Module B5

Exponential and logarithmic functions

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Introduction

Historians use it, banks use it, fish breeders use it, hospitals use it, even nuclear physicists use it. The exponential and its related function are often thought to be the most commonly occurring non-linear functions in nature. One type of exponential function is typified by its slow start followed by an ever increasing rise, while the other decreases quickly then slows down...if you have had the flu then you have experienced an exponential growth function in action. First by the rapid way the virus takes over your body, then when you take a painkiller how it rapidly relieves some of the symptoms for a while.

You will have previously studied exponential functions in *Mathematics tertiary preparation level A* or elsewhere. In this module we will refresh and build on this knowledge to develop a fuller understanding of the exponential function, its related function, the logarithmic function, and their uses.

More formally when you have successfully completed this module you should be able to:

- describe the pattern of exponential growth and decay in words, algebraic terms and using graphs
- recognize the occurrence of exponential growth and decay in real world situations
- demonstrate an understanding of the definition of a logarithm and its relationship to the exponential function
- use the logarithmic laws to simplify expressions and solve problems
- recognize the occurrence of logarithmic functions in real world situations
- solve problems involving exponential growth and decay, graphically and algebraically
- solve simple exponential equations.

5.1 Exponential functions

5.1.1 The function and its graph

If you have lived in Australia for any period of time, whether in the city or the bush, then you would have heard of our rabbit problem. In 1859 the European rabbit was introduced into Australia by some well meaning people and in 1995 the population was estimated at 300 million, causing approximately $600 million worth of damage to agriculture and native wildlife. This is a classic example of exponential growth in action. The jokes about reproducing like rabbits are not to be taken lightly. Rabbits reproduce very rapidly with 4 to 5 litters of 4 to 6 kittens each season. This means that they can increase 10 fold in each season. The joke below looks at the mathematics behind this from a rabbit's perspective.

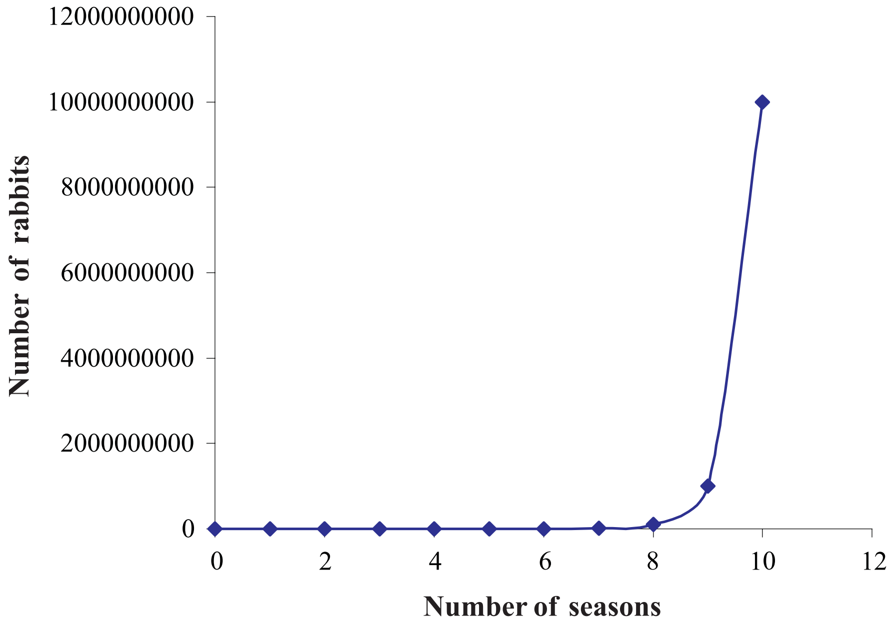
However, from our perspective we might think of it a little differently. Let's begin, hypothetically, with one pregnant female rabbit.

Season	Number of rabbits	In exponential notation
0	1	10^0
1	10	10^1
2	10×10	10^2
3	$10\times10\times10$	10^3
4	$10\times10\times10\times10$	10^4
5	$10\times10\times10\times10\times10$	10^5
6	$10\times10\times10\times10\times10\times10$	10^6
7	$10\times10\times10\times10\times10\times10\times10$	10^7
8	$10\times10\times10\times10\times10\times10\times10\times10$	10^8
9	$10\times10\times10\times10\times10\times10\times10\times10\times10$	10^9
10	$10\times10\times10\times10\times10\times10\times10\times10\times10\times10$	10^{10}

After ten seasons we could end up with 10^{10} or ten thousand million rabbits (this of course assumes there is unlimited food, no predators or diseases like Calicivirus).

The equation of this function would be $f(n)=10^n$, where n is the number of seasons. It would appear as below, if the number of seasons is the independent variable and number of rabbits the dependent variable.

Population growth of rabbits over ten seasons



Before making any generalizations let's draw some more exponential graphs.

But we first need to recall some terminology, especially words like index, exponent and base.

$P=4^x$ — x is the index or exponent; 4 is the base

Activity 5.1

1. Sketch the graph of $y = 2^x$
2. Sketch the graph of $f(x) = 3^x$

After drawing these graphs think about the similarities between them and list them below.

____

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Graphs which increase as the independent variable increases like the graphs in activity 5.1 are called **exponential growth functions**.

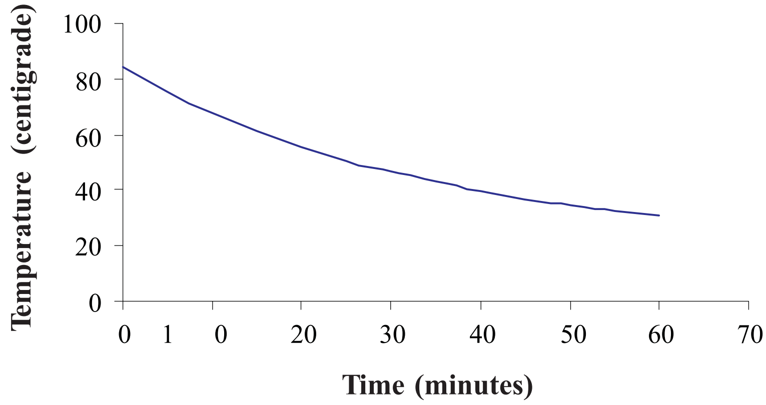
Other important characteristics of the graphs in activity 5.1 are that:

- they are all functions because there is only one value of the dependent variable for each value of the independent variable
- their domains are all unrestricted and include all real numbers
- the range of each function is restricted to values greater than zero
- the vertical intercept is one
- as the independent variable gets more negative (approaches negative **infinity**) the dependent variable gets closer and closer to zero
- as the independent variable increases (approaches infinity) the dependent variable increases very quickly.

Note, when a function gets very close to a value but does not reach it we say that we have an asymptote at that value. In this case the asymptote will be the straight line $y = 0$. So for exponential growth functions the horizontal axis is an **asymptote**.

Not all exponential graphs represent growth functions. Have you ever wondered why that cup of coffee you thought you just poured got cold so quickly? Think about the graph below which represents the temperature of a cooling cup of coffee.

Temperature of a cup of coffee over time



Before making any generalization let's draw some more exponential graphs.

Activity 5.2

1. Sketch the graph of $y = 2^{-x}$
2. Sketch the graph of $f(x) = 3^{-x}$

After drawing these graphs think about the similarities between them and list them below.

