity on the market reaches 5 tonnes, the rate at which the price is dropping is $1.65 for every extra 1 tonne in production.

4. We need to differentiate the equation $P = 0.05n^2 + 0.3n - 3.8$,

$$\begin{aligned} \frac{dP}{dn} &= 0.05 \times 2n - 0.3 \\ &= 0.1n - 0.3 \end{aligned}$$

When $n = 10$, the marginal profit is therefore,

$$\begin{aligned} \frac{dP}{dn} &= 0.1 \times 10 - 0.3 \\ &= 1 - 0.3 \\ &= 0.7 \end{aligned}$$

That is, for every increase in production at this point, there is an increase in profit of $0.70.

Activity 6.11

1. (a) We need to differentiate and find when the derivative has the value of zero.

$$\frac{dy}{dx} = 2x+1$$
$$0 = 2x+1$$
$$x = -\frac{1}{2}$$

The y-value of the turning point can be found be substituting $x = -\frac{1}{2}$ into the original equation.

$$\begin{aligned} y &= x^2 + x - 2 \\ &= \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) - 2 \\ &= -2.25 \end{aligned}$$

Therefore the turning point is (–0.5, –2.25)

(b) From your earlier work on quadratic functions, you should realise that this function will have a minimum turning point at (–0.5, –2.25). Therefore we need to find at which of the end points in this domain the function takes a maximum value.

When $x = -1,\ y = x^2 + x - 2 = (-1)^2 + (-1) - 2 = -2$

When $x = 4$, $y = x^2 + x - 2 = 4^2 + 4 - 2 = 18$

The global maximum is 10.

(c) The global minimum must be –2.25 as this is the value the function takes at its turning point.

2. (a) $f(x) = 0$ when $x = -1$ and $x = 1$. These are the x-intercepts.

(b) $f(0) = 1$. This is the y-intercept.

(c) $f'(x) = 0$ when $x \approx -0.33$ and $x = 1$. These are the turning points.

(d) Place a ruler on the graph at $x = 0.5$ the gradient is approximately –1. Therefore $f'(0.5) \approx -1$.

3. To find the turning points we need to firstly differentiate the function and find for what values the derivative is zero.

$$\begin{aligned} y &= 2\sin x \\ \frac{dy}{dx} &= 2\cos x \\ 0 &= 2\cos x \\ x &= \cos^{-1} 0 \\ &= \frac{\pi}{2} \text{ and } \frac{3\pi}{2} \\ &\approx 1.57 \text{ and } 4.71 \end{aligned}$$

To find the nature of the turning points we need to examine the derivative either side of $x = 1.57$ and $x = 4.71$. For convenience we will show these in the following table:

x-value	1.4	1.57	1.6	4.6	4.71	4.8
$\frac{dy}{dx}$	0.34	0	–0.06	–0.22	0	0.17

Notice that the gradient near $x = 1.57$ changes from positive to zero and then to negative. Consequently there must be a maximum turning point at $x = 1.57$. Similarly near $x = 4.71$ the gradient changes from negative to zero and then to positive. Consequently there must be a minimum turning point at $x = 4.71$.

4. We need to differentiate the function and then find where the derivative is zero.

$$
\begin{aligned}
y &= 2x^3 - 6x^2 - 18x + 1 \\
\frac{dy}{dx} &= 6x^2 - 12x - 18 \\
0 &= 6(x^2 - 2x - 3) \\
0 &= (x+1)(x-3) \\
x &= -1 \text{ and } 3
\end{aligned}
$$

To find the y-values of these points we need to substitute these values of x into the original equation.

