Module 3

MATRICES

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Introduction

If you have taken course TPP7182, Level B mathematics or studied matrices in the past much of this module will be revision. However there is some new material so make sure you locate these sections and complete the exercises before moving on to another module.

Every one of us has to organise data in a way which is meaningful and readily identifiable. e.g. the weekly outlays for the household, the cricket scores for the test series, the assessment marks for a unit of study. We do this organisation usually in the form of tables and now days people often use spreadsheets on their computers for such purposes. Tables such as these which organise data are called matrices in mathematics. (The singular of matrices is matrix.) Matrices and matrix algebra have wide applications in mathematics and are especially important in planning production schedules and predicting long term outcomes. We will develop matrix algebra using a production example.

Matrix Representation of Data

Example 3.1:

Consider a safety equipment company that produces three types of protective equipment – helmets, shoulder pads and hip pads. These are made from various amounts of plastic, foam and nylon cord using different amounts of labour.

The table below gives the amount of each material and the amount of labour needed to make one of each the pieces of equipment.

	Product		
Material	Helmet	Shoulder Pad	Hip Pad
Plastic	4	2	2
Foam	1	3	2
Nylon Cord	1	3	3
Labour	3	2	2

So to make one hip pad, 2 units of plastic, 2 units of foam, 3 units of nylon cord and 2 units of labour are required.

As long as we know what each row and column means we can reduce the table above to a matrix which we will call matrix A.

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix}$$

where each element a_{ij}, (i.e. entry) in A is the amount of material i needed to make one item j

See Note 1

e.g. $a_{2,3}$ is the amount of foam needed to make one hip pad. From the matrix, a_{ij} equals 2.

✪ Which element of A gives the amount of nylon cord needed to make a shoulder pad?

$a_{3,2}$ =

row column

Answer: a_{32} is 3.

Note that when describing an element of a matrix the row of the element is given firstly and the column of the element is given secondly. This is very important to avoid confusion.

Exercise Set 3.1

1. Using matrix A above, complete the following:

$a_{2,2} = \square$ $\qquad a_{2,2} = \square$

$a_{2,2} = \square$ $\qquad a_{2,2} = \square$

$\square = 4$

2. Describe in words what $a_{2,2}$ represents.

Notes

1. We conventionally use a capital letter for the name of a matrix and the corresponding small letter for an element of that matrix.

Each row and column in a matrix is called a **vector**. So a matrix is made up of a group of row vectors or a group of column vectors. The number of rows and columns in a matrix give the **dimension** or order of the matrix. Matrix A is a 4 row by 3 column matrix which we write as (4×3).

Again note that the order used to express the dimension of a matrix is important. It is **always** number of rows by number of columns.

In computing, especially in spreadsheets matrices are often called **arrays**. Data in arrays can consist of names, letters etc. and numbers. A matrix is a two dimensional array; a vector is a one dimensional array in which the data are numbers.

Exercise Set 3.2

Give the dimension of each of the following and identify the element specified.

(a)
$$P = \begin{bmatrix} 2 & 4 \\ 5 & 1 \\ 6 & 2 \end{bmatrix}$$
$p_{3,2} =$

(b)
$$Q = \begin{bmatrix} 1 & 2 & 4 \\ 0.2 & 8 & 3 \end{bmatrix}$$
$q_{2,2} =$

(c)
$$R = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & 2 & 0 \\ 0.2 & 4 & 6 & 0 \\ -0.2 & -1 & 8 & 4 \\ 8 & -2 & 2 & 8 \end{bmatrix}$$
$r_{4,1} =$

(d)
$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$s_{3,2} =$

Reconsider our safety equipment company example. We could have arranged the data in a matrix with the rows describing the products and the columns describing the amounts of material needed.

e.g.

	Material			
Product	Plastic	Foam	Nylon Cord	Labour
Helmet	4	1	1	3
Shoulder Pad	2	3	3	2
Hip Pad	2	2	3	2

The matrix form of this table is then $\begin{bmatrix} 4 & 1 & 1 & 3 \\ 2 & 3 & 3 & 2 \\ 2 & 2 & 3 & 2 \end{bmatrix}$

✪ Examine the original matrix A and this matrix. How are they related?

. . . .

. . . .

. . . .

Answer: You should have seen that this matrix has the rows of matrix A as its columns and the columns of matrix A as its rows.

■

We say that this matrix is the **transpose of matrix** $\boldsymbol{A}$ and use the symbol $\boldsymbol{A^T}$ to describe it.

So $A = \begin{bmatrix} 4 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix}$ and $A^T = \begin{bmatrix} 4 & 1 & 1 & 3 \\ 2 & 3 & 3 & 2 \\ 2 & 2 & 3 & 2 \end{bmatrix}$

and A is (4×3) and A^T is (3×4)

Exercise Set 3.3

Find the transpose of each of P, Q, R, S in Exercise Set 3.2 and give the dimensions of P^T, Q^T, R^T and S^T

Suppose now that manufacturing the safety equipment involves two distinct processes. In the first process the foam and plastic for each product type are cut out and in the second process, each product type is assembled and sewn with the nylon cord. The matrices B and C below describe these processes.

First Process

$B = \begin{bmatrix} 4 & 2 & 2 \\ 1 & 3 & 2 \\ 0 & 0 & 0 \\ 2.5 & 1 & 1 \end{bmatrix}$ where b_{ij} is the amount of material i used in the first process (cutting out) to make product j e.g. $b_{2,1}$ is the amount of foam cut out to make one helmet. $(b_{2,1} = 1)$

✪ Why are the entries in the third row of B all zero?

. . . .

Answer: Because the nylon cord is not involved in the first process where the foam and plastic are cut out.

Second Process:

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 3 & 3 \\ 0.5 & 1 & 1 \end{bmatrix}$$

where c_{ij} is the amount of material i used in the second process (assembly and sewing) to make product j e.g. $c_{4,3}$ is the amount of labour used to assemble and sew one hip pad. ($c_{4,3} = 1$)

✪ $c_{1,1}$ and $c_{2,1}$ are both zero. What does this mean? . . . .

. . . .

Answer: The plastic and foam components do not require assembly and sewing.

■

Addition and Subtraction of Matrices

Now B and C have the same dimensions (4×3) and it makes sense that their sum should equal A, the matrix which gives the total material requirements for each product.

i.e. $B + C = A$

$$\begin{bmatrix} 4 & 2 & 2 \\ 1 & 3 & 2 \\ 0 & 0 & 0 \\ 2.5 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 3 & 3 \\ 0.5 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix}$$

✪ Examine the previous matrix equation and complete the following

Matrix B has dimension (×)
Matrix C has dimension (×)
Matrix A has dimension (×)

$$b_{1,1} + c_{1,1} = a_{\boxed{}}$$

$$b_{4,3} + c_{4,3} = a_{\boxed{}}$$

and generally, $b_{ij} + c_{ij} = a_{\boxed{}}$

Answer: B, C and A are all (4×3) and $b_{11} + c_{11} = a_{11}$; $b_{43} + c_{43} = a_{43}$; and generally $b_{ij} + c_{ij} = a_{ij}$

Addition of Matrices

To add two matrices

- ensure that the dimension of each matrix is the same
- if the matrices result from a real world problem make sure the addition makes sense
- add each of the corresponding elements of the matrices.

Note: Matrix addition is

- Commutative. So if X and Y are both $(n \times m)$ matrices $X + Y = Y + X$

  and is

- Associative. So if X, Y and Z are all $(n \times m)$ matrices

  $(X + Y) + Z = X + (Y + Z)$

Subtraction of Matrices

Matrix **subtraction** also requires that the matrices have the same size.

To subtract matrix Y from matrix X

- ensure that the dimension of each matrix is the same
- subtract each of the corresponding elements of the matrices.

Example 3.2:

If $X = \begin{bmatrix} 3 & 8 \\ 4 & 2 \\ 1 & 8 \end{bmatrix}$ and $Y = \begin{bmatrix} -3 & 8 \\ 2 & 8 \\ 8 & 4 \end{bmatrix}$ find $Z = X - Y$

Solution:

$$Z = X - Y = \begin{bmatrix} 3-(-3) & 8-8 \\ 4-2 & 2-8 \\ 1-8 & 8-4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 2 & -6 \\ -7 & 4 \end{bmatrix}$$

Note: Matrix subtraction is not commutative so $X - Y \neq Y - X$

Exercise Set 3.4

1. Consider these matrices

$$A = \begin{bmatrix} 2 & 4 & -1 & 6 \\ -7 & 1 & 8 & 3 \\ 5 & 2 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 4 & 6 & 3 \\ 2 & 7 & -1 & 5 \\ 4 & 2 & -3 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 1 \\ 3 & 3 \\ 5 & 4 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 7 & 6 \\ 1 & 0 & 3 \end{bmatrix} \qquad E = \begin{bmatrix} 7 & 21 & 21 \\ 7 & 14 & 21 \\ 14 & 35 & 42 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the following if possible

(a) $A + B$
(b) $C - A$
(c) $E + D$
(d) $E + F$
(e) $C - F$
(f) $B - A$
(g) $A - B$

2. An experiment to test the yield of four different varieties of a grain from three different fertilizers was conducted on three equal sized blocks on a farm.

   The results were:

Block 1

Variety

$$\text{Fertiliser} \quad \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 10 & 13 & 11 & 8 \\ 12 & 14 & 11 & 9 \\ 11 & 11 & 13 & 12 \end{bmatrix}$$

Block 2

Variety

$$\text{Fertiliser} \quad \begin{bmatrix} 11 & 13 & 12 & 9 \\ 11 & 13 & 12 & 9 \\ 10 & 12 & 12 & 11 \end{bmatrix}$$

Block 3

Variety

$$\text{Fertiliser} \quad \begin{bmatrix} 11 & 13 & 12 & 9 \\ 12 & 14 & 12 & 9 \\ 11 & 12 & 13 & 12 \end{bmatrix}$$

[Don’t worry about the units of measurement]

(i) What was the total farm production for each variety of grain from each fertiliser?

(ii) What was the total farm production of each variety?

(iii) What was the total farm production?

Multiplication of a Matrix by a Scalar

A scalar is a single number. To multiply a matrix by a scalar simply multiply each element of the matrix by the scalar.

Example 3.3: Find the product of 2 and $\begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}$

Solution: $2 \times \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 \times 2 & 4 \times 2 \\ -3 \times 2 & 2 \times 2 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ -6 & 4 \end{bmatrix}$

Example 3.4: Find one-third of $\begin{bmatrix} 3 & -6 \\ 2 & 8 \\ 4 & 3 \end{bmatrix}$

Solution: $\frac{1}{3} \times \begin{bmatrix} 3 & -6 \\ 2 & 8 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 3 \times \frac{1}{3} & -6 \times \frac{1}{3} \\ 2 \times \frac{1}{3} & 8 \times \frac{1}{3} \\ 4 \times \frac{1}{3} & 3 \times \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ \frac{2}{3} & \frac{8}{3} \\ \frac{4}{3} & 1 \end{bmatrix}$

Note that a scalar is a (1×1) matrix.

Multiplication of a Matrix by a Vector

Suppose an order comes in to the safety equipment manufacturer for 35 helmets, 10 shoulder pads and 20 hip pads and the manager wants to determine how much raw material will be required to satisfy the order.

For 1 helmet, 4 units of plastic are needed $\therefore$ for 35 helmets 4×35 units of plastic are needed.

For 1 shoulder pad, 2 units of plastic are needed $\therefore$ for 10 shoulder pads 2×10 units of plastic are needed.

For 1 hip pad, 2 units of plastic are needed $\therefore$ for 20 hip pads 2×20 units of plastic are needed.

$\therefore$ the total amount of plastic needed for the order is
$(4 \times 35 + 2 \times 10 + 2 \times 20)$ units $= 200$

Similarly the total amount of foam needed is
$(1 \times 35 + 3 \times 10 + 2 \times 20)$ units $= 105$

the total amount of nylon cord needed is
$(1 \times 35 + 3 \times 10 + 3 \times 20)$ units $= 125$ and

the total amount of labour needed is
$(3 \times 35 + 2 \times 10 + 2 \times 20)$ units $= 165$

We can obtain the total amount of the different materials needed by multiplying our original matrix A by the vector which contains the number of items that have been ordered i.e. by the column vector D.

Let's define this column vector D as

$$D = \begin{bmatrix} 35 \\ 10 \\ 20 \end{bmatrix} \qquad \{D \text{ is } (3 \times 1)\}$$

then $AD = \begin{bmatrix} 4 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix} \times \begin{bmatrix} 35 \\ 10 \\ 20 \end{bmatrix}$ which we know equals $\begin{bmatrix} 200 & 105 & 125 & 165 \end{bmatrix}$

$$\begin{array}{cccc} 200 & 105 & 125 & 165 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \text{plastic needed} & \text{foam needed} & \text{nylon cord needed} & \text{labour needed} \end{array}$$

Let's call the resultant total materials needed row vector T. $\{T$ is $(1 \times 4)\}$

Now look back to where we calculated the amount of each material needed to fill the order.

Plastic	Foam	Nylon Cord	Labour
$(4 \times 35 + 2 \times 10 + 2 \times 20)$	$(1 \times 35 + 3 \times 10 + 2 \times 20)$	$(1 \times 35 + 3 \times 10 + 3 \times 20)$	$(3 \times 35 + 2 \times 10 + 2 \times 20)$

$$= \begin{array}{cccc} \left[200\right. & 105 & 125 & \left.165\right] \\ \text{Plastic} & \text{Foam} & \text{Nylon Cord} & \text{Labour} \end{array} = T$$

$\therefore AD = T$

i.e. $\begin{bmatrix} 4 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix} \times \begin{bmatrix} 35 \\ 10 \\ 20 \end{bmatrix}$

$$= \begin{bmatrix} (4 \times 35 + 2 \times 10 + 2 \times 20) & (1 \times 35 + 3 \times 10 + 2 \times 20) & (1 \times 35 + 3 \times 10 + 3 \times 20) & (3 \times 35 + 2 \times 10 + 2 \times 20) \end{bmatrix}$$

Use your fingers to follow through which element in A multiplies which element in D.