____

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Graphs which decrease as the independent variable increases like graphs in activity 5.2 are called **exponential decay functions**.

Important characteristics of exponential decay graphs in activity 5.2 are:

- they are all functions because there is only one value of the dependent variable for each value of the independent variable
- their domains are all unrestricted and include all real numbers
- the range of each function is restricted to values greater than zero
- the vertical intercept is one
- as the independent variable increases (approaches infinity) the dependent variable decreases slowing down as it approaches zero
- as the independent variable gets more negative (approaches negative infinity) the value of the dependent variable increases rapidly.

Note, for exponential decay functions the horizontal axis is an **asymptote**. In activity 5.2 the asymptote is $y = 0$.

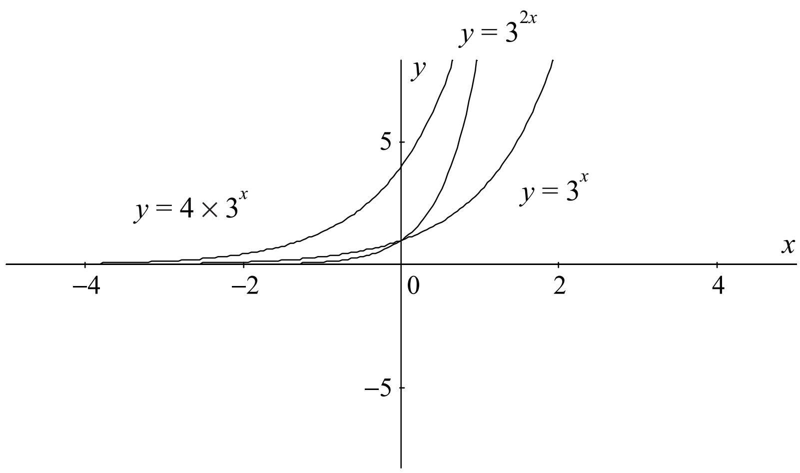
As we can see from the discussion above there are many similarities between exponential graphs, but there are also a number of differences. Sketch the three graphs below either on a graphing package or by hand and think about the differences between the graphs.

$y = 3^x$

$y = 3^{2x}$

$y = 4 \times 3^x$

You should have sketched something like this.



The differences you might have noticed are:

- Both $y = 3^x$ and $y = 3^{2x}$ intersect the y-axis at $y = 1$, while $y = 4 \times 3^x$ intersects the axis at $y = 4$
- $y = 3^{2x}$ increases at a rate faster than $y = 3^x$
- $y = 4 \times 3^x$ appears to increase at the same rate as $y = 3^x$, but is slower than $y = 3^{2x}$.

Recall from module 3 that if you want to find the vertical intercept of a function you have to find the value when the horizontal variable is zero.

Example

Find the y-intercept of $y = 4 \times (7.2)^t$.

To find the y-intercept we have to put $t = 0$

$y = 4 \times (7.2)^t$

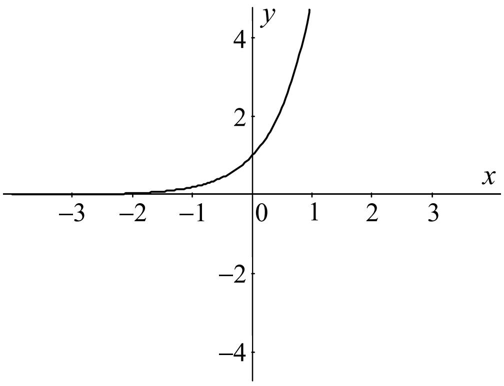
$y = 4 \times (7.2)^0$ Recall that $(7.2)^0 = 1$

$y = 4 \times 1$

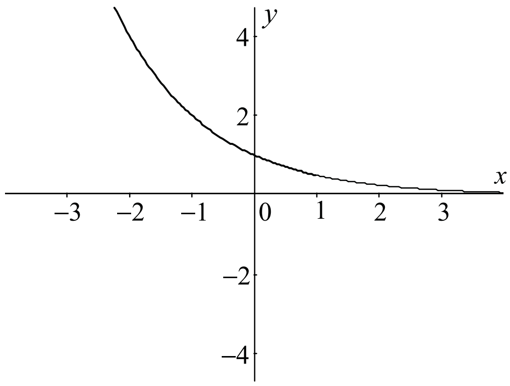
$y = 4$

Activity 5.3

1. Match the equations with the graphs of the following exponential functions.
   Equations: $y = 5^x$, $y = 5^{-x}$, $y = (0.5)^x$, $y = (0.5)^{-x}$



A



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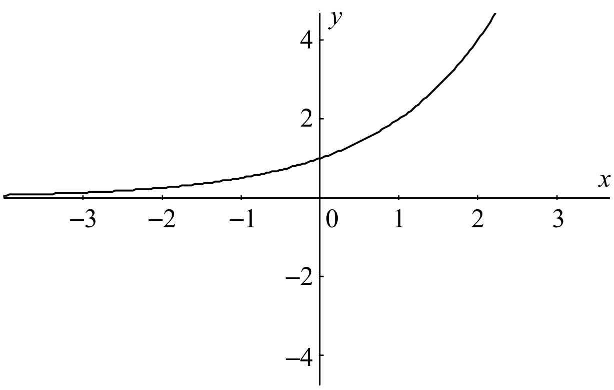

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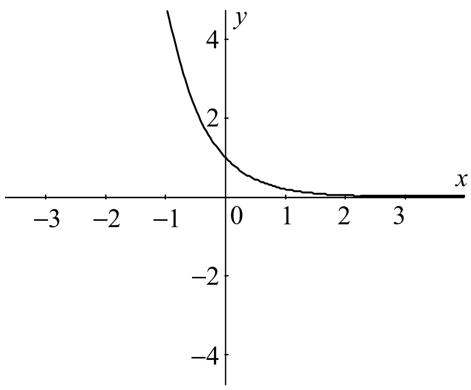


D

2. (a) On the same set of axes sketch the graphs of

   $y = 2^x$, $y = 10^x$ and $y = (\frac{1}{2})^x$.

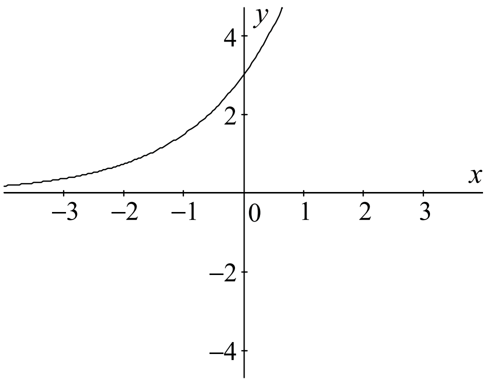
   (b) State the domain and range for each of the graphs.

   (c) Give the point of intersection.

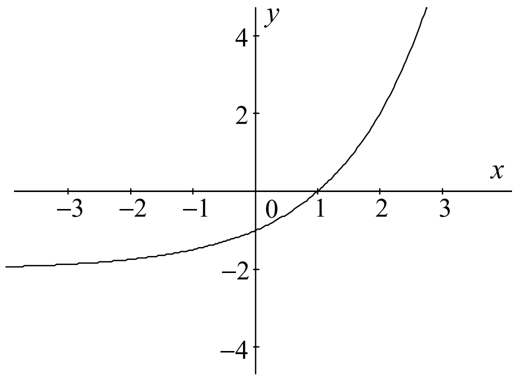
   (d) What is the horizontal asymptote of each of the graphs.