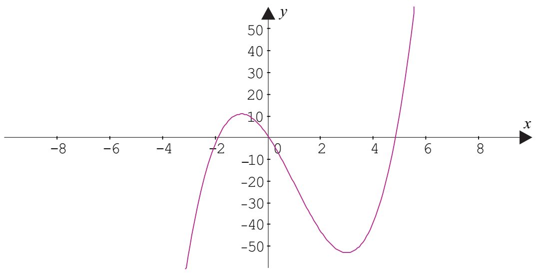
When $x = -1$, $y = 2(-1)^3 - 6(-1)^2 - 18(-1) + 1 = 11$

When $x = 3$, $y = 2(3)^3 - 6(3)^2 - 18 \times 3 + 1 = -53$

The turning points are therefore: (–1,11) and (3,–53). To determine the nature of these points we need to examine the derivative at points either side of $x = -1$ and $x = 3$.

x-value	–2	–1	0	2	3	4
$\frac{dy}{dx}$	30	0	–18	–18	0	30

The point (–1,11) is a maximum and the point (3,–53) is a minimum. In order to draw a sketch of the function we could determine a few more points. One in particular, which is easy to determine, is the y-intercept, in this case (0,1). Your graph should look like the one below:



5. An observation of the graph shows that the derivative has the following behaviour:

$-4 \le x < -3$	the gradient is positive, therefore the derivative is also positive.
$x = -3$	the derivative is zero.
$-3 < x < 1$	the derivative is negative.
$x = 1$	the derivative is zero.
$1 < x \le 4$	the derivative is positive.

6. We need to differentiate the function, equate the derivative to zero and then solve:

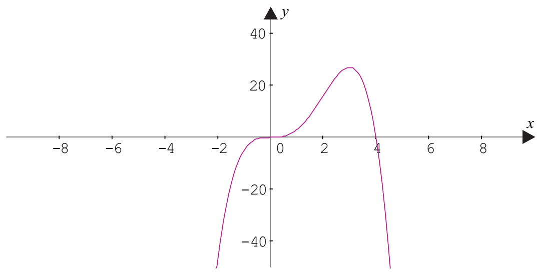
$$y = 4 + 2x^3 - x^4$$
$$\frac{dy}{dx} = 6x^2 - 4x^3$$
$$0 = 6x^2 - 4x^3$$
$$2x^2(3 - 2x) = 0$$
$$x = 0 \text{ or } x = 1.5$$

To find the nature of the stationary points, we need to examine the derivative either side of $x = 0$ and $x = 1.5$

x-value	-1	0	1	1.5	2
$\frac{dy}{dx}$	10	0	2	0	-8

This shows that there is a maximum at $x = \frac{3}{2}$ and a horizontal point of inflection at $x = 0$.

7. A suitable graph is shown below:



Activity 6.12

1. To find the minimum cost we need to find the value of x that satisfies $\frac{dC}{dx} = 0$.

$$\begin{aligned} C &= 0.5x + \frac{2500}{x} \\ &= 0.5x + 2500x^{-1} \\ \frac{dC}{dx} &= 0.5 - 2500x^{-2} \\ 0 &= 0.5 - \frac{2500}{x^2} \\ 2500 &= 0.5x^2 \\ x^2 &= 5000 \\ x &\approx \pm 71 \text{ km/h} \end{aligned}$$ (we can ignore the negative value)

The function has only one turning point and we should check that this gives a minimum cost. To do this examine the value of the derivative for speeds either side of 71 km/h.

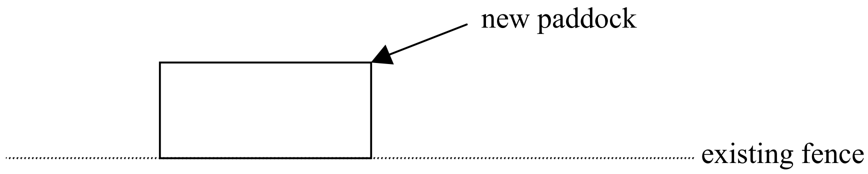
Speed	65 km/h	71 km/h	75 km/h
Value of derivative	–0.09	0	0.06

Therefore the cost is a minimum when the speed is 71 km/h. The cost at this speed will be:

$$\begin{aligned} C &= 0.5 \times 0.71 + \frac{2500}{71} \\ &\approx 71 \end{aligned}$$

Therefore the cost of the trip at 71 km/hr will be $71.