Exercise Set 3.5

Consider matrices A, D and T on the previous page and complete the following:

T has $\boxed{}$ row and $\boxed{}$ columns.

There are $\boxed{}$ elements in T

$t_{11} = \boxed{}$ $\quad t_{12} = \boxed{}$ $\quad t_{13} = \boxed{}$ $\quad t_{14} = \boxed{}$

To obtain t_{11} find the sum: $a_{11} \times d_{11} + a_{12} \times d_{21} + a_{13} \times d_{31}$

$$= \sum_{j=1}^{3} a_{1,j} d_{j,1}$$

To obtain t_{12} find the sum: $a_{21} \times d_{11} + a_{22} \times d_{21} + \boxed{} \times d_{31}$

$$= \sum_{j=\boxed{}}^{3} a_{2,j} d_{j,1}$$

To obtain t_{13} find the sum: $a_{31} \times \boxed{} + \boxed{} \times d_{21} + \boxed{} \times \boxed{}$

$$= \sum_{\boxed{}}^{\boxed{}} a_{\boxed{}} d_{j,1}$$

To obtain t_{14} find the sum: $\boxed{} \times \boxed{} + \boxed{} \times \boxed{} + \boxed{} \times \boxed{}$

$$= \sum_{\boxed{}}^{\boxed{}} \boxed{}\,\boxed{}$$

Multiplication of Two Matrices

So far we have seen that to multiply two matrices, the matrices do not have to be the same dimension (unlike addition and subtraction). However one condition about dimension that is required is that **the number of columns of the first matrix must equal the number of rows of the second matrix**.

If this is not the case the matrix product is not defined.

e.g.

$$\underset{(4\ \times\ 3)}{A} \quad \times \quad \underset{(3\ \times\ 1)}{D}$$

same ∴ product is defined

product will be (4×1)

The dimension of the matrix which results from the matrix multiplication is given by the 'outside' dimensions of the two matrices involved in the multiplication.

Let's consider our safety equipment manufacturer example further. Suppose the manager wants to forward plan for the next quarter's production. The expected demand from the manufacturer's two regular wholesale outlets for the next quarter are given in matrix E.

$$E = \begin{array}{l} \\ \text{Helmet} \\ \text{Shoulder Pad} \\ \text{Hip Pad} \end{array} \begin{array}{c} \text{Wholesaler} \\ \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 100 & 85 \\ 180 & 95 \\ 200 & 120 \end{bmatrix} \end{array}$$

We can use matrix multiplication to help the manager decide how much materials will need to be ordered to be able to satisfy the demands of the wholesalers. The product $R = A \times E$ will give the amount of materials needed for the demands of each wholesaler. (If you feel unsure of this, work out the requirements without using matrices.)

$A \times E =$ (material $\times$ product) $\times$ (product $\times$ demand)

$$(4 \times 3) \quad \times \quad (3 \times 2)$$

same ∴ product is defined

product, R will be (4×2)

The matrix resulting from this matrix multiplication, R, will have 4 rows and 2 columns. i.e. there are 8 elements r_{ij}, to be found. To find each of these elements we use the notion of matrix multiplication we met earlier. Any element r_{ij} in R will be given by the sum of the products of each element in the ith row of A and the corresponding element in the jth row of D.

$$A \times E = R$$

$$\begin{bmatrix} 4 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix} \times \begin{bmatrix} 100 & 85 \\ 180 & 95 \\ 200 & 120 \end{bmatrix} = \begin{bmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \\ r_{3,1} & r_{3,2} \\ r_{4,1} & r_{4,2} \end{bmatrix}$$

$$(4 \times 3) \qquad (3 \times 2) \qquad (4 \times 2)$$

To obtain $r_{1,1}$ find the sum:

$$\begin{aligned} & a_{1,1} \times e_{1,1} + a_{1,2} \times e_{2,1} + a_{1,3} \times e_{3,1} \\ = \; & 4 \times 100 + 2 \times 180 + 2 \times 200 \\ = \; & 1\,160 \end{aligned}$$

(Note the shading above)

Use a similar approach to find the remaining elements in R. If you need to, write in full the details as shown for $r_{1,1}$ otherwise use your fingers to identify which element in each row of A multiplies which element in each column of E.

$$A \times E = R$$

$$\begin{bmatrix} 4 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix} \times \begin{bmatrix} 100 & 85 \\ 180 & 95 \\ 200 & 120 \end{bmatrix} = \begin{bmatrix} (4 \times 100 + 2 \times 180 + 2 \times 200) & (\Box \times 85 + 2 \times \Box + 2 \times 120) \\ (1 \times 100 + 3 \times \Box + 2 \times 200) & (\Box \times 85 + 3 \times \Box + \Box \times 120) \\ (\Box \times \Box + \Box \times \Box + \Box \times \Box) & (\Box \times \Box + \Box \times \Box + \Box \times \Box) \\ (\quad + \quad + \quad) & (\quad) \end{bmatrix}$$

Now to find the total amount of material needed we can simply add the elements in each row of R.

$$R = \begin{bmatrix} 1160 & 770 \\ 1040 & 610 \\ 1240 & 730 \\ 1060 & 685 \end{bmatrix}$$

Total amount plastic needed = 1 160 + 770 = 1 930 units

Total amount of foam needed = 1 040 + 610 = 1 650 units

Total amount of nylon cord needed = 1 970 units

Total amount of labour needed = 1 745 units.

Some important points to note are

- matrix multiplication is generally **not** commutative. i.e. $XY \neq YX$
  so to avoid confusion read XY as 'Y is premultiplied by X'
- if X, Y and Z are of appropriate dimensions, matrix multiplication is **associative** i.e.
  $X(YZ) = (XY)Z$
- if X, Y and Z are of appropriate dimensions the distributive law applies to matrices i.e.
  $X(Y + Z) = XY + XZ$

Exercise Set 3.6

1. Find the average block production for each variety of grain for each type of fertiliser from Exercise Set 3.4 Question 2.

2. Consider the matrices given in Exercise Set 3.4 Question 1.
   Find the following if possible

   (a) $A \times B$
   (b) $B^T \times C$
   (c) $-D \times C$
   (d) $E^T \times F^T$

3. A toy manufacturing company makes small, medium and large teddy bears at each of three plants. Currently the production schedule per week is as shown in the matrix N below:

$$N = \begin{array}{l} \\ \text{Small} \\ \text{Medium} \\ \text{Large} \end{array} \begin{array}{c} \begin{array}{ccc} \textit{Plant 1} & \textit{Plant 2} & \textit{Plant 3} \end{array} \\ \begin{bmatrix} 150 & 300 & 80 \\ 50 & 100 & 150 \\ 200 & 100 & 500 \end{bmatrix} \end{array}$$

   (a) Because of Christmas orders the company needs to triple its production in October. What should the production schedule be for each week of October? (Give the answer in a table similar to that above)

   (b) Small bears cost \$4 each to make and \$2 each to transport; medium bears cost \$7 each to make and \$3 each to transport; large bears \$15 each to make and \$4 each to transport regardless of which plant makes them.

      (i) Form C, a (3×2) cost matrix for the bear types

      (ii) Multiply M, the matrix which gives the production schedule in October by a suitable matrix so that the resultant matrix N is (3×1) and gives the total number of small, medium and large bears to be made in October.

      (iii) Perform an appropriate matrix multiplication to yield a (1×2) matrix which gives the total production costs in October and the total transportation costs in October.

4. In a certain mathematics course there are six assignments worth 5% each, and essay worth 5% and an examination worth 65%. There are 215 marks for the assignments (35, 20, 50, 35, 40, 35 respectively), 5 marks for the essay and 125 marks for the examination. The marks obtained by six students are shown below.

Marks / **Student**	**Assignments**	**Essay**	**Exam**
A	21, 18, 40, 32, 30, 30	4.5	89
B	18, 14, 25, 22, 32, 12	4.0	42
C	30, 18, 42, 35, 39, 28	4.5	110
D	32, 15, 37, 28, 38, 22	3.5	70
E	29, 16, 12, 27, 10, 18	4.5	49
F	27, 19, 15, 32, 30, 30	5	80

Define suitable matrices and sums, differences or products so that the final matrix is (1×6) and gives the percentage for the course for each of the six students.

Special Matrices

1. A **square matrix** is one with the same number of rows and columns

2. The **transpose** of the transpose of a matrix is the original matrix

   e.g.

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 4 & 1 & 1 & 3 \\ 2 & 3 & 3 & 2 \\ 2 & 2 & 3 & 2 \end{bmatrix}$$

$$(A^T)^T = \begin{bmatrix} 4 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix}$$

3. The **zero matrix**, 0, is a matrix (of any dimension) all of whose elements are zero. If X and 0 are of appropriate dimension $X0 = 0$. (However sometimes when neither of two matrices A and B are the zero matrix, their product AB is the zero matrix.)

4. The **identity matrix** is a square matrix of any dimension in which all elements a_{ij} are zero **except** those when $i = j$. When $i = j$, the element is 1. We usually use I for an identity matrix and subscript n to show the size of the matrix.

   e.g. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   Note:

   The non zero elements (the one's) are on the leading (or primary) diagonal of the identity matrix.

   If any square matrix Z and the identity matrix are of the same dimension, $ZI = IZ = Z$

5. A matrix is **symmetric** if corresponding elements above and below the leading diagonal are the same. i.e. $A = A^T$

   e.g. $A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 1 & -2 \\ 8 & -2 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 1 & -2 \\ 8 & -2 & 0 \end{bmatrix}$

6. **Inverse of a matrix.** A square matrix, X, is said to be invertible if there exists another matrix of the same dimension, denoted A^{-1} such that $AA^{-1} = A^{-1}A = I$. *The matrix* A^{-1} *is called the inverse of the matrix* A.

   Note: To have an inverse a matrix must be a square **but** not all square matrices have inverses.

Example 3.6:

If $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, prove that $B = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$ is A^{-1}

Solution:

If B is the inverse of A, the product $A \times B$ (or the product $B \times A$) should equal I_2

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} (1 \times 3 + 2 \times -1) & (1 \times -2 + 2 \times 1) \\ (1 \times 3 + 3 \times -1) & (1 \times -2 + 3 \times 1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \therefore B = A^{-1}$$

Example 3.7:

If $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 2 & -2 & 1 \end{bmatrix}$, prove that $B = \begin{bmatrix} 2 & -\frac{3}{2} & 1 \\ 1 & -\frac{1}{2} & 0 \\ -2 & 2 & -1 \end{bmatrix}$ is the inverse of A.

Solution:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -\frac{3}{2} & 1 \\ 1 & -\frac{1}{2} & 0 \\ -2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} (2+1-2) & \left(-\frac{3}{2}-\frac{1}{2}+2\right) & (1+0-1) \\ (4+0-4) & (-3+0+4) & (2+0-2) \\ (4-2-2) & (-3+1+2) & (2+0-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \therefore B = A^{-1}$$

We will find how to actually find inverses shortly.

Exercise Set 3.7

Consider these matrices

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 2 & -2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix}$$

$$C = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} \qquad G = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

(a) Without doing any calculations decide which matrices cannot be the inverse of A.

(b) What is the matrix D called? Describe its features.

(c) What is the relationship between matrix E and matrix G?

(d) Find all the matrices which also have their inverses shown.

Linear Equations in Matrix Form

Any equation (or inequality) can be written in matrix notation using matrix multiplication.

e.g. $3x + 2y = 5$

Can be written as $\underset{(1\times 2)}{\begin{bmatrix} 3 & 2 \end{bmatrix}} \underset{(2\times 1)}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{(1\times 1)}{5}$

and if we define $A = \begin{bmatrix} 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 5 \end{bmatrix}$ we get $AX = B$

Similarly, the simultaneous equations

$5x_1 + 3x_2 = 15$

$4x_1 - 2x_2 = 12$

can be written as $\underset{(2\times 2)}{\begin{bmatrix} 5 & 3 \\ 4 & -2 \end{bmatrix}} \underset{(2\times 1)}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{(2\times 1)}{\begin{bmatrix} 15 \\ 12 \end{bmatrix}}$

and if we define $A = \begin{bmatrix} 5 & 3 \\ 4 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $B = \begin{bmatrix} 15 \\ 12 \end{bmatrix}$

we get $AX = B$

We can use the same idea for any number of linear equations.

Example 3.8:

Write these simultaneous equations in matrix form

$$\begin{aligned} x_1 - 2x_2 + 3x_4 + x_5 &= 100 \\ 2x_1 - 3x_3 + x_4 &= 60 \\ 4x_2 - x_3 + 2x_4 + x_5 &= 125 \end{aligned}$$

Let $A = \underset{(3\times 5)}{\begin{bmatrix} 1 & -2 & 0 & 3 & 1 \\ 2 & 0 & -3 & 1 & 0 \\ 0 & 4 & -1 & 2 & 1 \end{bmatrix}}$, $X = \underset{(5\times 1)}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}$ and $B = \underset{(3\times 1)}{\begin{bmatrix} 100 \\ 60 \\ 125 \end{bmatrix}}$

Then $AX = B$ summarises the set of equations.

✪ For the examples above pre-multiply the column vector of variables, X by the matrix of coefficients, A and check that the resulting column vector B does indeed equal the RHS of the equations.

Now we can express linear equations in matrix form we can use some of attributes and algebra of matrices to solve them.

Consider $AX = B$ where $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

This is the matrix representation of the system of linear equations

$$\begin{aligned} x + 2y &= 1 \\ x + 3y &= 3 \end{aligned}$$

and we want to solve for x and y, i.e. solve for the elements of vector X. **See Note** 1

We will demonstrate the technique with this small system of two equations in two unknowns and then with a system of three equations in three unknowns. Keep in mind that the same technique can be used for any number of equations.

Consider $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ i.e. $AX = B$ and we want to find X.