3. (a) Match the equations with the graphs of the following exponential functions.

   $y = 2^x$, $y = 3 \times 2^x$, $y = 2^x + 2$, $y = 2^x - 2$



A



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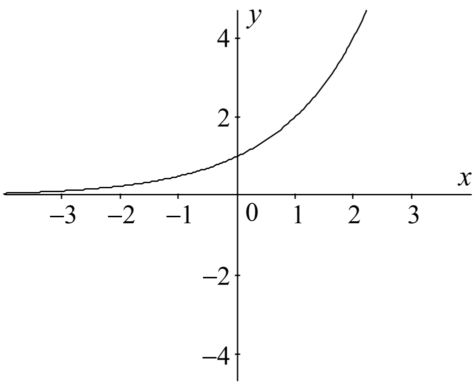

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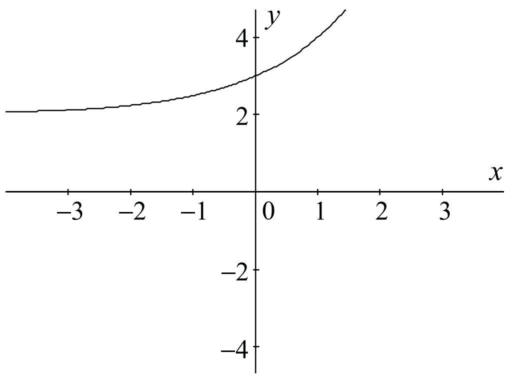


D

4. M (5,a) and P (b, $\frac{1}{27}$) are two points on the graph of $y = \frac{1}{3} \times 3^x$.

   (a) What is the value of a?

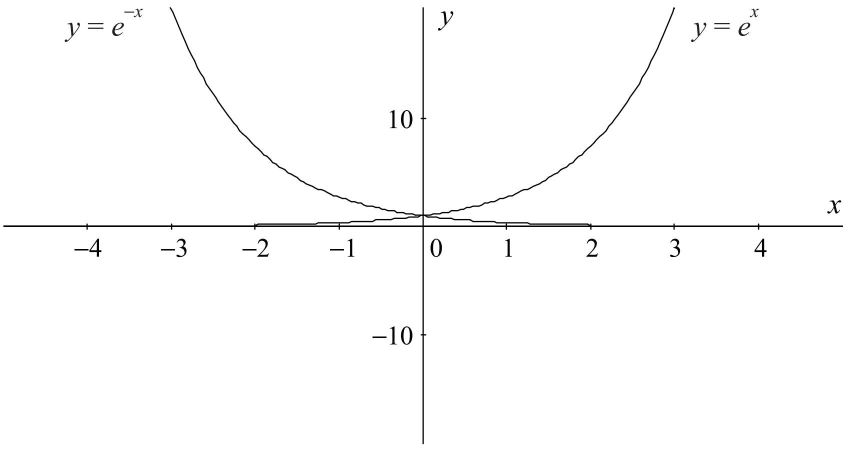
   (b) Find the value of b.

   (c) Determine the vertical intercept of the function.

5.1.2 The exponential function

So far we have looked at exponential functions that have a rational base, i.e. the base was either 2, 10 or 7.2 (a rational number). But a special irrational number called e exists. It is commonly used with exponential functions. This number does not have an exact value and is approximated by 2.718281828…(a non-terminating decimal). This number, e, is commonly used in higher mathematics, science, engineering and economics, as believe it or not, the use of e simplifies more complex calculations used in these areas.

The exponential functions generated using e look similar to other exponential growth and decay functions. They are depicted below. Recall that the value of e can be easily generated on your calculator using the e^x key. All you have to do is calculate the value of the function at $x = 1$. If you are unsure about doing this consult your tutor.



Note the graphs above are very similar to the curves graphed in activities 5.1 and 5.2, because the value of e lies between 2 and 3.

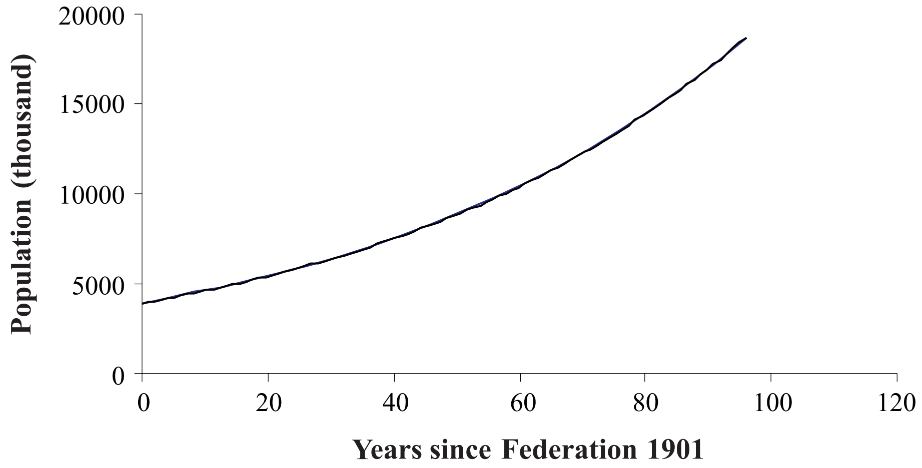
5.1.3 Case studies

Exponential functions are really best studied by examining applications in the real world. They often do not look like the stylized graphs we have practised above. Let's look at some more realistic exponential functions and then do some activities based on these examples.

Population growth

As discussed previously population growth is the classic example of an exponential growth curve. Australia's population was 18.5 million on 30 June 1997 almost five times the size of the population at the time of Federation (1 January 1901). The curve is approximated by the function $P(t) = 3913.4e^{0.0163t}$.

Australian population since Federation 1901 to 1997



Compound interest

Most people who have a savings account with a bank or other financial institution leave their deposits for a period of time expecting to accrue money as time passes. If the deposits are made in an account carrying **simple interest** (flat rate of interest) the interest received is calculated on the original deposit for the duration of the account.

This would mean that if you invested \$3 000 at a **flat interest rate** of 3.5% then in the first year you would have accrued,

$$\begin{aligned}
\text{Total earned} &= \text{principal} + 3.5\% \text{ of principal over 1 year} \\
&= 3000 + \frac{3.5}{100} \times 3000 \times 1 \\
&= 3000 + 0.035 \times 3000 \times 1 \\
&= 3105
\end{aligned}$$

or \$3 105.

We could perform the same calculations over five years, shown in the table below.

Year	Amount ($)
Year 0	3000
Year 1	3105
Year 2	3210
Year 3	3315
Year 4	3420
Year 5	3525

Most institutions today use **compound interest** rather than simple interest. In compound interest investments, the interest is calculated regularly on the **principal** (amount originally invested) plus interest. Using the example above, what would be the total amount accrued after the first year? So after year one we would have the original principal plus 3.5% of that principal. In fact this means that we would have 103.5% of the principal. We can work this out mathematically as follows.

$$
\begin{aligned}
\text{Total earned} &= \text{principal} + 3.5\% \text{ of principal} \\
&= 3000 + \frac{3.5}{100} \times 3000 \\
&= 3000 + 0.035 \times 3000 \\
&= 3000(1 + 0.035) \\
&= 3105
\end{aligned}
$$

or $3 105

We could perform the same calculations over five years, shown in the table below.