2. We should draw a diagram to model this situation and introduce some variables.



Let the length of the new paddock be l m and the width w m. Since the farmer is using the existing fence line, he only needs one length and two widths in order to make his fence. That is:

$$\begin{aligned} l + 2w &= 350 \\ l &= 350 - 2w \end{aligned}$$

We wish to produce a paddock with a maximum area, hence we need to find a function for area in terms of one of the variables l or w.

$$\begin{aligned} A &= l \times w && \text{(area of a rectangle)} \\ &= (350 - 2w) \times w && \text{(substituting the expression above)} \\ &= 350w - 2w^2 \end{aligned}$$

We need to find the value of w which satisfies $\frac{dA}{dw} = 0$.

$$\begin{aligned} A &= 350w - 2w^2 \\ \frac{dA}{dw} &= 350 - 4w \\ 0 &= 350 - 4w \\ 4w &= 350 \\ w &= 87.5 \end{aligned}$$

We need to examine the value of the derivative either side of 87.5 m to check that it is a maximum.

Width	80 m	87.5 m	95 m
Value of derivative	30	0	–30

Therefore the area is a maximum when the width is 87.5 m and when the length is:

$$\begin{aligned} l &= 350 - 2w \\ &= 350 - 2 \times 87.5 \\ &= 175 \text{ m} \end{aligned}$$

3. Let the length of each side of the square be x mm.
The length of the formed tray will be $500 - 2x$ mm.
The width of the formed tray will be $350 - 2x$ mm.
The height of the formed tray will be x mm.

We need to find an expression for the volume V of the tray in terms of x and then find the value of x that satisfies $\frac{dV}{dx} = 0$.

$$\begin{aligned} V &= \text{length} \times \text{width} \times \text{height} \\ &= (500 - 2x)(350 - 2x)x \\ &= (175\ 000 - 1700x + 4x^2)x \\ &= 175\ 000x - 1700x^2 + 4x^3 \\ \frac{dV}{dx} &= 175\ 000 - 3400x + 12x^2 \\ 0 &= 175\ 000 - 3400x + 12x^2 \end{aligned}$$

We need to solve the quadratic equation in order to find the turning points.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{3400 \pm \sqrt{(-3400)^2 - 4 \times 12 \times 175\ 000}}{24} \\ &\approx \frac{3400 \pm 1777.6}{24} \\ &\approx 215.7 \text{ or } 67.6 \end{aligned}$$

The first answer gives a negative width, consequently it is not relevant in this situation. We need to check that $x = 67.6$ mm gives a maximum volume.

Value of x	60 mm	67.6 mm	75 mm
Value of derivative	14200	0	–12500

Therefore a square of side 67.6 mm will ensure a maximum volume.

4. We need to find the value of t which satisfies the equation $\frac{di}{dt}=0$.

$$
\begin{aligned}
i &= 150\sin t + 150\cos t \\
\frac{di}{dt} &= 150\cos t - 150\sin t \\
0 &= 150\cos t - 150\sin t \\
\sin t &= \cos t \\
\frac{\sin t}{\cos t} &= 1 \\
\tan t &= 1 \\
t &= \tan^{-1} 1 \\
&= \frac{\pi}{4}, \pi + \frac{\pi}{4} \\
&\approx 0.79 \text{ ms and } 3.93 \text{ ms}
\end{aligned}
$$

To check which of these is in fact a minimum we should examine the value of the derivative either side of 0.79 ms and either side of 3.93 ms.

Value of t in ms	0.76	0.79	0.82	3.9	3.93	3.96
Value of derivative	5.39	0	–7.34	–5.72	0	7.00

Therefore the current is first a minimum after 3.93 ms, the value it takes at this point can be found by substituting $t = 3.93$ into the original equation.

$$
\begin{aligned}
i &= 150\sin t + 150\cos t \\
&= 150\sin 3.93 + 150\cos 3.93 \\
&\approx -212 \text{ A}
\end{aligned}
$$

The negative sign indicates that the current is flowing in the opposite direction.