If the RHS of the equation is premultiplied by the inverse of A, A^{-1}, then the LHS must also be premultiplied by A^{-1} to maintain a 'balanced' equation.

$A^{-1}A\,X = A^{-1}B$

Now we know that the product of a matrix and its inverse is the identity matrix

$$\begin{aligned} \therefore \quad A^{-1}A\,X &= A^{-1}B \\ \Rightarrow \quad IX &= A^{-1}B \end{aligned}$$

And we know that multiplying a matrix by the identity does not change the matrix.

$$\begin{aligned} \therefore \quad IX &= A^{-1}B \\ \Rightarrow \quad X &= A^{-1}B \end{aligned}$$

Thus we have found a way of finding the vector of unknowns i.e. we can obtain X.

What we need to do is to premultiply vector B (i.e. the RHSides of the original equations) by the **inverse** of the matrix of coefficients from the LHSides of the original equations.

Let's check that this does indeed work.

Notes

1. You could do this graphically or by substitution without much trouble. But say there were 10, 20 or 50 equations to solve – your existing techniques would indeed be tedious! Using matrix algebra we can find a simple technique which can easily be programmed for the computer so that the number of equations becomes irrelevant.

Example 3.9(a):

Solve
$$\begin{aligned} x + 2y &= 1 \quad \underline{\hspace{3em}} \quad ① \\ x + 3y &= 3 \quad \underline{\hspace{3em}} \quad ② \end{aligned}$$

using substitution or some other technique you know.

Solution: Subtract equation ① from equation ②

$\rightarrow$ $y = 2$

and substituting in ① for y and then solving for x

If $y = 2$, $x + 2y = 1$ gives
$$x + 4 = 1$$
$$\therefore \quad x = -3$$

So the solution to the set of equations is $x = -3$, $y = 2$. **See Note** 1

Example 3.9(b):

Use matrix algebra to solve the equations in Example 3.9(a) above.

Solution: We showed earlier (page 3.16) that the inverse of

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \text{ is } A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$\therefore$ If $X = A^{-1}B$

$$X = \underbrace{\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}}_{(2 \times 2)} \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{(2 \times 1)}$$

same $\therefore$ product defined $\rightarrow$ (2×1)

$$X = \begin{bmatrix} (3 \times 1 + -2 \times 3) \\ (-1 \times 1 + 1 \times 3) \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Now X is the vector of variables $\begin{bmatrix} x \\ y \end{bmatrix}$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \quad \therefore \quad x = -3 \text{ and } y = 2$$

Notes

1. Check solution by substituting in each of the original equations.

Exercise Set 3.8

1. (a) Write each set of simultaneous equations in matrix form. (Define all matrices and vectors)

(i) $\begin{aligned} -4x + 3y &= 8 \\ 3x - 2y &= 12 \end{aligned}$ (ii) $\begin{aligned} -x - 2y + 3z &= -4 \\ y - z &= 6 \\ 2x + y - 2z &= 2 \end{aligned}$

(b) Use **matrix algebra** to find the solution to each set of equations. [**Hint**: Use the inverses you found in Exercise Set 3.7]

Solution of a System of Linear Equations by Row Reduction

Before moving on to find the inverse of a matrix we need to introduce the Gauss-Jordan or row reduction technique for solving a system of linear equations as we will use this technique to find inverses of matrices.

We'll demonstrate the technique with a small system of two equations.

$$x + y = 300$$

$$3x + 2y = 700$$

In matrix notation, we can write this system as

$$\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 300 \\ 700 \end{bmatrix}$$

And then write the augmented matrix

$$\left[\begin{array}{cc:c} 1 & 1 & 300 \\ 3 & 2 & 700 \end{array}\right]$$

See Note 1

Notes

1. The augmented matrix just saves us carrying the x's and y's through the manipulations.

Now we want to perform a series of row operations on both sides of the augmented matrix until the LHS is the identity matrix and some numbers (actually the solution to the problem) are on the RHS. i.e. we end up with

$$\left[\begin{array}{cc:c} 1 & 0 & ?_1 \\ 0 & 1 & ?_2 \end{array}\right]$$

Then, if we reverse the augmentation process.

$$\left[\begin{array}{cc:c} 1 & 0 & ? \\ 0 & 1 & ? \end{array}\right] \quad \text{we get} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ?_1 \\ ?_2 \end{bmatrix}$$

i.e. $1 \times x + 0 \times y = ?_1$
$0 \times x + 1 \times y = ?_2$

i.e. $x = ?_1$
$y = ?_2$ and the solution to the system of equations has been found.

There are **three types of row operations** that may be applied to the augmented matrix.

See Note 1

- Divide (or multiply) a row by a number other than zero
- Subtract (or add) a multiple of one row from another
- Exchange two rows

To keep track of all the row operations

- we label each row of the augmented matrix starting with the top row as R_1
- we note the actual row operation that has been performed
- we indicate a new row e.g. a new row 2 with a dash (′) as a **superscript** e.g.

  $R_2 + R_1 \rightarrow R_2'$ means the new row two (R_2') was obtained by adding the old row one (R_1) to the old row two (R_2).

  $R_3 - 2R_2' \rightarrow R_3'$ means the new row three (R_3') was obtained by subtracting twice the new row two (R_2') from the old row three (R_3).

Reconsider our example.

$$\begin{aligned} x + y &= 300 \\ 3x + 2y &= 700 \end{aligned} \qquad \text{i.e.} \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 300 \\ 700 \end{bmatrix}$$

Notes

1. These operations are actually what you do when you solve simultaneous equations using the ordinary algebraic method.

In augmented matrix form

$$\left[\begin{array}{cc:c} 1 & 1 & 300 \\ 3 & 2 & 700 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \end{array}$$

We are aiming to use row reduction to get the form $\left[\begin{array}{cc:c} 1 & 0 & ?_1 \\ 0 & 1 & ?_2 \end{array}\right]$

Step 1: Generate a 1 in the **first** diagonal position by swapping rows or multiplying or dividing $\boldsymbol{R_1}$ by a constant. (We already have a 1 in this position so no need for step 1.)

Step 2: Generate a zero for each element under the **first** diagonal element by subtracting or adding an appropriate multiple or fraction of $\boldsymbol{R_1}$ from each of the rows **below** $\boldsymbol{R_1}$

$$\left[\begin{array}{cc:c} 1 & 1 & 300 \\ 0 & -1 & -200 \end{array}\right]\begin{array}{l} R_1 \\ R_2 - 3R_1 \to R_2' \end{array}$$

See Note 1

Step 3: Generate a 1 in the **second** diagonal position by swapping the second row with another row below it or multiplying or dividing $\boldsymbol{R_2}$ by a constant.

$$\left[\begin{array}{cc:c} 1 & 1 & 300 \\ 0 & 1 & 200 \end{array}\right]\begin{array}{l} R_1 \\ (-1)R_2 \to R_2' \end{array}$$

Step 4: Generate a zero for each element under the **second** diagonal element by subtracting or adding an appropriate multiple or fraction of $\boldsymbol{R_2}$ from each of the rows **below** R_2

There are no rows below R_2.

Step 5: The bottom LH corner of the LH matrix now has the correct form but the upper RH corner still needs modification. This involves working upwards and generating a zero **above** each of the diagonal elements.

We need to make the element **above** the **second** diagonal element a zero by using $\boldsymbol{R_2}$ as the operating row.

$$\left[\begin{array}{cc:c} 1 & 0 & 100 \\ 0 & 1 & 200 \end{array}\right]\begin{array}{l} R_1 - R_2 \to R_1' \\ R_2 \end{array}$$

See Note 2

Now we have the identity matrix on the LHS so row reduction is complete.

Notes

1. $(3\ 2 \ \vdots\ 700) - 3(1\ 1\ \vdots\ 300) \to (3\ 2\ \vdots\ 700) - (3\ 3\ \vdots\ 900) \to (0\ {-1}\ \vdots\ {-200})$
2. $(1\ 1\ \vdots\ 300) - (0\ 1\ \vdots\ 200) \to (1\ 0\ \vdots\ 100)$

Step 6: Reverse the augmentation process

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$$

$$\begin{aligned} \therefore \quad 1 \times x + 0 \times y &= 100 \qquad \text{i.e.} \quad x = 100 \\ 0 \times x + 1 \times y &= 200 \qquad \qquad \quad y = 200 \end{aligned}$$

Step 7: Check solution by substituting for x and y in the original equations.

If $x = 100$ and $y = 200$

$x + y = 100 + 200 = 300$ ✓

$3x + 2y = 3 \times 100 + 2 \times 200 = 700$ ✓

For larger systems of linear equations we simply repeat the steps 2–5 systematically as shown below.

Example 3.10:

Solve the system.

$$\begin{aligned} 2x + 3y - z &= 8 \\ x + y + z &= 2 \\ 3x - 4y + 5z &= -3 \end{aligned}$$

Solution:

In matrix form this becomes

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ -3 \end{bmatrix}$$

In augmented matrix form we get

$$\begin{array}{c} \left[\begin{array}{ccc:c} 2 & 3 & -1 & 8 \\ 1 & 1 & 1 & 2 \\ 3 & -4 & 5 & -3 \end{array}\right] \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Step 1: Generate a **1** in the **first** diagonal position. Swapping R_1 and R_2 achieves this.

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 2 \\ 2 & 3 & -1 & 8 \\ 3 & -4 & 5 & -3 \end{array}\right] \begin{array}{l} R_2 \to R_1' \\ R_2 \to R_2' \\ R_3 \end{array}$$

Step 2: Generate zeroes **below** the **first** diagonal position using $\boldsymbol{R_1}$ as the operating row.

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 4 \\ 0 & -7 & 2 & -9 \end{array}\right] \begin{array}{l} R_1 \\ R_2 - 2R_1 \to R_2' \\ R_3 - 3R_1 \to R_3' \end{array}$$

Step 3(a): Generate a **1** in the **second** diagonal position by multiplying or dividing R_2 by a constant. We already have a 1 in this position.

Step 4(a): Generate zeroes **below** the **second** diagonal position using $\boldsymbol{R_2}$ as the operating row.

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & -19 & 19 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 + 7R_2 \to R_3' \end{array}$$

Step 3(b): Generate a **1** in the **third** diagonal position by multiplying or dividing R_3 by a constant.

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \div (-19) \to R_3' \end{array}$$

Step 4(b): There are no elements below the third diagonal element.

Step 5(a): Generate zeroes **above** the **third** diagonal position using $\boldsymbol{R_3}$ as the operating row.

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 - R_3 \to R_1' \\ R_2 + 3R_3 \to R_2' \\ R_3 \end{array}$$

Step 5(b): Generate zeroes **above** the **second** diagonal position using $\boldsymbol{R_2}$ as the operating row.

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 - R_2 \to R_1' \\ R_2 \\ R_3 \end{array}$$

Now have I_3 on LHS of augmented matrix so row reduction is complete.

Step 6: Reverse the augmentation process

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned} \therefore\ x &= 2 \\ y &= 1 \\ z &= -1 \end{aligned}$$

Step 7: Check solution

$$\begin{aligned} \text{If}\quad x = 2,\ y &= 1 \text{ and } z = -1 \\ 2x + 3y - z &= 4 + 3 + 1 = 8 \ \checkmark \\ x + y + z &= 2 + 1 - 1 = 2 \ \checkmark \\ 3x - 4y + 5z &= 6 - 4 - 5 = -3 \ \checkmark \end{aligned}$$

As you become more familiar with this technique you will be able to condense the number of tableaux required. This has been done in the example below which you are to complete.

It is often tempting to omit the row operations notation at each step but their inclusion enables you to keep track of the changes you make and to detect errors.

Example 3.11:

Solve:

$$\begin{aligned} 2u + 3v - 2w &= 5 \\ 2u + v + w &= 11 \\ 3u + 2v - 3w &= 0 \end{aligned}$$

In matrix form

$$\begin{bmatrix} 2 & 3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix}$$

In augmented matrix form

$$\left[\begin{array}{ccc:c} & & & 5 \\ & & & 11 \\ & & & 0 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & \frac{3}{2} & -1 & \frac{5}{2} \\ 0 & -2 & 3 & 6 \\ & & & -\frac{15}{2} \end{array}\right] \begin{array}{l} R_1 \div \square \to R_1' \\ R_2 - 2R_1' \to R_2' \\ R_3 - 3R_1' \to R_3' \end{array}$$

Note: You must use the same operating row for all operations in the one tableau. Here R_1' is used.

See Note 1

Notes

1. The ~ symbol means 'is equivalent to' in this work.

$$\sim \left[\begin{array}{ccc:c} 1 & & & 7 \\ 0 & 1 & -\frac{3}{2} & \\ 0 & & & \frac{15}{2} \end{array}\right] \left.\begin{array}{l} R_1 - \frac{3}{2}R_2' \to R_1' \\ R_2 \div (-2) \to \boxed{} \\ R_3 + \frac{5}{2}R_2' \to R_3' \end{array}\right\} \text{Here } R_2' \text{ is the operating row}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array}\right] \left.\begin{array}{l} R_1 \boxed{} \boxed{} \to R_1' \\ R_2 + \frac{3}{2}R_3' \to R_2' \\ R_3 \times \boxed{} \to R_3' \end{array}\right\} \text{Here } R_3' \text{ is the operating row}$$

Row reduction is complete because $\boxed{}$ is on LHS

$\therefore\ u =$
$v =$
$w =$

Check solution

If $u = 2,\ v = 3$ and $w = 4$

$2u + 3v - 2w = \boxed{} + \boxed{} - \boxed{} = 5$ ✓

$\boxed{} + \boxed{} + \boxed{} = 4 + 3 + 4 = 11$ ✓

$\boxed{} + \boxed{} - \boxed{} = \boxed{} + \boxed{} - \boxed{} = \boxed{}$ ✓

How did you go with this activity? If you are not completely happy with your understanding, work through the steps of the procedure (with a separate tableau for each row reduction) to solve this simpler set of equations.

$$2x + 3y = 7$$
$$3x - 2y = -5$$

Now try to reduce the number of tableaux you need by doing several operations with the same operating row on the one tableau.