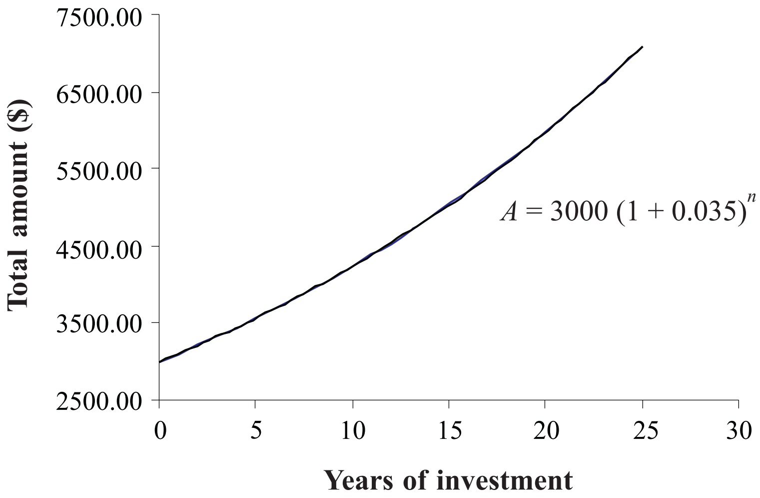
Year	Total in account ($)	
Year 0	3000	3000
Year 1	(1+0.035) of amount earned in year 0	$3000(1+0.035)$ $=3105$
Year 2	(1+0.035) of amount earned in year 1	$(1+0.035)\times 3000\times(1.035)$ $=3000(1+0.035)^2$ $=3213.68$
Year 3	(1+0.035) of amount earned in year 2	$(1+0.035)\times 3000(1+0.035)^2$ $=3000\times(1+0.035)^3$ $=3326.15$
Year 4	(1+0.035) of amount earned in year 3	$(1+0.035)\times 3000(1+0.035)^3$ $=3000\times(1+0.035)^4$ $=3442.57$
Year 5	(1+0.035) of amount earned in year 4	$(1+0.035)\times 3000(1+0.035)^4$ $=3000\times(1+0.035)^5$ $=3563.06$

Is the pattern of the calculation starting to look familiar? It is a special case of the compound interest formula you might have come across before.

$A = P(1+\frac{r}{100})^n$, **where A is the total amount returned, P is the principal (initial amount), r is the rate as a percentage returned in each investment period and n is the number of investment periods.**

If we were to graph the function determined in this example, we would see that it is a typical exponential growth shape if taken over 30 years. Notice, however, that in the early years the shape is close to a linear shape. This might explain why, in the early years, the returns you get from compound interest are only marginally better than the returns you get from simple interest. Look back at the two tables and see what the values were.

Total amount accrued applying compound interest



A bit of history… **For interest only**

The importance of e was first recognised by the Swiss mathematician Leonhard Euler. He gave it its name, derived many relationships using it and developed several different ways to calculate it. You might like to think about this way of looking at the derivation of e.

If we invested \$1 at 100% compound interest each year then the returns (A) on our investment would be represented by the formula $A = P(1+\frac{r}{100}) = (1+\frac{1}{n})^n$, since $P = 1$, $r = 100/\text{number of periods}$

If interest is paid yearly you receive $A = (1+\frac{1}{n})^n = (1+\frac{1}{1})^1 = 2$

If interest is paid monthly you receive $A = (1+\frac{1}{n})^n = (1+\frac{1}{12})^{12} = 2.61$
(12 is months per year)

If interest is paid daily you receive $A = (1+\frac{1}{n})^n = (1+\frac{1}{365})^{365} = 2.71$
(365 is days per year)

If interest is paid hourly you receive $A = (1+\frac{1}{n})^n = (1+\frac{1}{8760})^{8760} = 2.7181235$
(8760 is hours per year)

So it appears that we can get very close to the value of e if we find the value of $A = (1+\frac{1}{n})^n$ as n gets very large. **Do not learn this.**

Depreciation

Businesses and individuals usually own assets. Over time these assets sometimes lose value because of age, use or obsolescence. This fall in value is called **depreciation**. One type of depreciation is called diminishing value depreciation. It is like compound interest in reverse because the value of the asset is highest at the time of purchase and then continually reduces over time.

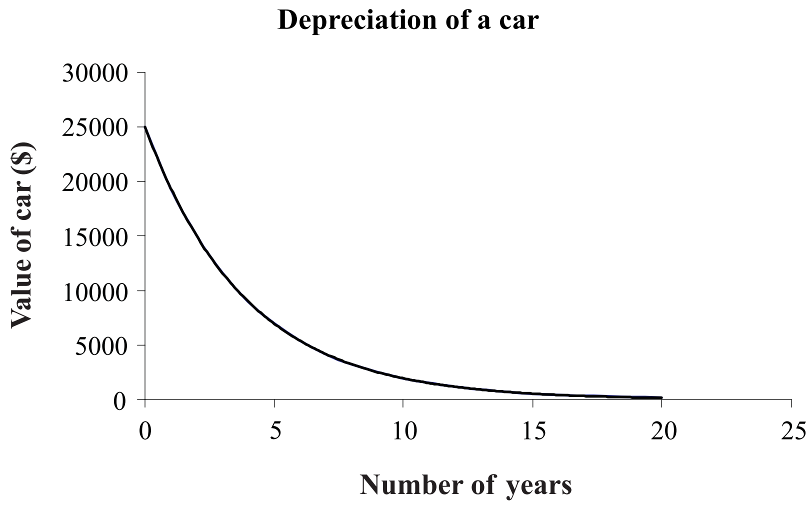
Suppose we purchased a car valued at $25 000 and we know that cars depreciate at 22.5% each year, what would the value of the car be after a period of time. Examine the table below for calculations for 5 years. Notice that they are similar to those for calculation of compound interest.

Year	**Total value of car ($)**	
Year 0	25 000	25 000
Year 1	(1–0.225) of year 0	$25000(1-0.225)$ $=19375$
Year 2	(1–0.225) of year 1	$(1-0.225)\times 25000(1-0.225)$ $=25000(1-0.225)^2$ $=15015.63$
Year 3	(1–0.225) of year 2	$(1-0.225)\times 25000(1-0.225)^2$ $=25000\times(1-0.225)^3$ $=11637.11$
Year 4	(1–0.225) of year 3	$(1-0.225)\times 25000\times(1-0.225)^3$ $=25000\times(1-0.225)^4$ $=9018.76$
Year 5	(1–0.225) of year 4	$(1-0.225)\times 25000\times(1-0.225)^4$ $=25000\times(1-0.225)^5$ $=6989.54$

This pattern also might be familiar because it is a special case of the depreciation formula,

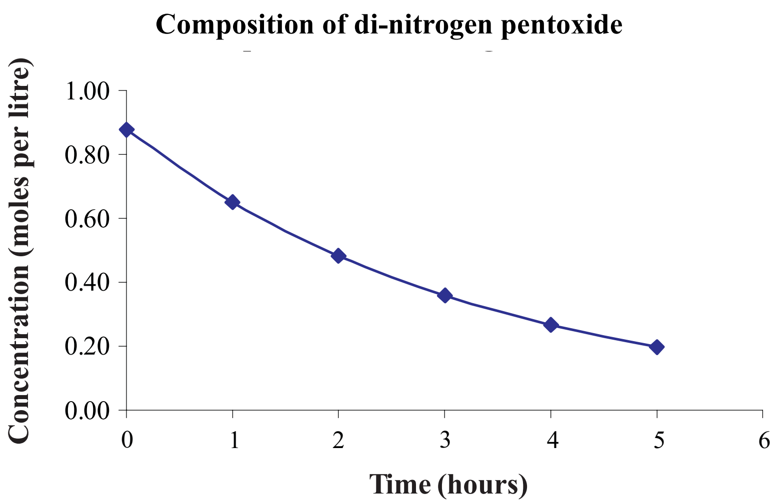
$D=P(1-\frac{r}{100})^n$ **where D is final value of the asset, P is the initial value of the asset, r is the rate of depreciation per period and n is the number of depreciation periods.**

If we were to graph the function determined in the example above we would see that it is a typical exponential decay shape if taken over 20 years.