5. In this case $T = 500$ and therefore the equation for power becomes $P = 500x - 0.0002x^3$.

In order to find the maximum power we need to find the value of x for which $\frac{dP}{dx}=0$.

$$
\begin{aligned}
P &= 500x - 0.0002x^3 \\
\frac{dP}{dx} &= 500 - 0.0006x^2 \\
0 &= 500 - 0.0006x^2 \\
0.0006x^2 &= 500 \\
x^2 &\approx 833333 \\
x &\approx 913
\end{aligned}
$$

To check that this value is actually a maximum, we should examine the value of the derivative either side of $x = 913$ rpm.

Value of x	900 rpm	913 rpm	920 rpm
Value of derivative	14	0	–7.8

Consequently a maximum power will occur when the rotational speed is 913 rpm.

6. Since the box has a square base, let the length of that square be x centimetres. Therefore the girth will be $4x$ cms. Also let the length of the parcel be l cms.

From the Post Office regulations we have:

$$\begin{aligned} l + 4x &= 190 \\ l &= 190 - 4x \end{aligned}$$

The volume of the parcel can be expressed:

$$\begin{aligned} V &= x^2 \times l \\ &= x^2 \times (190 - 4x) \\ &= 190x^2 - 4x^3 \end{aligned}$$

We need to find the value(s) of x for which $\frac{dV}{dx} = 0$.

$$\begin{aligned} V &= 190x^2 - 4x^3 \\ \frac{dV}{dx} &= 380x - 12x^2 \\ 0 &= 380x - 12x^2 \\ 0 &= x(380 - 12x) \\ x &= 0 \text{ or } 31.7 \end{aligned}$$

Of course in the context of this question a zero value has a trivial meaning. We need to check that a value of $x = 31.7$ cm does in fact produce a maximum volume.

Value of x	25 cm	31.7 cm	35 cm
Value of derivative	2000	0	–1400

A box with size 31.7 cm by 31.7 cm by 63.2 cm should provide a maximum volume, yet satisfy the Post Office regulations.

7. Let the radius of the can be r cm and its height h cm.

   The surface area can be written: $S = \pi r^2 + 2\pi rh$.

   Since we know the surface area is 150 cm², we can obtain an expression for h.

$$\begin{aligned} 150 &= \pi r^2 + 2\pi rh \\ 2\pi rh &= 150 - \pi r^2 \\ h &= \frac{150 - \pi r^2}{2\pi r} \end{aligned}$$

   Now the volume of a cylinder is given by the formula: $V = \pi r^2 h$ and in this case this is:

$$\begin{aligned} V &= \pi r^2 \left(\frac{150 - \pi r^2}{2\pi r} \right) \\ &= r\left(\frac{150 - \pi r^2}{2} \right) \\ &= 75r - \frac{\pi r^3}{2} \end{aligned}$$

   To find the maximum volume we need to find the value(s) of r for which $\frac{dV}{dr} = 0$.

$$\begin{aligned} V &= 75r - \frac{\pi r^3}{2} \\ \frac{dV}{dr} &= 75 - \frac{3\pi r^2}{2} \\ 0 &= 75 - \frac{3\pi r^2}{2} \\ \frac{3\pi r^2}{2} &= 75 \\ 3\pi r^2 &= 150 \\ r^2 &= \frac{50}{\pi} \\ r &\approx \pm 4 \end{aligned}$$

   A negative answer has no meaning in this context therefore the result is a radius of 4 cm. To check that it produces a maximum volume we should examine the value of the derivative either side of r = 4 cm.

Value of r	3 cm	4 cm	5 cm
Value of derivative	32.6	0	–42.8

Given that $r = 4$ we can find the height:

$$\begin{aligned} h &= \frac{150 - \pi \times 4^2}{2 \times \pi \times 4} \\ &= 4 \end{aligned}$$

Therefore the dimensions of the can will be a radius of 4 cm and a height of 4 cm.