Exercise Set 3.9

1. Solve each system of equations using the Gauss-Jordan row reduction technique

(i) $\begin{aligned} -4x + 3y &= 8 \\ 3x - 2y &= 12 \end{aligned}$

(ii) $\begin{aligned} -x - 2y + 3z &= -4 \\ y - z &= 6 \\ 2x + y - 2z &= 2 \end{aligned}$

[Note: You solved these in Exercise Set 3.8 using $X = A^{-1}B$]

2. Use the Gauss-Jordan row reduction technique to solve the following

(a) $\begin{aligned} x + 2y &= 12 \\ 4x - 3y &= 4 \end{aligned}$

(b) $\begin{aligned} 2x - y &= 7 \\ -x + 2y &= -5 \end{aligned}$

(c) $\begin{aligned} x + y + z &= 6 \\ 3x + 4y + 2z &= 17 \\ -x + 2y + z &= 6 \end{aligned}$

(d) $\begin{aligned} x - y + z &= 3 \\ 2x + 3y - 4z &= 9 \\ -x + 2y - z &= 0 \end{aligned}$

(e) $\begin{aligned} -x + 2y + 3z &= 10 \\ 3x - 2y + z &= -16 \\ -2x - y - 4z &= 5 \end{aligned}$

(f) $\begin{aligned} x + y + z &= 1 \\ 2x - y + 4z &= -2 \\ -x + 3y - 3z &= -7 \end{aligned}$

3. If the Gauss-Jordan row reduction technique is successful in finding a solution a system of linear equations what does this tell you about the matrix of coefficients? Justify your answer.

Solution of Linear Equations Using the Inverse of the Coefficient Matrix

We showed earlier that a system of linear equations

e.g. $$\begin{aligned} x + y &= 300 \\ 3x + 2y &= 700 \end{aligned}$$

can be represented as the matrix equation

$$AX = B \quad \text{where} \quad A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 300 \\ 700 \end{bmatrix}$$

and that we can solve these equations using

$$X = A^{-1}B$$

There is an alternate method to the one just covered for solving linear systems. This method relies on us being able to find the inverse matrix, A^{-1} of the matrix of coefficients A. (Recall that we said earlier that not all matrices have inverses. However the problems that you will meet in this level of mathematics will generally involve A matrices which do have inverses.)

So our task is to find the inverse of the square matrix of coefficients, A.

Inverse Matrices

One way of finding the inverse is to use the Gauss-Jordan technique you have recently mastered.

If B is the inverse of A, then $AB = I$

Say A is (2×2), then B is (2×2) and I is I_2

i.e. $$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then we have to find the values of b_{11}, b_{12}, b_{21} and b_{22}. This is similar to finding x and y when we solved

$$\begin{aligned} x + y &= 300 \\ 3x + 2y &= 700 \end{aligned}$$

i.e. $$\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 300 \\ 700 \end{bmatrix}$$

To find x and y we formed the augmented matrix of the first matrix on the LHS of the equation and the matrix on the RHS of the equation.

$$\left[\begin{array}{cc:c} 1 & 1 & 300 \\ 3 & 2 & 700 \end{array}\right]$$

and performed row operations until the LHS of the augmented matrix was reduced to the identity matrix. Then the reduced RHS of the augmented matrix gave the values of x and y.

To find the b_{11}, b_{12}, b_{21} and b_{22} in the inverse matrix we follow the same procedure.

The matrix equation we have is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So we form the augmented matrix of the first matrix on the LHS of the equation and the matrix on the RHS of the equation

$$\left[\begin{array}{cc:cc} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \end{array}\right]$$

and perform row operations until the LHS of the augmented matrix reduces to the identity matrix. Then the reduced RHS of the augmented matrix will give the values of b_{11}, b_{12}, b_{21} and b_{22} i.e. the elements of the matrix B which is the inverse of A.

Let's try the method on the matrix of coefficients, A from

$$\begin{aligned} x + y &= 300 \\ 3x + 2y &= 700 \end{aligned}$$

Example 3.12:

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$

We know that $A\,A^{-1} = I_2$

Form $\left[A \;\vdots\; I_2\right]$

$$\left[\begin{array}{cc:cc} 1 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \end{array}$$

$$\sim \left[\begin{array}{cc:cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right]\begin{array}{l} R_1 \\ R_2 - 3R_1 \to R_2' \end{array} \qquad \sim \left[\begin{array}{cc:cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & -1 \end{array}\right]\begin{array}{l} R_1 - R_2' \to R_1' \\ R_2 \times (-1) \to R_2' \end{array}$$

LHS of augmented matrix has reduced to I_2 so RHS of augmented matrix gives A^{-1}, the inverse of A.

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$$

Check solution by showing $A\,A^{-1} = I_2$ (or $A^{-1}A = I_2$)

$$A\,A^{-1} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (1 \times -2 + 1 \times 3) & (1 \times 1 + 1 \times -1) \\ (3 \times -2 + 2 \times 3) & (3 \times 1 + 2 \times -1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \quad \checkmark$$

Now we can use the inverse matrix to solve our system of linear equations.

$$\begin{aligned} x + y &= 300 \\ 3x + 2y &= 700 \end{aligned}$$

i.e. $AX = B$ where $A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 300 \\ 700 \end{bmatrix}$

$$X = A^{-1}B$$

$$= \underbrace{\begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}}_{(2 \times 2)}\underbrace{\begin{bmatrix} 300 \\ 700 \end{bmatrix}}_{(2 \times 1)} \qquad = \begin{bmatrix} -2 \times 300 + 1 \times 700 \\ 3 \times 300 + -1 \times 700 \end{bmatrix}$$

Same $\therefore$ product defined

$\downarrow$

(2×1)

$$= \begin{bmatrix} 100 \\ 200 \end{bmatrix}$$

i.e. $x = 100$ and $y = 200$ which we know is correct

Let's solve one of the three equation systems (which we have previously solved using elimination) using the inverse coefficient matrix method.

Example 3.13:

Solve the following for u, v and w

$$\begin{aligned} 2u + 3v - 2w &= 5 \\ 2u + v + w &= 11 \\ 3u + 2v - 3w &= 0 \end{aligned}$$

Solution:

$$AX = B \quad \text{where} \quad A = \begin{bmatrix} 2 & 3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & -3 \end{bmatrix}, \quad X = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 \\ 11 \\ 0 \end{bmatrix}$$

If $\quad AX = B$
$\quad\quad X = A^{-1}B$

$$\left[A \mid I_3\right] = \left[\begin{array}{ccc|ccc} 2 & 3 & -2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & 2 & -3 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{3}{2} & -1 & \frac{1}{2} & 0 & 0 \\ 0 & -2 & 3 & -1 & 1 & 0 \\ 0 & -\frac{5}{2} & 0 & -\frac{3}{2} & 0 & 1 \end{array}\right] \begin{array}{l} R_1/2 \to R_1' \\ R_2 - 2R_1' \to R_2' \\ R_3 - 3R_1' \to R_3' \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{3}{2} & -1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{15}{4} & -\frac{1}{4} & -\frac{5}{4} & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2/(-2) \to R_2' \\ R_3 + \frac{5}{2}R_2' \to R_3' \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{3}{2} & 0 & \frac{17}{30} & \frac{1}{3} & -\frac{4}{15} \\ 0 & 1 & 0 & \frac{3}{5} & 0 & -\frac{2}{5} \\ 0 & 0 & 1 & \frac{1}{15} & \frac{1}{3} & -\frac{4}{15} \end{array}\right] \begin{array}{l} R_1 + R_3' \to R_1' \\ R_2 + \frac{3}{2}R_3' \to R_2' \\ R_3 \times (-\frac{4}{15}) \to R_3' \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{3}{5} & 0 & -\frac{2}{5} \\ 0 & 0 & 1 & \frac{1}{15} & \frac{1}{3} & -\frac{4}{15} \end{array}\right] \begin{array}{l} R_1 - \frac{3}{2}R_2 \to R_1' \\ R_2 \\ R_3 \end{array}$$

Now the LHS of the augmented matrix is I_3, so the RHS must be A^{-1}, assuming we've made no errors! Before proceeding to find X, let's check that A^{-1} **is** correct by showing $AA^{-1} = I_3$

$$AA^{-1} = \begin{bmatrix} 2 & 3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & -3 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{3}{5} & 0 & -\frac{2}{5} \\ \frac{1}{15} & \frac{1}{3} & -\frac{4}{15} \end{bmatrix} = \begin{bmatrix} (-\frac{2}{3}+\frac{9}{5}-\frac{2}{15}) & (\frac{2}{3}+0-\frac{2}{3}) & (\frac{2}{3}-\frac{6}{5}+\frac{8}{15}) \\ (-\frac{2}{3}+\frac{3}{5}+\frac{1}{15}) & (\frac{2}{3}+0+\frac{1}{3}) & (\frac{2}{3}-\frac{2}{5}-\frac{4}{15}) \\ (1-\frac{4}{5}+\frac{4}{5}) & (1+0-1) & (1-\frac{4}{5}+\frac{4}{5}) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Thus we know A^{-1} is correct.

$$\therefore \quad X = A^{-1}B = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{3}{5} & 0 & -\frac{2}{5} \\ \frac{1}{15} & \frac{1}{3} & -\frac{4}{15} \end{bmatrix} \begin{bmatrix} 5 \\ 11 \\ 0 \end{bmatrix} = \begin{bmatrix} (-\frac{5}{3}+\frac{11}{3}+0) \\ (3+0+0) \\ (\frac{1}{3}+\frac{11}{3}+0) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$(3 \times 3) \quad (3 \times 1)$

same $\rightarrow$ Product defined

(3×1)

i.e. $u = 2$; $v = 3$ and $w = 4$ which is the same solution we obtained earlier.

Inverses of matrices have wide application in mathematics (not just for solving simple linear systems) so I suggest you spend some time on this section before proceeding. Make sure you do all the exercises.

Exercise Set 3.10

1. For each problem in Question 2 of Exercise Set 3.9

   (i) Find the inverse of the matrix of coefficients, A. Check your inverses by showing $AA^{-1} = I$ (or $A^{-1}A = I$)

   (ii) Rewrite each inverse matrix so it contains integers only. (You'll need to take out a common factor from the elements of the matrix and use it as a pre-multiplier in fractional form.)

   (iii) Perform an appropriate multiplication using the inverse matrix to again verify the solution to each problem.

2. (i) Solve this set of simultaneous equations using the inverse of the matrix of coefficients

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 4y + 5z &= 9 \\ 3x + 5y + 6z &= 1 \end{aligned}$$

   (ii) Verify your solution

Determinant of a Square Matrix

Before closing this chapter I want to introduce you to a characteristic of a square matrix called the **determinant**. If a square matrix has a determinant which is **zero**, it will **not** have an **inverse**. So before doing the work to find the inverse of a matrix it is a good idea to check if the determinant of the matrix is non zero. There are several methods for finding the determinant of a matrix. We will use the following method.

Note: The shorthand way of writing 'the determinant of A' is det A or $|A|$. (Do not confuse with the absolute value symbol)

If A is a (2×2) matrix

i.e. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

determinant of A i.e. $\det A = a_{11}\, a_{22} - a_{12}\, a_{21}$

Example 3.14:

If $A = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$, find the determinant of A. Will A have an inverse?

Solution:

$$\begin{aligned} \det A &= (3 \times 1) - (-2 \times -1) \\ &= 3 - 2 \\ &= 1 \end{aligned}$$

$\therefore$ as $\det A$ is non zero, A will have and inverse.

We showed in **Example 3.9(b)** *that the inverse of* A *is* $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

If A is a (3×3) matrix

i.e. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 3.15:

If $A = \begin{bmatrix} 2 & 3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & -3 \end{bmatrix}$, find the determinant of A. Will A have an inverse?

Solution:

$$\det A = 2 \times \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} - 3 \times \begin{vmatrix} 2 & 1 \\ 3 & -3 \end{vmatrix} + (-2)\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$

Now use the method for a (2×2) matrix to work out the determinants of the (2×2) matrices

$$\begin{aligned} \det A &= 2(1 \times -3 - 1 \times 2) - 3(2 \times -3 - 1 \times 3) - 2(2 \times 2 - 1 \times 3) \\ &= 2 \times -5 - 3 \times -9 - 2 \times 1 \\ &= -10 + 27 - 2 \\ &= 15 \end{aligned}$$

$\therefore$ as $\det A$ is non zero, A will have an inverse.