Chemical reactions

Some chemicals when in solution break down into different components so that the concentration of the original compound changes over time. In 1864 Guildberg and Waage recognized that at a constant temperature the rate of this type of reaction followed an exponential decay curve. An example of this is the breakdown of di-nitrogen pentoxide into nitrogen oxide and oxygen. When this decomposition is graphed we get the following figure.



The equation for this function is:

$C(t) = 0.87e^{-0.30t}$, where t is in hours and $C(t)$ is concentration in moles per litre. The initial amount of chemical in the solution was 0.87 moles per litre.

The above show some specific applications of exponential functions. Let's look as some examples using these types of applications.

Example

Peter invested $8 000 in a fixed term deposit for 3 years attracting 12% pa interest compounded quarterly. What would be his total return?

We can use the formula $A = P(1 + \frac{r}{100})^n$ to calculate the amount returned but first we need to determine the components of the formula. *P*, the principal is the amount invested and is $8 000. *n* is the number of interest gathering periods, so because the interest is accrued quarterly for 3 years then *n* must be 12. *r* is the interest rate as a percentage at each accruing period. Since the interest is defined as 12% pa (per annum) it should be divided by 4 to get the interest rate at each quarter. Therefore *r* is 3%.

So the amount A is given by, $A = P(1 + \frac{r}{100})^n$, $P = 8000, r = 3$ and $n = 12$, substituting into the formula,

$$\begin{aligned} A &= 8000(1 + \frac{3}{100})^{12} \\ &= 11406.09 \end{aligned}$$

The amount returned is $11 406.09.

Example

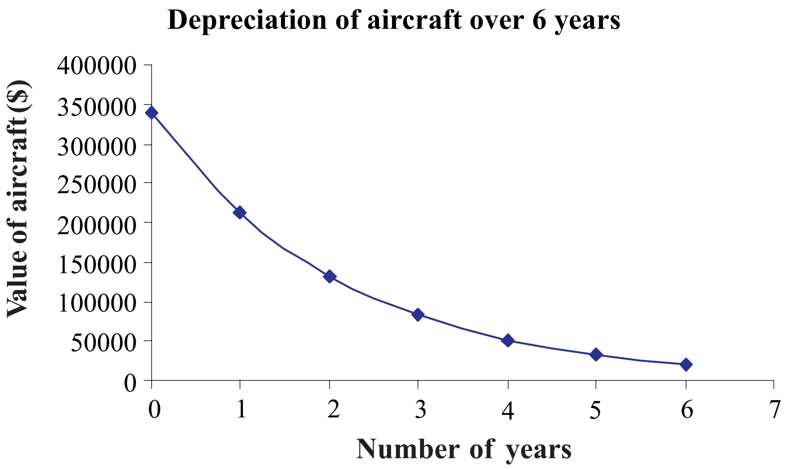
Aircraft used in agricultural spraying and dusting are eligible for 37.5% depreciation allowance (diminishing value). If the aeroplane originally cost $340 000, construct a depreciation schedule (to the nearest dollar) for the first 6 years. Present this as a table and a graph.

A depreciation schedule involves calculating the value of the aircraft every year for six years.

The depreciation formula to calculate the depreciated value, *D*, is $D = P(1 - \frac{r}{100})^n$, where *P* is $340 000, *r* is 37.5% (depreciation is only calculated yearly in this instance) and *n* changes from 0 to 6.

The depreciation schedule will thus be:

Year	Value of aircraft ($)
0	340 000
1	212 500
2	132 813
3	83 008
4	51 880
5	32 425
6	20 266



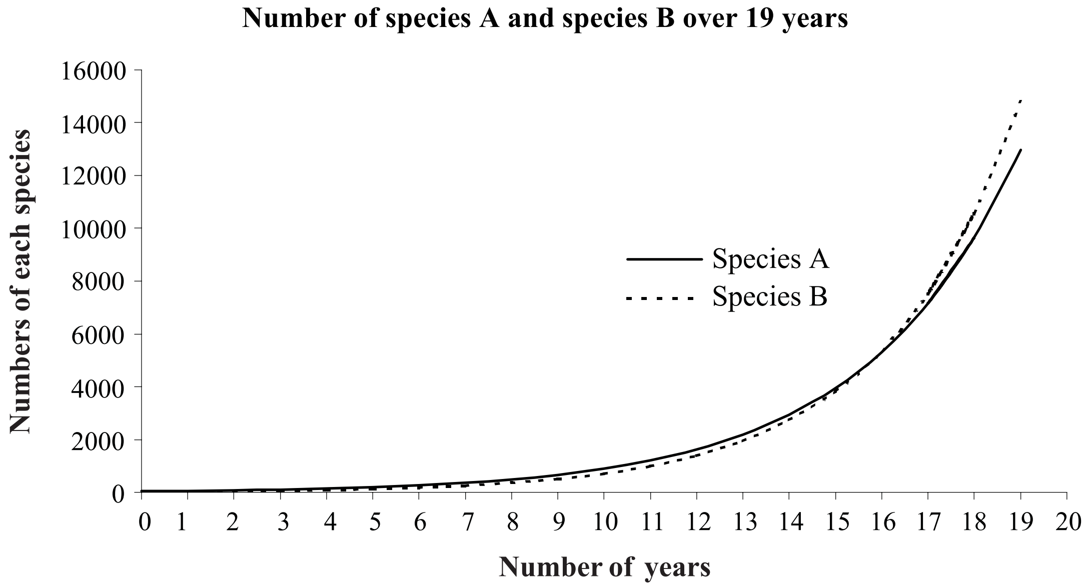
Example

In a newly created wildlife sanctuary, it is estimated that the numbers of the population of Species A will triple every 3.7 years. The population growth will follow the exponential function $N_A(t) = 46 \times 3^{\frac{t}{3.7}}$. The population numbers of Species B will quadruple every 4.1 years and its growth will follow the function $N_B(t) = 24 \times 4^{\frac{t}{4.1}}$. Sketch the graphs of each function and estimate how long it will take before the numbers of the two species are equal. (N_A and N_B represent the number of each species, t is the time in years).

From the formulas of each graph, we know that both are exponential growth functions and increase as time increases. You could graph each function on a graphing package or by hand. To graph the functions by hand it will be important to first calculate a table of values for each function and use these to help plot the curves. Such a table of values is presented below.