8. We need to find the value of p for which $\frac{dW}{dp} = 0$.

$$\begin{aligned} W &= 115 + 15.7p - 2.81p^2 \\ \frac{dW}{dp} &= 15.7 - 5.62p \\ 0 &= 15.7 - 5.62p \\ 5.62p &= 15.7 \\ p &\approx 2.8 \end{aligned}$$

Since the original function is a quadratic with a negative coefficient of x^2, its turning point must be a maximum. Consequently maximum power occurs when there is an internal pressure of about 2.8 kPa.

Solutions to a taste of things to come

Capacitance in an Alternating Current (AC) circuit

Since the voltage is $v = 12\sin t$, we can find the current from the following expression:

$$i = C\frac{dv}{dt} \quad \text{------------ equation (1)}$$

However, we need to firstly find an expression for $\frac{dv}{dt}$.

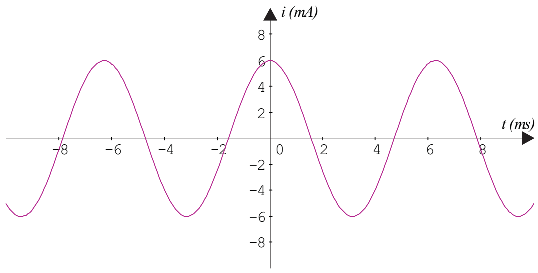
$$\frac{dv}{dt} = 12\cos t$$

Substituting this expression and $C = 0.5$ into equation (1) gives:

$$\begin{aligned} i &= 0.5 \times 12\cos t \\ &= 6\cos t \end{aligned}$$

(Note: normally the units of capacitance would have to be converted into Farads before substituting into equation (1) and the resulting current would be in Amperes. I have chosen to leave them in milli-Farads and consequently the current will be in milli-Amps.)

That is, the current is given by a cosine curve, see below:



After 1 ms the current will be:

$$\begin{aligned} i &= 6\cos t \\ &= 6\cos 1 \\ &\approx 3.24 \text{ mA} \end{aligned}$$

Multi-variable functions and partial differentiation

In order to find $\frac{\partial h}{\partial x}$ we treat the variable y as if it were a constant.

$$\begin{aligned} h &= x\ln y + x^2 y \\ \frac{\partial h}{\partial x} &= \ln y + 2xy \end{aligned}$$

In order to find $\frac{\partial h}{\partial y}$ we treat the variable x as if it were a constant.

$$\begin{aligned} h &= x\ln y + x^2 y \\ \frac{\partial h}{\partial y} &= x \times \frac{1}{y} + x^2 \times 1 \\ &= \frac{x}{y} + x^2 \end{aligned}$$

Solutions to post-test

1. Find the derivative of the following functions:

(a) $\frac{dy}{dx} = 15x^4 - 6x^2$

(b) $y = \frac{2}{x^2} = 2x^{-2}$

$$\frac{dy}{dx} = -4x^{-3}$$
$$= \frac{-4}{x^3}$$

(c) $y = (x+7)^2 = x^2 + 14x + 49$

$$\frac{dy}{dx} = 2x + 14$$

(d) $y = 3\sqrt{x} - 2x + 1$

$$= 3x^{\frac{1}{2}} - 2x + 1$$
$$\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}} - 2$$
$$= \frac{3}{2\sqrt{x}} - 2$$

(e) $y = 12\cos x$

$$\frac{dy}{dx} = -12\sin x$$

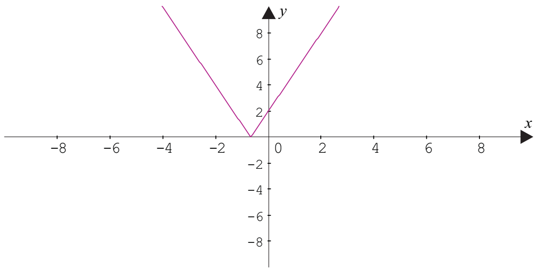
2. We need to differentiate the function and then substitute $x = 2.5$ into its derivative.