We found the inverse of A *in* **Example 3.13**

Exercise Set 3.11

1. For each problem of Question 2 of Exercise Set 3.9, find the determinant of the matrix of coefficients, A.

2. Find the determinant to determine which of the following matrices have an inverse.

(a) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -1 \\ -3 & -5 \end{bmatrix}$

(d) $\begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -4 & 5 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}$

(g) $\begin{bmatrix} 6 & -4 & 4 \\ 4 & 2 & -2 \\ -3 & 2 & 5 \end{bmatrix}$

(h) $\begin{bmatrix} 0 & 4 & 6 & 3 \\ 2 & 7 & -1 & 5 \\ 4 & 2 & -3 & 1 \end{bmatrix}$

Solutions to Exercise Sets

Solutions Exercise Set 3.1 page 3.2

1. $a_{2,2} = 3$ $\quad a_{4,2} = 2$

   $a_{4,1} = 3$ $\quad a_{2,4}$ does not exist as matrix A has only 3 columns

   $a_{1,1} = 4$

2. $a_{1,3}$ is the amount of plastic needed to make one hip pad

Solutions Exercise Set 3.2 page 3.3

1. (a) P is 3 rows $\times$ 2 columns; i.e. (3×4) $\quad p_{3,2} = 2$

   (b) Q is 2 rows $\times$ 3 columns; i.e. (2×3) $\quad q_{2,2} = 8$

   (c) R is 4 rows $\times$ 4 columns; i.e. (4×4) $\quad r_{4,1} = 8$

   (d) S is 3 rows by 3 columns; i.e. (3×3) $\quad s_{3,2} = 0$

Solutions Exercise Set 3.3 page 3.4

(a) $P^T = \begin{bmatrix} 2 & 5 & 6 \\ 4 & 1 & 2 \end{bmatrix}$ $\quad P^T$ is (2×3)

(b) $Q^T = \begin{bmatrix} 1 & 0.2 \\ 2 & 8 \\ 4 & 3 \end{bmatrix}$ $\quad Q^T$ is (3×2)

(c) $R^T = \begin{bmatrix} \frac{1}{4} & 0.2 & -0.2 & 8 \\ \frac{1}{8} & 4 & -1 & -2 \\ 2 & 6 & 8 & 2 \\ 0 & 0 & 4 & 8 \end{bmatrix}$ $\quad R^T$ is (4×4)

(d) $S^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad S^T$ is (3×3)

Solutions Exercise Set 3.4 page 3.7

1. (a) $A + B$

Check A and B have the same dimension. A is (3×4) and B is (3×4)

$\therefore \quad A + B$ is possible

$$A + B = \begin{bmatrix} 2+0 & 4+4 & -1+6 & 6+3 \\ -7+2 & 1+7 & 8+(-1) & 3+5 \\ 5+4 & 2+2 & 0+(-3) & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 8 & 5 & 9 \\ -5 & 8 & 7 & 8 \\ 9 & 4 & -3 & 2 \end{bmatrix}$$

(b) $C - A$

C is (3×2) and A is (3×4)

$\therefore \quad C - A$ is not possible

(c) $E + D$

E is (3×3) and D is (2×3)

$\therefore \quad E + D$ is not possible

(d) $E + F$

E is (3×3) and F is (3×3)

$\therefore \quad E + F$ is possible

$$E + F = \begin{bmatrix} 7+1 & 21+0 & 21+0 \\ 7+0 & 14+1 & 21+0 \\ 14+0 & 35+0 & 42+1 \end{bmatrix} = \begin{bmatrix} 8 & 21 & 21 \\ 7 & 15 & 21 \\ 14 & 35 & 43 \end{bmatrix}$$

(e) $C - F$

C is (3×2) and F is (3×3)

$\therefore \quad C - F$ is not possible

(f) $B - A$

We know B and A have the same dimension

$$B - A = \begin{bmatrix} 0-2 & 4-4 & 6-(-1) & 3-6 \\ 2-(-7) & 7-1 & -1-8 & 5-3 \\ 4-5 & 2-2 & -3-0 & 1-1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 7 & -3 \\ 9 & 6 & -9 & 2 \\ -1 & 0 & -3 & 0 \end{bmatrix}$$

(g) $$A - B = \begin{bmatrix} 2-0 & 4-4 & -1-6 & 6-3 \\ -7-2 & 1-7 & 8-(-1) & 3-5 \\ 5-4 & 2-2 & 0-(-3) & 1-1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -7 & 3 \\ -9 & -6 & 9 & -2 \\ 1 & 0 & 3 & 0 \end{bmatrix}$$

Note carefully $A - B \neq B - A$. Just as it is with numbers and algebra, matrix subtraction is not commutative.

Solutions Exercise Set 3.4 cont.

2. (i) Each set of results may be seen as a matrix, so the total farm production for each variety from each fertilizer is given by the sum of the three matrices.

$$\begin{bmatrix} 10 & 13 & 11 & 8 \\ 12 & 14 & 11 & 9 \\ 11 & 11 & 13 & 12 \end{bmatrix} + \begin{bmatrix} 11 & 13 & 12 & 9 \\ 11 & 13 & 12 & 9 \\ 10 & 12 & 12 & 11 \end{bmatrix} + \begin{bmatrix} 11 & 13 & 12 & 9 \\ 12 & 14 & 12 & 9 \\ 11 & 12 & 13 & 12 \end{bmatrix} = \begin{array}{c} \\ \text{F1} \\ \text{F2} \\ \text{F3} \end{array} \begin{array}{c} \begin{array}{cccc} \text{VA} & \text{VB} & \text{VC} & \text{VD} \end{array} \\ \begin{bmatrix} 32 & 39 & 35 & 26 \\ 35 & 41 & 35 & 27 \\ 32 & 35 & 38 & 35 \end{bmatrix} \end{array}$$

So, for example, the total farm production for variety D when fertilizer 3 was used was 35 as shown by element (3, 4).

(ii) Total production for Variety A = element (1, 1) + element (2,1) + element (3,1)
= 32 + 35 + 32 = 99

Total production for Variety B = (1,2) + (2,2) + (2,3)
= 39 + 41 + 35 = 115

Total production for Variety C = (1,3) + (2,3) + (3,3)
= 35 + 35 + 38 = 108

Total production for Variety D = (1,4) + (2,4) + (3,4)
= 26 + 27 + 35 = 88

(iii) Total farm production = 99 + 115 + 108 + 88 = 410

Solutions Exercise Set 3.5 page 3.10

1. (a) T has 1 row and 4 columns

(b) There are 4 elements in T

$t_{1,1} = 200$; $t_{1,2} = 105$; $t_{1,3} = 125$; $t_{1,4} = 165$

(c) $$t_{1,1} = a_{1,1} \times d_{1,1} + a_{1,2} \times d_{2,1} + a_{1,3} \times d_{3,1} = \sum_{j=1}^{3} a_{1,j}\ d_{j,1}$$

$$t_{1,2} = a_{2,1} \times d_{1,1} + a_{2,2} \times d_{2,1} + a_{2,3} \times d_{3,1} = \sum_{j=1}^{3} a_{2,j}\ d_{j,1}$$

$$t_{1,3} = a_{3,1} \times d_{1,1} + a_{3,2} \times d_{2,1} + a_{3,3} \times d_{3,1} = \sum_{j=1}^{3} a_{3,j}\ d_{j,1}$$

$$t_{1,4} = a_{4,1} \times d_{1,1} + a_{4,2} \times d_{2,1} + a_{4,3} \times d_{3,1} = \sum_{j=1}^{3} a_{4,j}\ d_{j,1}$$

Solutions Exercise Set 3.6 page 3.13

1. Average $= \frac{1}{3}$ of total production of each variety

$$= \frac{1}{3} \times \begin{array}{c} \begin{array}{cccc} \text{VA} & \text{VB} & \text{VC} & \text{VD} \end{array} \\ \begin{bmatrix} 99 & 115 & 108 & 88 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{cccc} \text{VA} & \text{VB} & \text{VC} & \text{VD} \end{array} \\ \begin{bmatrix} 33 & 38\frac{1}{3} & 36 & 29\frac{1}{3} \end{bmatrix} \end{array}$$

2. (a) $A \times B$

A is (3×4) and B is (3×4)

different $\therefore A \times B$ is not defined

(b) $B^T \times C$

B^T is (4×3) and C is (3×2)

same

$A \times B$ is defined and is (4×2)

$$B^T \times C = \begin{bmatrix} 0 & 2 & 4 \\ 4 & 7 & 2 \\ 6 & -1 & -3 \\ 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 3 & 3 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} (0 \times 3 + 2 \times 3 + 4 \times 5) & (0 \times 1 + 2 \times 3 + 4 \times 4) \\ (4 \times 3 + 7 \times 3 + 2 \times 5) & (4 \times 1 + 7 \times 3 + 2 \times 4) \\ (6 \times 3 + (-1) \times 3 + (-3) \times 5 & (6 \times 1 + (-1) \times 3 + (-3) \times 4) \\ (3 \times 3 + 5 \times 3 + 1 \times 5) & (3 \times 1 + 5 \times 3 + 1 \times 4) \end{bmatrix}$$

$$= \begin{bmatrix} 26 & 22 \\ 43 & 33 \\ 0 & -9 \\ 29 & 22 \end{bmatrix}$$

Always check the dimension of your final matrix. Is it what you predicted? We expected $B^T \times C$ to give a (4×2) matrix and this is indeed the size of the product.

Solutions Exercise Set 3.6 cont.

2. (c) $-D \times C$

$-D$ is $(-1)\,D \;\therefore\; -D$ is (2×3) and C is (3×2)

same

$\rightarrow$ $-D \times C$ is defined and is (2×2)

$$-D \times C = \begin{bmatrix} -2 & -7 & -6 \\ -1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 3 & 3 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} ((-2)\times 3 + (-7)\times 3 + (-6)\times 5) & ((-2)\times 1 + (-7)\times 3 + (-6)\times 4) \\ ((-1)\times 3 + 0\times 3 + (-3)\times 5) & ((-1)\times 1 + 0\times 3 + (-3)\times 4) \end{bmatrix}$$

$$= \begin{bmatrix} -57 & -47 \\ -18 & -13 \end{bmatrix}$$

Check is the matrix (2×2) ?✓

(d) $E^T \times F^T$

E^T is (3×3) and F^T is (3×3)

same

$\rightarrow$ $ET \times FT$ is defined and is (3×3)

$$E^T \times F^T = \begin{bmatrix} 7 & 7 & 14 \\ 21 & 14 & 35 \\ 21 & 21 & 42 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} (7\times 1 + 0 + 0) & (0 + 7\times 1 + 0) & (0 + 0 + 14\times 1) \\ (21\times 1 + 0 + 0) & (0 + 14\times 1 + 0) & (0 + 0 + 35\times 1) \\ (21\times 1 + 0 + 0 & (0 + 21\times 1 + 0) & (0 + 0 + 42\times 1) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 7 & 14 \\ 21 & 14 & 35 \\ 21 & 21 & 42 \end{bmatrix}$$

Check! (3×3) ✓

Solutions Exercise Set 3.6 cont.

3. (a) In October the production must be tripled

$$\therefore \quad 3 \times \begin{bmatrix} 150 & 300 & 80 \\ 50 & 100 & 150 \\ 200 & 100 & 500 \end{bmatrix} = \begin{bmatrix} 3\times 150 & 3\times 300 & 3\times 80 \\ 3\times 50 & 3\times 100 & 3\times 150 \\ 3\times 200 & 3\times 100 & 3\times 500 \end{bmatrix}$$

$$= \begin{bmatrix} 450 & 900 & 240 \\ 150 & 300 & 450 \\ 600 & 300 & 1500 \end{bmatrix}$$

$\therefore$ the production schedule in October should be

$$\begin{array}{l} \\ \text{Small} \\ \text{Medium} \\ \text{Large} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Plant 1} & \text{Plant 2} & \text{Plant 3} \end{array} \\ \begin{bmatrix} 450 & 900 & 240 \\ 150 & 300 & 450 \\ 600 & 300 & 1500 \end{bmatrix} \end{array}$$

(b) (i) Costs for each bear are the making cost and the transport cost

$$C = \begin{array}{l} \\ \text{Small} \\ \text{Medium} \\ \text{Large} \end{array} \begin{array}{c} \begin{array}{cc} \text{Making Cost} & \text{Transport Cost} \end{array} \\ \begin{bmatrix} 4 & 2 \\ 7 & 3 \\ 15 & 4 \end{bmatrix} \end{array}$$

(ii) M is (3×3) and the resultant matrix is to be (3×1)

$\therefore$ The matrix product must be

$(3 \times 3) \quad \times \quad (? \times ?)$

must be the same

$\rightarrow (3 \times 1)$

must be 'outside' dimension
of unknown matrix

$\therefore$ Unknown matrix will be (3×1), i.e. a column vector. This vector must have the effect in the product of adding up the rows of M. (i.e. giving the total number of each type of bear).

The vector required is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Solutions Exercise Set 3.6 cont.

3. (b) (ii) continued

$$N = \begin{bmatrix} 450 & 900 & 240 \\ 150 & 300 & 450 \\ 600 & 300 & 1500 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (450 \times 1 + 900 \times 1 + 240 \times 1) \\ (150 \times 1 + 300 \times 1 + 450 \times 1) \\ (600 \times 1 + 300 \times 1 + 1500 \times 1) \end{bmatrix} = \begin{array}{cl} \begin{bmatrix} 1590 \\ 900 \\ 2400 \end{bmatrix} & \begin{array}{l} \text{Total Small} \\ \text{Total Medium} \\ \text{Total Large} \end{array} \end{array}$$

Note that multiplying by a row or column vector where each element is one is how you 'add up' to get totals when the data is in a matrix. You have to be careful to get the order of the multiplication correct.

(iii) N is (3 × 1) and C is (C × 2), so to multiply them we need to take the transpose of either N or C.

N^T is (1 × 3)

$\therefore \; N^T \times C$

(1 × 3) (3 × 2)

same

N^TC is defined and is (1 × 2) {as required}

Note: To get a (1 × 2) resultant matrix N **had** to be transposed and used as the premultiplier.

$$N^T \times C = \begin{bmatrix} 1590 & 900 & 2400 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 7 & 3 \\ 15 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (1590 \times 4 + 900 \times 7 + 2400 \times 15) & (1590 \times 2 + 900 \times 3 + 2400 \times 4) \end{bmatrix}$$

$$= \begin{bmatrix} (6360 + 6300 + 3600) & (3180 + 2700 + 9600) \end{bmatrix}$$

$$= \begin{bmatrix} 48660 & 15480 \end{bmatrix}$$

$\therefore$ in October the total costs for production will be \$48 660 and the total costs for transport will be \$15 480.

If you wanted to overall costs obviously you simply add these two figures. However applying the principle we used earlier we can't premultiply $\begin{bmatrix} 48660 & 15480 \end{bmatrix}$ by a column vector of one's.

$$\underset{(1 \times 2)}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \; \underset{(2 \times 1)}{\begin{bmatrix} 48660 & 15480 \end{bmatrix}} = \underset{(1 \times 1)}{\begin{bmatrix} (1 \times 48660 + 1 \times 15480) \end{bmatrix}} = \begin{bmatrix} 64140 \end{bmatrix}$$

The overall costs for October will be \$64140.

Solutions Exercise Set 3.6 cont.