Number of years	Number of species A	Number of species B
0	46	24
1	62	34
2	83	47
3	112	66
4	151	93
5	203	130
6	273	183
7	368	256
8	495	359
9	666	503
10	896	706
11	1 206	990
12	1 622	1 388
13	2 183	1 946
14	2 938	2 729
15	3 954	3 827
16	5 321	5 367
17	7 160	7 526
18	9 636	10 554
19	12 967	14 799

The curves of each function are shown below.



From the curves it appears that the species numbers will be equal between 15 and 17 years.

Example

The amount of certain elements that decay over time is modelled by the function

$N(t) = N_0 e^{-kt}$, where N is the amount of the element in grams, t is time in the units given and N_0 the initial amount of the element in grams (k is a constant specific for each element). Find how much carbon and iodine are present after a set period of time (t) given the information provided in the following table.

Element	***k***	N_0	***t***
Carbon	1.203×10^{-4}	3	5760 years
Iodine	0.08666	5	8 days

Using the decay function, $N(t) = N_0 e^{-kt}$:

For Carbon, the amount left after t = 5 760 years is

$$\begin{aligned} N &= 3 \times e^{-1.203\times10^{-4}\times5760} \\ &\approx 3 \times e^{-0.6929} \\ &\approx 1.500 \text{ g} \end{aligned}$$

For Iodine, the amount left after t = 8 days is

$$\begin{aligned} N &= 5 \times e^{-0.08666\times8} \\ &\approx 5 \times e^{-0.6933} \\ &\approx 5 \times 0.4999 \\ &\approx 2.5 \text{ g} \end{aligned}$$

It is interesting to note that in each of the cases above the resultant mass is half of the initial mass of each element. This is an important notion in nuclear research. The time taken for a quantity of a specific element to be reduced to one half of its original mass is known as the **half-life** of the element. The half-life of carbon is 5 760 years and the half-life of iodine is 8 days.

Activity 5.4

1. The taxation department allows depreciation of 25% pa on the diminishing value of computers. If a business installs computers valued at $120 000, construct a depreciation schedule for the next five years presenting the information in both table and graphical format.
2. It has been projected that inflation over the next 8 years will run at 4% (compounded annually). How much would you expect to pay for a litre of milk in 8 years time if its cost today is $1.25?
3. Melvil has just sold his house for $90 000 and decides to invest the proceeds for a fixed term of 1 year. His bank offers 2 investment packages: The first attracts 6.5% compounded half yearly and the second 6.4% compounded daily. Which investment package would you advise Melvil to take?
4. A particular bacteria culture doubles every 20 minutes and follows the exponential function $N(t) = 200 \times 2^{3t}$, where $N(t)$ is the number of bacteria in the culture after t hours. Sketch the graph of the function and use it to estimate how long it will be before the number of bacteria in the culture reaches 1 000 000.
5. A certain substance decays exponentially over time and is modelled by the function $N(t) = 4e^{(-\frac{t}{5771})}$, where $N(t)$ is measured in grams and t in years. Find how much of the substance is present initially and how much is present 4 000 years later. Use your findings to comment on the half-life of this particular substance.
6. A small rodent grows in weight 10% per month for the first 10 months of its life. What would be the weight of the animal at 10 months if it weighed 50 grams at birth? (Hint: This can be treated like compound interest.)

5.1.4 Average rate of change

We have previously made statements about exponential functions in terms of how quickly one variable is changing with respect to another.

Statements such as:

…as the independent variable increases the dependent variable increases rapidly…

…as the independent variable increases the dependent variable decreases more slowly…

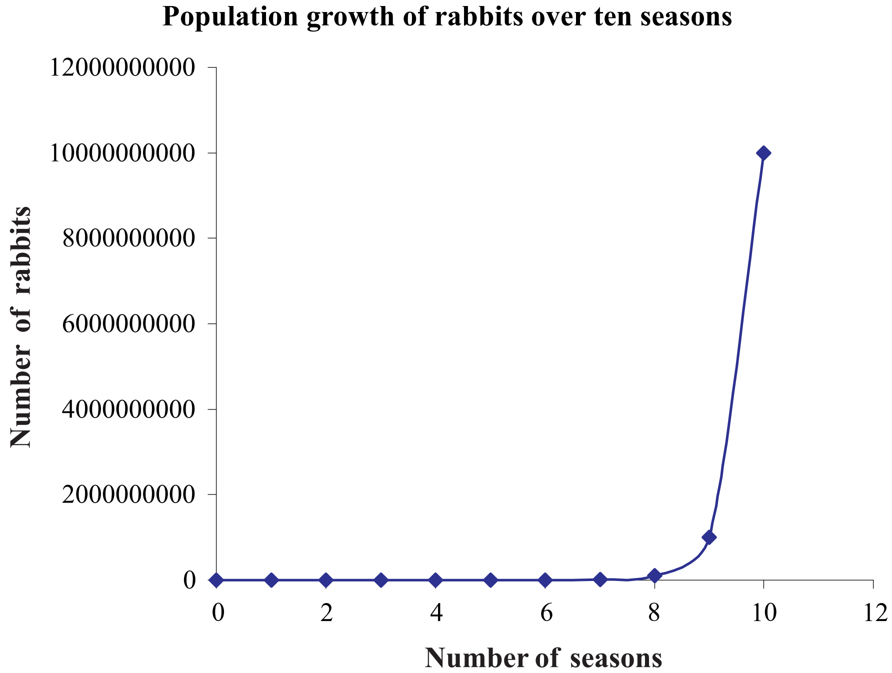
... $y = 3^{2x}$ *increases at a rate faster than* $y = 3^x$ *...*

... $y = 4 \times 3^x$ *appears to increase at the same rate at* $y = 3^x$*, but is slower than* $y = 3^{2x}$ *...*

But what do these statements actually mean and how can we quantify them? Recall from module 5 that average rates of change were measured in parabolic curves by considering the gradient of a straight line joined between two points on the curves. We can use the same technique to get an average rate of change of an exponential function.

Let's consider an example for exponential growth and one example for exponential decay.

The growth in Australia's rabbit population was depicted in the graph below.



How can we compare the growth rates of the population over the first two seasons with the last two seasons?

To find the average rate of change between the first two seasons we would have to know the gradient of the straight line that connected these two points, $n = 0$ and $n = 2$.

Recall from module 5 that the gradient of a straight line is the difference in the height divided by the difference in the horizontal values,

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Using the equation for the function for this curve, $f(n) = 10^n$, where n is the number of seasons and $f(n)$ is the number of rabbits, we can generate a table of values from which we can calculate the gradients between each of the points.