$$g(x) = 2\sin x + 2\ln x$$
$$g'(x) = 2\cos x + \frac{2}{x}$$
$$g'(2.5) = 2\cos(2.5) + \frac{2}{2.5}$$
$$\approx -0.8$$

3. When $V = 2 + 3e^x - 2\cos x$

$$\frac{dV}{dx} = 3e^x + 2\sin x$$

4. A sketch of the function $y = |3x + 2|$ is shown below:



The function is not differentiable where the point occurs, which in this case is at $x = -\frac{2}{3}$.

5. The turning points occur when the derivative is zero.

$$\begin{aligned} y &= x^4 - 32x + 48 \\ \frac{dy}{dx} &= 4x^3 - 32 \\ 0 &= 4x^3 - 32 \\ 4x^3 &= 32 \\ x^3 &= 8 \\ x &= 2 \end{aligned}$$

To find the y-value of this point, substitute $x = 2$ into the original equation.

$$y = 2^4 - 32 \times 2 + 48 = 0$$

Consequently the only turning point is at (2,0). To find whether it is a maximum or a minimum, we should examine the value of the derivative near this point.

Value of x	1	2	3
Value of derivative	–28	0	76

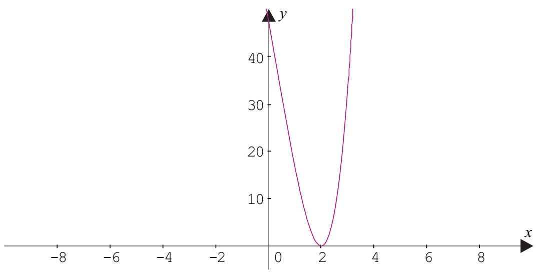
Therefore the point (2,0) is a minimum turning point.

The y-intercept of the function is (0,48).

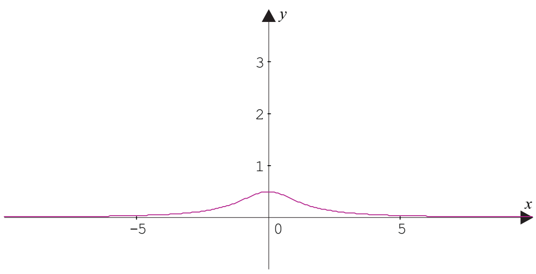
For large values of x the function will get very large, that is: $\lim_{x \to \infty} f(x) = \infty$.

For small values of x the function will get very large, that is: $\lim_{x \to -\infty} f(x) = \infty$.

The sketch of the function is shown below:



6. The function may look like the one shown below:



7. We need to determine the values of v for which $\dfrac{dC}{dv} = 0$

$$
\begin{aligned}
C &= v\left(\frac{1500}{v^2} + 5\right) \\
&= \frac{1500}{v} + 5v \\
&= 1500v^{-1} + 5v \\
\frac{dC}{dv} &= -1500v^{-2} + 5 \\
0 &= \frac{-1500}{v^2} + 5 \\
1500 &= 5v^2 \\
v &= \sqrt{300} \\
&\approx 17.3
\end{aligned}
$$

To determine whether this is in fact a minimum, we need to examine values of the derivative near $v = 17.3$.

Value of v	16	17.3	18
Value of derivative	–0.86	0	0.37

Therefore the minimum cost will occur at a cruising speed of 17.3 knots.

The cost of running the ship at this cruising speed can be found by substituting $v = 17.3$ into the original formula.

$$\begin{aligned} C &= v\left(\frac{1500}{v^2}+5\right) \\ &= 17.3\left(\frac{1500}{17.3^2}+5\right) \\ &= 173.2 \end{aligned}$$

That is the cost will be approximately \$173 200.