4. As each assignment is marked out of a different total, the marks must be adjusted before totalling.

$$5\times\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}\begin{array}{c}\begin{array}{cccccc} \text{ASS1} & \text{ASS2} & \text{ASS3} & \text{ASS4} & \text{ASS5} & \text{ASS6}\end{array}\\ \begin{bmatrix} 21 & 18 & 40 & 32 & 30 & 30 \\ 18 & 14 & 25 & 22 & 32 & 12 \\ 30 & 18 & 42 & 35 & 39 & 28 \\ 32 & 15 & 37 & 28 & 38 & 22 \\ 29 & 16 & 12 & 27 & 10 & 18 \\ 27 & 19 & 15 & 32 & 30 & 30 \end{bmatrix}\end{array}\begin{bmatrix} \frac{1}{35} \\ \frac{1}{20} \\ \frac{1}{50} \\ \frac{1}{35} \\ \frac{1}{40} \\ \frac{1}{35} \end{bmatrix} = 5\times\begin{bmatrix} (\frac{21}{35}+\frac{18}{20}+\frac{40}{50}+\frac{32}{35}+\frac{30}{40}+\frac{30}{35}) \\ (\frac{18}{35}+\frac{14}{20}+\frac{25}{50}+\frac{22}{35}+\frac{32}{40}+\frac{12}{35}) \\ (\frac{30}{35}+\frac{18}{20}+\frac{42}{50}+\frac{35}{35}+\frac{39}{40}+\frac{28}{50}) \\ (\frac{32}{35}+\frac{15}{20}+\frac{37}{50}+\frac{28}{35}+\frac{38}{40}+\frac{22}{50}) \\ (\frac{29}{35}+\frac{16}{20}+\frac{12}{50}+\frac{27}{35}+\frac{10}{40}+\frac{18}{50}) \\ (\frac{27}{35}+\frac{19}{20}+\frac{15}{50}+\frac{32}{35}+\frac{30}{40}+\frac{30}{50}) \end{bmatrix}$$

$$= 5\times\begin{bmatrix} 4.82 \\ 3.82 \\ 5.13 \\ 4.59 \\ 3.25 \\ 4.29 \end{bmatrix} = \begin{bmatrix} 24.1 \\ 19.1 \\ 25.7 \\ 23.0 \\ 16.3 \\ 21.5 \end{bmatrix}\begin{array}{l} \text{Assignment \% for Student A} \\ \text{Assignment \% for Student B} \\ \text{Assignment \% for Student C} \\ \text{Assignment \% for Student D} \\ \text{Assignment \% for Student E} \\ \text{Assignment \% for Student F} \end{array}$$

Now before totalling the % from each part of the assessment, the exam marks must be modified.

$$\underset{(1\times 3)}{\begin{bmatrix} 1 & 1 & \frac{65}{125} \end{bmatrix}} \underset{(3\times 6)}{\begin{bmatrix} 24.1 & 19.1 & 25.7 & 23.0 & 16.3 & 21.5 \\ 4.5 & 4.0 & 4.5 & 3.5 & 4.5 & 5 \\ 89 & 42 & 110 & 70 & 49 & 80 \end{bmatrix}}\begin{array}{l} \text{Assignment} \\ \text{Essay} \\ \text{Exam} \end{array}$$

$$\begin{bmatrix}(24.1+4.5+46.3) & (19.1+4+21.8) & (25.7+4.5+57.2) & (23+3.5+36.4) & (16.3+4.5+25.5) & (21.5+5+41.6)\end{bmatrix}$$

$$\begin{bmatrix}74.9 & 44.9 & 87.4 & 62.9 & 46.3 & 68.1\end{bmatrix}$$

$\therefore$ The results for the students are A = 75%, B = 45%, C = 87%, D = 63%, E = 46%, F = 68%.

There are many ways of doing this problem.

Solutions Exercise Set 3.7 page 3.17

(a) A is a (3×3) matrix so its inverse is (3×3). This means C, E and G cannot be its inverse. Matrix D is (3×3) but it is the identity matrix so multiplying matrix A by it can only give A, so D cannot be the inverse of A.

(b) Matrix D is the (3×3) identity matrix i.e. I_3. Its features are

- square (3×3)
- symmetric (i.e. $D = D^T$)
- all elements $d_{ij} = 0$ except when i = j, then $d_{ij} = 1$

(c) $E = G^T$ or $G = E^T$ or $E = G$

(d) If $B = A^{-1}$, then $AB = I_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -5 & 6 \end{bmatrix}$$

This is not I_3 $\therefore$ $B \neq A^{-1}$

If $F = A^{-1}$, then $AF = I_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 7 & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$$

No need to continue multiplication because 1st diagonal element is not 1.

This is not I_3 $\therefore$ $F \neq A^{-1}$

If $B = F^{-1}$, then $BF = I_3$

$$\begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is I_3 $\therefore$ $B = F^{-1}$ (or $F = B^{-1}$)

If $C = E^{-1}$, then $CE = I_2$

$$\begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This is I_2 $\therefore$ $C = E^{-1}$ (or $E = C^{-1}$)

As $E = F$, C is also F^{-1}.

Solutions Exercise Set 3.8 page 3.21

1. (a) (i) Let $A = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$ $\quad X = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad$ and $\quad B = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$

Then $AX = B$

(ii) Let $A = \begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix}$ $\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad$ and $\quad B = \begin{bmatrix} -4 \\ 6 \\ 2 \end{bmatrix}$

Then $AX = B$

(b) (i) From Exercise Set 3.7, $\quad A^{-1} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

$\therefore\ X = A^{-1}B$

$$= \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \end{bmatrix} = \begin{bmatrix} (16 + 36) \\ (24 + 48) \end{bmatrix} = \begin{bmatrix} 52 \\ 72 \end{bmatrix}$$

$\therefore\ x = 52$ and $y = 72$

Checking: when $x = 52$ and $y = 72$

$-4x + 3y = -4 \times 52 + 3 \times 72 = 8$ ✓
$3x - 2y = 3 \times 52 - 2 \times 72 = 12$ ✓

(ii) From Exercise Set 3.7, $\quad A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$

$\therefore\ X = A^{-1}B$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} (-4 + 6 + 2) \\ (-8 + 24 + 2) \\ (-8 + 18 + 2) \end{bmatrix} = \begin{bmatrix} 4 \\ 18 \\ 12 \end{bmatrix}$$

$\therefore\ x = 4,\ y = 18$ and $y = 12$

Check by substituting in each of the original equations.

Solutions Exercise Set 3.9 page 3.28

1. (i) $-4x + 3y = 8$
$3x - 2y = 12$

In augmented matrix form

$$\left[\begin{array}{cc|c} -4 & 3 & 8 \\ 3 & -2 & 12 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \end{array} \sim \left[\begin{array}{cc|c} 1 & -\frac{3}{4} & -2 \\ 3 & -2 & 12 \end{array}\right]\begin{array}{l} R_1 \div (-4) \to R' \\ R_2 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & -\frac{3}{4} & -2 \\ 0 & \frac{1}{4} & 18 \end{array}\right]\begin{array}{l} R_1 \\ R_2 - 3R_1 \to R_2' \end{array} \qquad (3 \;\; -2 \,\vdots\, 12) - (3 \;\; -\tfrac{9}{4} \,\vdots\, -6) \to (0 \;\; \tfrac{1}{4} \,\vdots\, 18)$$

$$\sim \left[\begin{array}{cc|c} 1 & -\frac{3}{4} & -2 \\ 0 & 1 & 72 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \times 4 \to R_2' \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 52 \\ 0 & 1 & 72 \end{array}\right]\begin{array}{l} R_1 + \frac{3}{4}R_2 \to R_1' \\ R_2 \end{array} \qquad (1 \;\; -\tfrac{3}{4} \,\vdots\, -2) - (0 \;\; \tfrac{3}{4} \,\vdots\, 54) \to (1 \;\; 0 \,\vdots\, 52)$$

$\therefore\ x = 52$ and $y = 72$ (The same result we obtained before)

(ii) $-x - 2y + 3z = -4$
$y - z = 6$
$2x + y - 2z = 2$

In augmented matrix form

$$\left[\begin{array}{ccc|c} -1 & -2 & 3 & -4 \\ 0 & 1 & -1 & 6 \\ 2 & 1 & -2 & 2 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -1 & 6 \\ 2 & 1 & -2 & 2 \end{array}\right]\begin{array}{l} R_1 \div (-1) \to R_1' \\ R_2 \\ R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -1 & 6 \\ 0 & -3 & 4 & -6 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 - 2R_1' \to R_3' \end{array}$$

$\{R_2$ already has a zero under the 1st diagonal element$\}$

$$(2 \;\; 1 \;\; -2 \,\vdots\, 2) - (2 \;\; 4 \;\; -6 \,\vdots\, 8) \to (0 \;\; -3 \;\; 4 \,\vdots\, -6)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 1 & 12 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 + 3R_2 \to R_3' \end{array}$$

$\{R_2$ already has a 1 in the 2nd diagonal element$\}$

$$(0 \;\; -3 \;\; 4 \,\vdots\, -6) + (0 \;\; 3 \;\; -3 \,\vdots\, 18) \to (0 \;\; 0 \;\; 1 \,\vdots\, 12)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 40 \\ 0 & 1 & 0 & 18 \\ 0 & 0 & 1 & 12 \end{array}\right]\begin{array}{l} R_1 + 3R_3 \to R_1' \\ R_2 + R_3 \to R_2' \\ R_3 \end{array}$$

$\{R_3$ already has a 1 in the 3rd diagonal element$\}$

$$(1 \;\; 2 \;\; -3 \,\vdots\, 4) + (0 \;\; 0 \;\; 0 \,\vdots\, 36) \to (1 \;\; 2 \;\; 0 \,\vdots\, 40)$$
$$(0 \;\; 1 \;\; -1 \,\vdots\, 6) + (0 \;\; 0 \;\; 1 \,\vdots\, 12) \to (0 \;\; 1 \;\; 0 \,\vdots\, 18)$$

Solutions Exercise Set 3.9 cont.

1. (ii) continued

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 18 \\ 0 & 0 & 1 & 24 \end{array}\right]\begin{array}{l} R_1 - 2R_2 \to R_1' \\ R_2 \\ R_3 \end{array} \qquad (1\ \ 2\ \ 0 \,\vdots\, 40) - (0\ \ 2\ \ 0 \,\vdots\, 36) \to (1\ \ 0\ \ 0 \,\vdots\, 4)$$

$\therefore\ x = 4,\ y = 18$ and $z = 24$ (The same result we obtained before)

2. (a) $x + 2y = 12$
$4x - 3y = 4$

In augmented matrix form

$$\left[\begin{array}{cc:c} 1 & 2 & 12 \\ 4 & -3 & 4 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \end{array} \sim \left[\begin{array}{cc:c} 1 & 2 & 12 \\ 0 & -11 & -44 \end{array}\right]\begin{array}{l} R_1 \\ R_2 - 4R_1 \to R_2' \end{array}$$

$$\sim \left[\begin{array}{cc:c} 1 & 2 & 12 \\ 0 & 1 & 4 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \div (-11) \to R_2' \end{array} \sim \left[\begin{array}{cc:c} 1 & 0 & 4 \\ 0 & 1 & 4 \end{array}\right]\begin{array}{l} R_1 - 2R_2 \to R_1' \\ R_2 \end{array}$$

$\therefore\ x = 4$ and $y = 4$

Check: when $x = 4$ and $y = 4$

$x + 2y = 4 + 2 \times 4 = 4 + 8 = 12$ ✓

$4x - 3y = 4 \times 4 - 3 \times 4 = 16 - 12 = 4$ ✓

(b) $2x - y = 7$
$-x + 2y = -5$

In augmented matrix form

$$\left[\begin{array}{cc:c} 2 & -1 & 7 \\ -1 & 2 & -5 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \end{array} \sim \left[\begin{array}{cc:c} 1 & -\frac{1}{2} & \frac{7}{2} \\ 0 & \frac{3}{2} & -\frac{3}{2} \end{array}\right]\begin{array}{l} R_1 \div 2 \to R_1' \\ R_2 + R_1' \to R_2' \end{array}$$

$$\sim \left[\begin{array}{cc:c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right]\begin{array}{l} R_1 + \frac{1}{2}R_2' \to R_1' \\ R_2 \times \frac{2}{3} \to R_2' \end{array}$$

Note: I've short cutted here

$\therefore\ x = 3$ and $y = 1$

Check: when $x = 3$ and $y = 1$

$2x - y = 2 \times 3 - (-1) = 6 + 1 = 7$ ✓

$-x + 2y = -3 + 2 \times (-1) = -3 - 2 = -5$ ✓

Solutions Exercise Set 3.9 cont.

2. (c)

$$\begin{aligned} x + y + z &= 6 \\ 3x + 4y + 2z &= 17 \\ -x + 2y + z &= 6 \end{aligned}$$

In augmented matrix form

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 3 & 4 & 2 & 17 \\ -1 & 2 & 1 & 6 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & 2 & 12 \end{array}\right]\begin{array}{l} R_1 \\ R_2 - 3R_1 \to R_2' \\ R_3 + R_1 \to R_3' \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 5 & 15 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 - 3R_2 \to R_3' \end{array} \sim \left[\begin{array}{ccc:c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]\begin{array}{l} R_1 - R_3' \to R_1' \\ R_2 + R_3' \to R_2' \\ R_3 \div 5 \to R_3' \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]\begin{array}{l} R_1 - R_2 \to R_1' \\ R_2 \\ R_3 \end{array}$$

$\therefore\ x = 1,\ y = 2$ and $z = 3$

Check: when $x = 1, y = 2$ and $z = 3$

$x + y + z = 1 + 2 + 3 = 6$ ✓

$3x + 4y + 2z = 3 \times 1 + 4 \times 2 + 2 \times 3 = 17$ ✓

$-x + 2y + z = -1 + 2 \times 2 + 3 = 6$ ✓

(d)

$$\begin{aligned} x - y + z &= 3 \\ 2x + 3y - 4z &= 9 \\ -x + 2y - z &= 0 \end{aligned}$$

In augmented matrix form

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & 3 \\ 2 & 3 & -4 & 9 \\ -1 & 2 & -1 & 0 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & 3 \\ 0 & 5 & -6 & 3 \\ 0 & 1 & 0 & 3 \end{array}\right]\begin{array}{l} R_1 \\ R_2 - 2R_1 \to R_2' \\ R_3 + R_1 \to R_3' \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 5 & -6 & 3 \end{array}\right]\begin{array}{l} R_1 \\ R_3 \to R_2 \\ R_2 \to R_3 \end{array} \sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -6 & -12 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 - 5R_2 \to R_3' \end{array}$$

Solutions Exercise Set 3.9 cont.