Seasons (n)	**Number of rabbits ($f(n)$)**
0	1
2	100
8	$10^8 = 100\,000\,000$
10	$10^{10} = 10\,000\,000\,000$

Average rate of change over the first two seasons is

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}} = \frac{f(n_2) - f(n_1)}{n_2 - n_1}$$

$$m = \frac{100 - 1}{2 - 0} \approx 50$$

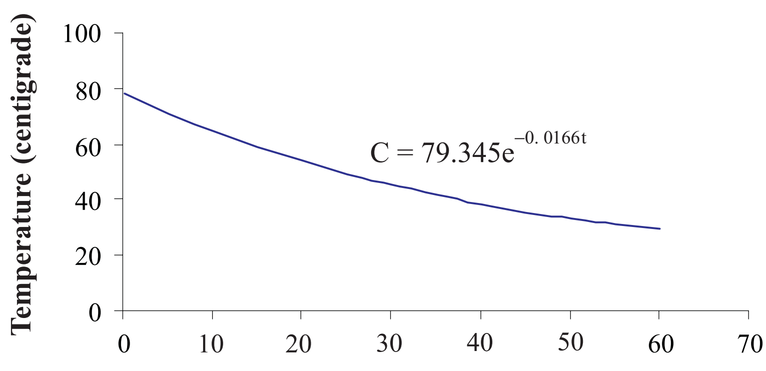
Average rate of change over the last two seasons is

$$m = \frac{10\,000\,000\,000 - 100\,000\,000}{10 - 8} \approx 4\,950\,000\,000$$

In real terms, because the gradients are both positive, the function is increasing and population is growing at a rate of 50 rabbits per season over the first two seasons compared with 4 950 000 000 rabbits per season over the last two seasons. This means that population growth rate was small at the start but increased very rapidly over the last two seasons, and is thus increasing.

Our cooling cup of coffee has a different exponential pattern. But how is the temperature changing between 0 and 5 minutes compared with between 20 and 25 minutes? We could read these values off the graph if it was detailed enough. Otherwise we could calculate the rate of change from the function values. The equation of this function is $C = 79.345e^{-0.0166t}$, where C is the temperature in centigrade and t is time in minutes.

Temperature of a cup of coffee over time



Using the equation for the function for this curve, $C = 79.345e^{-0.0166t}$, we can generate a table of values from which we can calculate the gradients over each time period.

Time (min)	Temperature (C°)
0	$C(0) = 79.345e^{-0.0166\times 0} \approx 79$
5	$C(5) = 79.345e^{-0.0166\times 5} \approx 73$
20	$C(20) = 79.345e^{-0.0166\times 20} \approx 57$
25	$C(25) = 79.345e^{-0.0166\times 25} \approx 52$

Average rate of change between 0 and 5 minutes is

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}} = \frac{C(t_2) - C(t_1)}{t_2 - t_1}$$

$$m = \frac{73 - 79}{5 - 0} = -1.2$$

Average rate of change between 20 and 25 minutes is

$$m = \frac{52 - 57}{25 - 20} = -1$$

In real terms, because the gradients are both negative, the function is decreasing. The temperature is dropping at an average rate of 1.2 degrees per minute during the first five minutes and 1 degree per minute between 20 and 25 minutes. This means that the cooling rate is decreasing or getting slower i.e. the coffee is cooling more slowly as time passes.

Activity 5.5

1. For each of the functions complete the table, sketch the graph and find the average rate of change over the stated intervals.

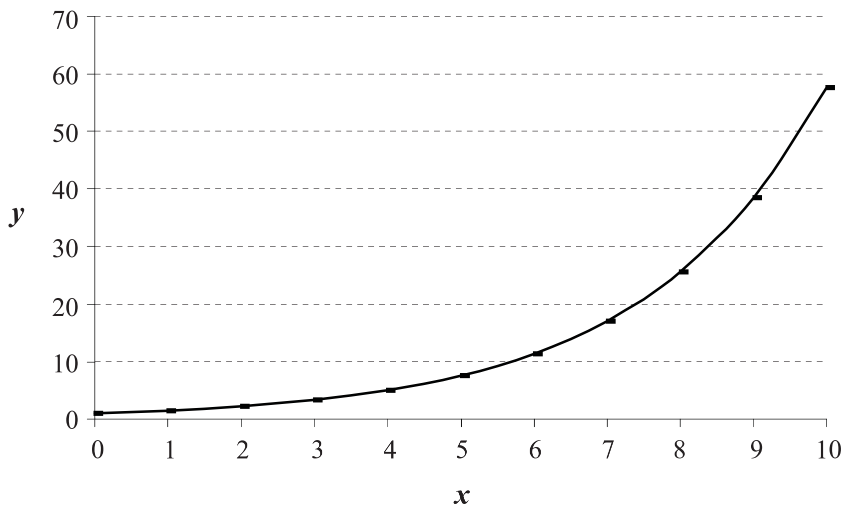
   (a) $y = e^x$ intervals from –2 to –1 and from 2 to 3

x	***y***
–2	
–1	
0	
1	
2	
3	

   (b) $f(x) = 2^{-x}$ intervals from –3 to –1 and from 1 to 3

x	***f(x)***
–3	
–2	
–1	
0	
1	
2	
3	

2. Using the graph below answer the following questions.



(a) Read off the approximate values (to the nearest 10) from the graph to find the average gradient between the points $x = 9$ and $x = 10$.

(b) Describe in your own words what the rate of change is between the points in (a)

3. A car bought for \$32 000 depreciates during 6 years at 25% per year on the diminishing value. What is the average rate of change of the value of the car (to the nearest dollar)

(a) during the first 2 years

(b) over the 6 years?

4. Ice melts at a rate proportional to its size and shape. At room temperature the volume (cm^3) of an iceblock changes according to the exponential function $V = 30e^{-0.05672t}$, where t is time in minutes.

(a) Find the rate at which the iceblock melts during the first five minutes.

(b) At what average rate is it melting from the 30th to the 35th minute.

(c) What does this mean about the melting rate?

5. Producing in bulk usually results in decreased production costs per unit and consequently an increased profit margin on each of the items. For a certain line of goods produced in 1000 unit lots, the profit may be expressed exponentially as $P = e^{0.0198(n-1)}$, where P is the profit per unit (\$) and n is the number of 1000 unit lots produced. Use the equation to generate values to find the average rate of change in profit when increasing production from 4 000 to 5 000 units.

5.1.5 The inverse of the exponential function

In previous sections of this module we set up a function from which we could calculate the number of rabbits after a given number of seasons. This function was $f(n) = 10^n$ where $f(n)$ is the number of rabbits and n is the number of seasons. Writing the function in this way shows that we are thinking of the number of rabbits as a function of seasons.

Now suppose that instead of wanting to calculate the number of rabbits for any season, we were given the number of rabbits and wanted to know how many seasons it had taken to reach this number…a very reasonable question for an ecologist to ask. How could we do this?

Well firstly, we could use our table of values calculated previously. It allowed us to calculate the number of rabbits given the season and is represented by the function, $f(n) = 10^n$. If we wanted to do the reverse and find the season given the number of rabbits, we could still use this table but read it backwards. So if we had 1 000 rabbits it must be season 3.

Season (*n*)	**Number of rabbits *f(n)***
0	10^0
1	10^1
2	10^2
3	10^3
4	10^4
5	10^5
6	10^6
7	10^7
8	10^8
9	10^9
10	10^{10}

Or more conventionally, as we have done in module 4, we would just switch the columns to get the table below.