2. (d) continued

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 - R_3' \to R_1' \\ R_2 \\ R_3 \div (-6) \to R_3' \end{array} \sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 + R_2 \to R_1' \\ R_2 \\ R_2 \end{array}$$

$\therefore\ x = 4,\ y = 3$ and $z = 2$

Check: when $x = 4, y = 3$ and $z = 2$

$x - y + z = 4 - 3 + 2 = 3$ ✓

$2x + 3y - 4z = 2 \times 4 + 3 \times 3 - 4 \times 2 = 9$ ✓

$-x + 2y - z = -4 + 2 \times 3 - 2 = 0$ ✓

(e)
$$\begin{aligned} -x + 2y + 3z &= 10 \\ 3x - 2y + z &= -16 \\ -2x - y - 4z &= 5 \end{aligned}$$

In augmented matrix form

$$\left[\begin{array}{ccc:c} -1 & 2 & 3 & 10 \\ 3 & -2 & 1 & -16 \\ -2 & -1 & -4 & 5 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \sim \left[\begin{array}{ccc:c} 1 & -2 & -3 & -10 \\ 0 & 4 & 10 & 14 \\ 0 & -5 & -10 & -15 \end{array}\right] \begin{array}{l} R_1 \times (-1) \to R_1' \\ R_2 - 3R_1' \to R_2' \\ R_3 + 2R_1' \to R_3' \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & -2 & -3 & -10 \\ 0 & 1 & \frac{5}{2} & \frac{7}{2} \\ 0 & 0 & \frac{5}{2} & \frac{5}{2} \end{array}\right] \begin{array}{l} R_1 \\ R_2 \div 4 \to R_2' \\ R_3 + 5R_2' \to R_3' \end{array} \sim \left[\begin{array}{ccc:c} 1 & -2 & 0 & -7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 + 3R_3' \to R_1' \\ R_2 - \frac{2}{5}R_3' \to R_2' \\ R_3 \times \frac{2}{5} \to R_3' \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 + 2R_2 \to R_1' \\ R_2 \\ R_3 \end{array}$$

$\therefore\ x = -5,\ y = 1$ and $z = 1$

Check: when $x = -5, y = 1$ and $z = 1$

$-x + 2y + 3z = -(-5) + 2 \times 1 + 3 \times 1 = 10$ ✓

$3x - 2y + z = 3 \times (-5) - 2 \times 1 + 1 = -16$ ✓

$-2x - y - 4z = -2 \times (-5) - 1 - 4 \times 1 = 5$ ✓

Solutions Exercise Set 3.9 cont.

2. (f)

$$\begin{aligned} x + y + z &= 1 \\ 2x - y + 4z &= -2 \\ -x + 3y - 3z &= -7 \end{aligned}$$

In augmented matrix form

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 2 & -1 & 4 & -2 \\ -1 & 3 & -3 & -7 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & -3 & 2 & -4 \\ 0 & 4 & -2 & -6 \end{array}\right]\begin{array}{l} R_1 \\ R_2 - 2R_1 \to R_2' \\ R_3 + R_1 \to R_3' \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{2}{3} & \frac{4}{3} \\ 0 & 0 & \frac{2}{3} & -\frac{34}{3} \end{array}\right]\begin{array}{l} R_1 \\ R_2 \div (-3) \to R_2' \\ R_3 - 4R_2' \to R_3' \end{array} \sim \left[\begin{array}{ccc:c} 1 & -1 & 0 & 18 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & -17 \end{array}\right]\begin{array}{l} R_1 - R_3' \to R_1' \\ R_2 + \frac{2}{3}R_3' \to R_2' \\ R_3 \times \frac{3}{2} \to R_3' \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 28 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & -17 \end{array}\right]\begin{array}{l} R_1 - R_2 \to R_1' \\ R_2 \\ R_2 \end{array}$$

$\therefore\ x = 28,\ y = -10$ and $z = -17$

Check: when $x = 28, y = -10$ and $z = -17$

$$\begin{aligned} x + y + z &= 28 - 10 - 17 = 1 \ \checkmark \\ 2x - y + 4z &= 2 \times 28 - (-10) + 4 \times (-17) = -2 \ \checkmark \\ -x + 3y - 3z &= -28 + 3 \times (-10) - 3 \times (-17) = -7 \ \checkmark \end{aligned}$$

3. Being successful in finding a solution using the Gauss Jordan row reduction technique means that the matrix of coefficients has an inverse. (Recall that not all square matrices have inverses.)

We know if $AX = B$

$X = A^{-1}B$

By finding X using row reduction $\Rightarrow A^{-1}$ must exist.

So using Q1(f) as an example we know

$$\begin{bmatrix} 28 \\ -10 \\ -17 \end{bmatrix} = A^{-1}\begin{bmatrix} 1 \\ -2 \\ -7 \end{bmatrix}$$

We will find out how to find inverses using the same row reduction technique in the next section of work.

Solutions Exercise Set 3.10 page 3.34

1. (a) (i) $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$

$$A \,\vdots\, I_2 = \left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 4 & -3 & 0 & 1 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \end{array} \sim \left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & -11 & -4 & 1 \end{array}\right]\begin{array}{l} R_1 \\ R_2 - 4R_1 \to R_2' \end{array}$$

$$\sim \left[\begin{array}{cc:cc} 1 & 0 & \frac{3}{11} & \frac{2}{11} \\ 0 & 1 & \frac{4}{11} & -\frac{1}{11} \end{array}\right]\begin{array}{l} R_1 - 2R_2' \to R_1' \\ R_2 \div (-11) \to R_2' \end{array}$$

$$\therefore\ A^{-1} = \begin{bmatrix} \frac{3}{11} & \frac{2}{11} \\ \frac{4}{11} & -\frac{1}{11} \end{bmatrix}$$

Checking:

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} \frac{3}{11} & \frac{2}{11} \\ \frac{4}{11} & \frac{-1}{11} \end{bmatrix} = \begin{bmatrix} (\frac{3}{11} + \frac{8}{11}) & (\frac{2}{11} - \frac{2}{11}) \\ (\frac{12}{11} - \frac{12}{11}) & (\frac{8}{11} + \frac{3}{11}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}_2 \checkmark$$

(ii) $A^{-1} = \begin{bmatrix} \frac{3}{11} & \frac{2}{11} \\ \frac{4}{11} & -\frac{1}{11} \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$

(iii) $X = A^{-1}B$

$$= \frac{1}{11}\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 12 \\ 4 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} (3 \times 12 + 2 \times 4) \\ (4 \times 12 - 1 \times 4) \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 44 \\ 44 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$\therefore\ x = 4$ and $y = 4$ as found before.

Solutions Exercise Set 3.10 cont.

1. (b) (i) $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$A \,\vdots\, I_2 = \left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \end{array} \sim \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & 1 \end{array}\right]\begin{array}{l} R_1 \div 2 \to R_1' \\ R_2 + R_1' \to R_2' \end{array}$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \end{array}\right]\begin{array}{l} R_1 + \frac{1}{2}R_2' \to R_1' \\ R_2 \times \frac{2}{3} \to R_2' \end{array}$$

$$\therefore\ A^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

Checking:

$$AA^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} (\frac{4}{3} - \frac{1}{3}) & (\frac{2}{3} - \frac{2}{3}) \\ (\frac{-2}{3} + \frac{2}{3}) & (-\frac{1}{3} + \frac{4}{3}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \checkmark$$

(ii) $$A^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(iii) $X = A^{-1}B$

$$= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} (14-5) \\ (7-10) \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$\therefore\ x = 3$ and $y = -1$ as found before.

Solutions Exercise Set 3.10 cont.

1. (c) (i) $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix}$

$$A \,\vdots\, I_3 = \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 4 & 2 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \sim \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & 3 & 2 & 0 & 0 & 1 \end{array}\right]\begin{array}{l} R_1 \\ R_2 - 3R_1 \to R_2' \\ R_3 + R_1 \to R_3' \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & 0 & 5 & 10 & -3 & 1 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 - 3R_2 \to R_3' \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 1 & 0 & -1 & \frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & -1 & \frac{2}{5} & \frac{1}{5} \\ 0 & 0 & 1 & 2 & -\frac{3}{5} & \frac{1}{5} \end{array}\right]\begin{array}{l} R_1 - R_3' \to R_1' \\ R_2 + R_3' \to R_2' \\ R_3 \div 5 \to R_3' \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & -1 & \frac{2}{5} & \frac{1}{5} \\ 0 & 0 & 1 & 2 & -\frac{3}{5} & \frac{1}{5} \end{array}\right]\begin{array}{l} R_1 - R_2 \to R_1' \\ R_2 \\ R_3 \end{array}$$

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{5} & -\frac{2}{5} \\ -1 & \frac{2}{5} & \frac{1}{5} \\ 2 & -\frac{3}{5} & \frac{1}{5} \end{bmatrix}$$

Checking:

$$AA^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{5} & -\frac{2}{5} \\ -1 & \frac{2}{5} & \frac{1}{5} \\ 2 & -\frac{3}{5} & \frac{1}{5} \end{bmatrix}$$

Solutions Exercise Set 3.10 cont.

1. (c) (i) continued

$$= \begin{bmatrix} (0-1+2) & (\frac{1}{5}+\frac{2}{5}-\frac{3}{5}) & (-\frac{2}{5}+\frac{1}{5}+\frac{1}{5}) \\ (0-4+4) & (\frac{3}{5}+\frac{8}{5}-\frac{6}{5}) & (-\frac{6}{5}+\frac{4}{5}+\frac{2}{5}) \\ (0-2+2) & (-\frac{1}{5}+\frac{4}{5}-\frac{3}{5}) & (\frac{2}{5}+\frac{2}{5}+\frac{1}{5}) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \checkmark$$

(ii) $A^{-1} = \begin{bmatrix} 0 & \frac{1}{5} & -\frac{2}{5} \\ -1 & \frac{2}{5} & \frac{1}{5} \\ 2 & -\frac{3}{5} & \frac{1}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0 & 1 & -2 \\ -5 & 2 & 1 \\ 10 & -3 & 1 \end{bmatrix}$

(iii $X = A^{-1}B$

$$= \frac{1}{5}\begin{bmatrix} 0 & 1 & -2 \\ -5 & 2 & 1 \\ 10 & -3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 17 \\ 6 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} (0+17-12) \\ (-30+34+6) \\ (60-51+6) \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore\ x = 1,\ y = 2$ and $z = 3$ as found before.

Solutions Exercise Set 3.10 cont.

1. (d) (i) A $= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & -4 \\ -1 & 2 & -1 \end{bmatrix}$

$$A \,\vdots\, I_3 = \left[\begin{array}{rrr:rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 3 & -4 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr:rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -6 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 - 2R_1 \to R_2' \\ R_3 + R_1 \to R_3' \end{array}$$

$$\sim \left[\begin{array}{rrr:rrr} 1 & 0 & -\frac{1}{5} & \frac{3}{5} & \frac{1}{5} & 0 \\ 0 & 1 & -\frac{6}{5} & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & \frac{6}{5} & \frac{7}{5} & -\frac{1}{5} & 1 \end{array}\right] \begin{array}{l} R_1 + R_2' \to R_1' \\ R_2 \div 5 \to R_2' \\ R_3 - R_2' \to R_3' \end{array}$$

$$\sim \left[\begin{array}{rrr:rrr} 1 & 0 & 0 & \frac{25}{30} & \frac{5}{30} & \frac{1}{6} \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{7}{6} & -\frac{1}{6} & \frac{5}{6} \end{array}\right] \begin{array}{l} R_1 + \frac{1}{5}R_3' \to R_1' \\ R_2 + \frac{6}{5}R_3' \to R_2' \\ R_3 \times \frac{5}{6} \to R_3' \end{array}$$

$$\therefore\ A^{-1} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ 1 & 0 & 1 \\ \frac{7}{6} & -\frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

Checking:

$$AA^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & -4 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ 1 & 0 & 1 \\ \frac{7}{6} & -\frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

Solutions Exercise Set 3.10 cont.

1. (d) (i) continued

$$= \begin{bmatrix} (\frac{5}{6} - 1 + \frac{7}{6}) & (\frac{1}{6} + 0 - \frac{1}{6}) & (\frac{1}{6} - 1 + \frac{5}{6}) \\ (\frac{10}{6} + 3 - \frac{28}{6}) & (\frac{2}{6} + 0 + \frac{4}{6}) & (\frac{2}{6} + 3 - \frac{20}{6}) \\ (-\frac{5}{6} + 2 - \frac{7}{6}) & (-\frac{1}{6} + 0 + \frac{1}{6}) & (-\frac{1}{6} + 2 - \frac{5}{6}) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \checkmark$$

(ii) $$A^{-1} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ 1 & 0 & 1 \\ \frac{7}{6} & -\frac{1}{6} & \frac{5}{6} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 & 1 & 1 \\ 6 & 0 & 6 \\ 7 & -1 & 5 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= \frac{1}{6}\begin{bmatrix} 5 & 1 & 1 \\ 6 & 0 & 6 \\ 7 & -1 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 9 \\ 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} (15 + 9 + 0) \\ (18 + 0 + 0) \\ (21 - 9 + 0) \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 24 \\ 18 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$\therefore$ $x = 4$, $y = 3$ and $z = 2$ as found before.

Solutions Exercise Set 3.10 cont.

1. (e) (i) $A = \begin{bmatrix} -1 & 2 & 3 \\ 3 & -2 & 1 \\ -2 & -1 & -4 \end{bmatrix}$

$$A \,\vdots\, I_3 = \left[\begin{array}{ccc:ccc} -1 & 2 & 3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ -2 & -1 & -4 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & -2 & -3 & -1 & 0 & 0 \\ 0 & 4 & 10 & 3 & 1 & 0 \\ 0 & -5 & -10 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \times (-1) \to R_1' \\ R_2 - 3R_1' \to R_2' \\ R_3 + 2R_1' \to R_3' \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{5}{2} & \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{5}{2} & \frac{7}{4} & \frac{5}{4} & 1 \end{array}\right] \begin{array}{l} R_1 + 2R_2' \to R_1' \\ R_2 \div 4 \to R_2' \\ R_3 + 5R_2' \to R_3' \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -\frac{9}{10} & -\frac{1}{2} & -\frac{4}{5} \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & \frac{7}{10} & \frac{1}{2} & \frac{2}{5} \end{array}\right] \begin{array}{l} R_1 - 2R_3' \to R_1' \\ R_2 - \frac{5}{2}R_3' \to R_2' \\ R_3 \times \frac{2}{5} \to R_3' \end{array}$$

$$\therefore\ A^{-1} = \begin{bmatrix} -\frac{9}{10} & -\frac{1}{2} & -\frac{4}{5} \\ -1 & -1 & -1 \\ \frac{7}{10} & \frac{1}{2} & \frac{2}{5} \end{bmatrix}$$

Checking:

$$AA^{-1} = \begin{bmatrix} -1 & 2 & 3 \\ 3 & -2 & 1 \\ -2 & -1 & -4 \end{bmatrix} \begin{bmatrix} -\frac{9}{10} & -\frac{1}{2} & -\frac{4}{5} \\ -1 & -1 & -1 \\ \frac{7}{10} & \frac{1}{2} & \frac{2}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \checkmark$$

Solutions Exercise Set 3.10 cont.

1. (e) continued

(ii) $A^{-1} = \begin{bmatrix} -\frac{9}{10} & -\frac{1}{2} & -\frac{4}{5} \\ -1 & -1 & -1 \\ \frac{7}{10} & \frac{1}{2} & \frac{2}{5} \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -9 & -5 & -8 \\ -10 & -10 & -10 \\ 7 & 5 & 4 \end{bmatrix}$ or $-\frac{1}{10}\begin{bmatrix} 9 & 5 & 8 \\ 10 & 10 & 10 \\ -7 & -5 & -4 \end{bmatrix}$

(iii) $X = A^{-1}B$

$$= \frac{1}{10}\begin{bmatrix} -9 & -5 & -8 \\ -10 & -10 & -10 \\ 7 & 5 & 4 \end{bmatrix}\begin{bmatrix} 10 \\ -16 \\ 5 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} (-90+80-40) \\ (-100+160-50) \\ (70-80+20) \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -50 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore\ x = -5,\ y = 1$ and $z = 1$ as found before.

(f) (i) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ -1 & 3 & -3 \end{bmatrix}$

$$A \,\vdots\, I_3 = \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 4 & 0 & 1 & 0 \\ -1 & 3 & -3 & 0 & 0 & 1 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 2 & -2 & 1 & 0 \\ 0 & 4 & -2 & 1 & 0 & 1 \end{array}\right]\begin{array}{l} R_1 \\ R_2 - 2R_1 \to R_2' \\ R_3 + R_1 \to R_3' \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & -\frac{5}{3} & \frac{4}{3} & 1 \end{array}\right]\begin{array}{l} R_1 - R_2' \to R_1' \\ R_2 \div (-3) \to R_2' \\ R_3 - 4R_2' \to R_3' \end{array}$$

Solutions Exercise Set 3.10 cont.

1. (f) (i) continued

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{27}{6} & -3 & -\frac{5}{2} \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{5}{2} & 2 & \frac{3}{2} \end{array}\right] \begin{array}{l} R_1 - \frac{5}{3}R_3' \to R_1' \\ R_2 + \frac{2}{3}R_3' \to R_2' \\ R_3 \times \frac{3}{2} \to R_3' \end{array}$$

$$\therefore\ A^{-1} = \begin{bmatrix} \frac{9}{2} & -3 & -\frac{5}{2} \\ -1 & 1 & 1 \\ -\frac{5}{2} & 2 & \frac{3}{2} \end{bmatrix}$$

Checking:

$$AA^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ -1 & 3 & -3 \end{bmatrix} \begin{bmatrix} \frac{9}{2} & -3 & -\frac{5}{2} \\ -1 & 1 & 1 \\ -\frac{5}{2} & 2 & \frac{3}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \checkmark$$

(ii) $$A^{-1} = \begin{bmatrix} \frac{9}{2} & -3 & -\frac{5}{2} \\ -1 & 1 & 1 \\ -\frac{5}{2} & 2 & \frac{3}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 9 & -6 & -5 \\ -2 & 2 & 2 \\ -5 & 4 & 3 \end{bmatrix} \text{ or } -\frac{1}{2}\begin{bmatrix} -9 & 6 & 5 \\ 2 & -2 & -2 \\ 5 & -4 & -3 \end{bmatrix}$$

(iii) $X = A^{-1}B$

$$= \frac{1}{2}\begin{bmatrix} 9 & -6 & -5 \\ -2 & 2 & 2 \\ -5 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -7 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} (9+12+35) \\ (-2-4-14) \\ (-5-8-21) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 56 \\ -20 \\ -34 \end{bmatrix} = \begin{bmatrix} 28 \\ -10 \\ -17 \end{bmatrix}$$

$\therefore\ x = 28,\ \ y = -10$ and $z = -17$ as found before.

Solutions Exercise Set 3.10 cont.

2. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 6 \\ 9 \\ 1 \end{bmatrix}$

Then $AX = B$

$\therefore\ X = A^{-1}B$

(i) Find A^{-1} first.

$$A \,\vdots\, I_3 = \left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 - 2R_1 \to R_2' \\ R_3 - 3R_1 \to R_3' \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \to R_3 \\ R_3 \to R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \times (-1) \to R_2' \\ R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 2 & 0 & -5 & 3 & 0 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right] \begin{array}{l} R_2 - 3R_3' \to R_1' \\ R_2 - 3R_3' \to R_2' \\ R_3 \times (-1) \to R_3' \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right] \begin{array}{l} R_1 - 2R_2 \to R_1' \\ R_2 \\ R_3 \end{array}$$

$$\therefore\ A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

Solutions Exercise Set 3.10 cont.

2 (i) continued

Check inverse by finding AA^{-1}.

$$AA^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{I}_3 \checkmark$$

Now $X = A^{-1}B$

$$= \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \\ 1 \end{bmatrix} = \begin{bmatrix} (6 - 27 + 2) \\ (-18 + 27 - 1) \\ (12 - 9 + 0) \end{bmatrix} = \begin{bmatrix} -19 \\ 8 \\ 3 \end{bmatrix}$$

$\therefore\ x = -19,\ \ y = 8$ and $z = 3$.

(ii) Verification of solution.

Method A:

Substitution in original equations.

When $x = -19,\ \ y = 8$ and $z = 3$

$$\begin{aligned} x + 2y + 3z &= -19 + 2 \times 8 + 3 \times 3 = 6 \ \checkmark \\ 2x + 4y + 5z &= 2 \times (-19) + 4 \times 8 + 5 \times 3 = 9 \ \checkmark \\ 3x + 5y + 6z &= 3 \times (-19) + 5 \times 8 + 6 \times 3 = 1 \ \checkmark \end{aligned}$$

Method B:

Show A$X = B$ (which is really the same thing as the substitution method above)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} -19 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} (-19 + 16 + 9) \\ (-38 + 32 + 15) \\ (-57 + 40 + 18) \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 1 \end{bmatrix} = B \ \checkmark$$

Solutions Exercise Set 3.11 page 3.36

1. (a) $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$

$\det A = 1 \times (-3) - 2 \times 4 = -3 - 8 = -11$

(b) $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$\det A = 2 \times 2 - (-1) \times (-1) = 4 - 1 = 3$

(c) $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix}$

$$\det A = 1 \times \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} - 1 \times \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} + 1 \times \begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix}$$

$= 1 \times (4 \times 1 - 2 \times 2) - 1 \times (3 \times 1 - 2 \times (-1)) + 1(3 \times 2 - 4 \times (-1))$

$= (4 - 4) - (3 + 2) + (6 + 4) = 0 - 5 + 10 = 5$

(d) $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & -4 \\ -1 & 2 & -1 \end{bmatrix}$

$$\det A = 1 \times \begin{vmatrix} 3 & -4 \\ 2 & -1 \end{vmatrix} - (-1) \times \begin{vmatrix} 2 & -4 \\ -1 & -1 \end{vmatrix} + 1 \times \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}$$

$= 1 \times (3 \times (-1) - (-4) \times 2) + 1 \times (2 \times (-1) - (-4) \times (-1)) + 1 \times (2 \times 2 - 3 \times (-1))$

$= (-3 + 8) + (-2 - 4) + (4 + 3) = 5 - 6 + 7 = 6$

Solutions Exercise Set 3.11 cont.

1. (e) $A = \begin{bmatrix} -1 & 2 & 3 \\ 3 & -2 & 1 \\ -2 & -1 & -4 \end{bmatrix}$

$$\det A = -1 \times \begin{vmatrix} -2 & 1 \\ -1 & -4 \end{vmatrix} - 2 \times \begin{vmatrix} 3 & 1 \\ -2 & -4 \end{vmatrix} + 3 \times \begin{vmatrix} 3 & -2 \\ -2 & -1 \end{vmatrix}$$

$$= -1 \times ((-2) \times (-4) - 1 \times (-1)) - 2 \times (3 \times (-4) - 1 \times (-2)) + 3 \times (3 \times (-1) - (-2) \times (-2))$$

$$= -(8 + 1) - 2 \times (-12 + 2) + 3 \times (-3 - 4)$$

$$= -9 + 20 - 21 = -10$$

(f) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ -1 & 3 & -3 \end{bmatrix}$

$$\det A = 1 \times \begin{vmatrix} -1 & 4 \\ 3 & -3 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 4 \\ -1 & -3 \end{vmatrix} + 1 \times \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix}$$

$$= 1 \times ((-1) \times (-3) - 4 \times 3) - 1 \times (2 \times (-3) - 4 \times (-1)) + 1 \times (2 \times 3 - (-1) \times (-1))$$

$$= (3 - 12) - (-6 + 4) + (6 - 1) = -9 + 2 + 5 = -2$$

You may care to look at the value of the determinants in these problems and compare them with the factor outside the matrix of integer values that you obtained in part (ii) of Q1 in Exercise 3.10. The factors are the reciprocals of the respective determinants.

In further courses in mathematics you will discover another method to find inverse matrices which uses the reciprocal of the determinant. We will not study it in this course.

Solutions Exercise Set 3.11 cont.

2. (a) Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$\det A = (2 \times 3) - (0 \times 0) = 6$ which is nonzero $\therefore$ A will have an inverse.

(b) Let $B = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$

$\det B = (0 \times 3) - (2 \times 0) = 0$ $\therefore$ B will not have an inverse

(c) Let $C = \begin{bmatrix} 2 & -1 \\ -3 & -5 \end{bmatrix}$

$\det C = (2 \times (-5)) - ((-1) \times (-3)) = -10 - 3 = -13$ which is nonzero

$\therefore$ C will have an inverse.

(d) Let $D = \begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$

$\det D = (6 \times 2) - (4 \times 3) = 12 - 12 = 0$ $\therefore$ D will not have an inverse

(e) Let $E = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -4 & 5 \end{bmatrix}$

$$\det E = 2 \times \begin{vmatrix} 1 & 1 \\ -4 & 5 \end{vmatrix} - 3 \times \begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} + (-1) \times \begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix}$$

$$= 2 \times (1 \times 5 - 1 \times (-4)) - 3 \times (1 \times 5 - 1 \times 3) - 1 \times (1 \times (-4) - 1 \times 3)$$

$$= 2 \times (5 + 4) - 3 \times (5 - 3) - (-4 - 3)$$

$= 18 - 6 + 7 = 19$ which is nonzero $\therefore$ E will have an inverse

Solutions Exercise Set 3.11 cont.

2. (f) Let $F = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}$

$$\begin{aligned}
\det F &= 1 \times \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - 3 \times \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} + (-2) \times \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} \\
&= ((-1) \times (-1) - 1 \times 2) - 3 \times (2 \times (-1) - 1 \times 3) - 2 \times (2 \times 2 - (-1) \times 3) \\
&= (1 - 2) - 3 \times (-2 - 3) - 2 \times (4 + 3) \\
&= -1 + 15 - 14 \\
&= 0 \qquad \therefore\ F \text{ will not have an inverse}
\end{aligned}$$

(g) Let $G = \begin{bmatrix} 6 & -4 & 4 \\ 4 & 2 & -2 \\ -3 & 2 & 5 \end{bmatrix}$

$$\begin{aligned}
\det G &= 6 \times \begin{vmatrix} 2 & -2 \\ 2 & 5 \end{vmatrix} - (-4) \times \begin{vmatrix} 4 & -2 \\ -3 & 5 \end{vmatrix} + 4 \times \begin{vmatrix} 4 & 2 \\ -3 & 2 \end{vmatrix} \\
&= 6 \times (2 \times 5 - (-2) \times 2) + 4 \times (4 \times 5 - (-2) \times (-3)) + 4 \times (4 \times 2 - 2 \times (-3)) \\
&= 6 \times (10 + 4) + 4 \times (20 - 6) + 4 \times (8 + 6) \\
&= 84 + 56 + 56 = 196 \qquad \text{which is nonzero } \therefore\ G \text{ has an inverse.}
\end{aligned}$$

(h) Let $H = \begin{bmatrix} 0 & 4 & 6 & 3 \\ 2 & 7 & -1 & 5 \\ 4 & 2 & -3 & 1 \end{bmatrix}$

H is not a square matrix $\therefore$ it does not have a determinant nor an inverse.