Number of rabbits *(n)*	**Seasons *g(n)***
$10^0 = 1$	0
$10^1 = 10$	1
$10^2 = 100$	2
$10^3 = 1\ 000$	3
$10^4 = 10\ 000$	4
$10^5 = 100\ 000$	5
10^6	6
10^7	7
10^8	8
10^9	9
10^{10}	10

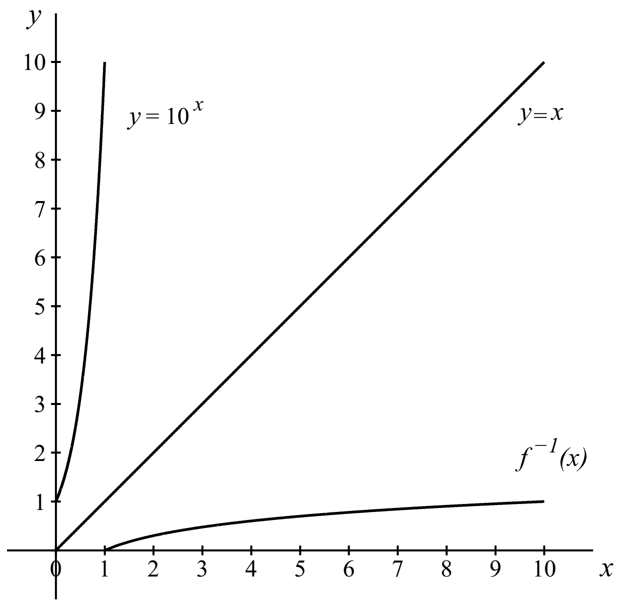
Do you recall this process of switching the place of the independent and dependent variables from module 4? It is called finding the inverse of a function. So $g(n)$ is really the inverse of $f(n)$ and could be written as $f^{-1}(n)$.

But what is this inverse function? Well, if you look at the above table you might notice a pattern. The value of the function is the power to which we have raised 10. So if we want to find the value of the inverse function at $n = 100$. First think what power of 10 is 100...it will of course be 2. So the value of the inverse function is 2.

In summary we could say

$f^{-1}(n)$ = the power to which we raise 10 to get n

If we used some of the values above and graphed the two functions on the same set of axes we would get the following. As is usual with inverse functions one should be a reflection of the other about the line, $y = x$



In the above diagram if x is 1 then $f(x)$ is 10 and $f^{-1}(x)$ is 0, since the power to which we raised 10 to get 1 is 0. We will investigate this further in the next section but first complete the following activity.

Activity 5.6

1. In the compound interest example in section 5.1.3 we looked at an investment of $3 000 at 3.5% compounded annually and found that it had amounted to $3 563 in 5 years.

   (a) Set up a table showing number of years *(n)* and amount earned *A(n)*.

   (b) Use this table to produce the inverse of this function.

   (c) Describe the use of the inverse function.

   (d) Determine approximately how long it will take for $3 000 to grow to $4 000.

2. By the year 2000 Australia's population in expected to be 19.5 million and after that growing exponentially as $P(t) = 19.5e^{0.0163t}$, where P is millions of people and t is years.

   (a) Set up a table of value for this function.

   (b) Use this table to set up a table of values for the inverse of the P function.

   (c) Use this table to find when the population will reach 22.5 million.

3. (a) On the same set of axes draw

   - $y = x, \quad 0 \le x \le 10$
   - $y = 10^{0.1x}$

   (b) Use the two graph above to roughly sketch the inverse of $y = 10^{0.1x}$

4. The decay of radium is modelled by the function $R = R_0e^{-0.077t}$, where R is the amount remaining (g), t is time (weeks) and R_0 is the original amount. Generate a table of values to find the half-life of 10 g of radium. Remember that half-life means time to reach half of the original amount.

5. Carbon dating involves the measurement of concentration of carbon remaining in an object. The decay function $C = 100 \times 2^{-0.1786t}$ is used to determine the age of a bone taken from an archaeological dig, where C is the concentration remaining and t is time in thousands of years. It is found that 60% of the original carbon remains in the samples. Estimate the age of the bone. (Hint: Develop a table of values for the inverse function and find when $C = 60$).

5.2 Logarithmic functions

5.2.1 What is a logarithm?

Let's experiment. Using the 10^x key on your calculator, evaluate the following powers.

$10^2 =$

$10^3 =$

$10^{2.5} =$

$10^{1.5} =$

$10^{0.5} =$

$10^0 =$

$10^{-0.5} =$

$10^{-2} =$

Now associated with the 10^x key on your calculator is the **log** key. Use that key to complete the following table

$10^2 = 100$	$\log 100 =$
$10^3 = 1000$	$\log 1000 =$
$10^{2.5} \approx 316.22777$	$\log 316.22777 =$
$10^{1.5} \approx 31.622777$	$\log 31.622777 =$
$10^{0.5} \approx 3.1622777$	$\log 3.1622777 =$
$10^0 = 1$	$\log 1 =$
$10^{-0.5} \approx 0.3162277$	$\log 0.31622777 =$
$10^{-2} = 0.01$	$\log 0.01 =$

What did you discover?

You will have noticed that if you take the logarithm of the answer to 10^2, you get 2. This same pattern should have happened in every case.

Taking the logarithm to the base 10 of 100 asks us the question what power of 10 do we use to get 100.

The logarithmic function is actually the inverse of the exponential function.

Now let's do some more calculations and think about what the numbers mean and how to say them. The table below has been completed for you but it would be useful for you to go to your calculator now and try to calculate the values shown.

Value of x	**Value of log$x$**	**How to write it**	**What it means**	**How to say it**
1	0	$\log_{10} 1 = 0$	What power do we raise 10 to, to get 1	log 1 to the base 10, is 0
10	1	$\log_{10} 10 = 1$	What power do we raise 10 to, to get 10	log 10 to the base 10, is 1
100	2	$\log_{10} 100 = 2$	What power do we raise 10 to, to get 100	log 100 to the base 10, is 2
1000	3	$\log_{10} 1000 = 3$	What power do we raise 10 to, to get 1 000	log 1 000 to the base 10, is 3

You might ask at this stage why bother with inventing a new function when we could have just as easily calculated the answers to the above using trial and error and knowledge of powers of 10.

Well not all numbers are whole powers of 10 but luckily the calculator has been programmed to easily evaluate the logarithm of any real number. Before we had calculators we had special books filled with logarithms of every possible number. They were called log tables. We don't bother with these now, so let's use the calculator to evaluate the logarithm of numbers which are not whole number powers of 10. We have done some for you below. Try them for yourself.

Value of x	**Value of log$x$**	**How to write it**	**What it means**
2	0.3010	$\log_{10} 2 \approx 0.3010$	What power do we raise 10 to, to get 2
7567	3.8789	$\log_{10} 7567 \approx 3.8789$	What power do we raise 10 to, to get 7 567

Logarithms to the base 10 are called **common logarithms**.

Logarithms can be to any base we like, but we only have calculator keys for base e and 10. Previously, it was noted that exponential functions often took the form of $f(x) = e^x$. The inverse of this function is a logarithmic function, this time to the base e instead of 10. It is often abbreviated to $f(x) = \ln x$ (pronounced as 'ell en x'). This type of logarithm is called the **natural or naperian logarithm**.

Now try some calculations on your calculator using the ln key and think about what the numbers mean and how to say them.

Value of x	**Value of $\ln x$**	**How to write it**	**What it means**	**How to say it**
1	0	$\ln 1 = 0$	What power do we raise e to, to get 1	log 1 to the base e, is 0
$e \approx 2.7183$	1	$\ln e = 1$	What power do we raise e to, to get 2.7183	log e to the base e, is 1
$e^2 \approx 7.3891$	2	$\ln 7.3891 \approx 2$	What power do we raise e to, to get 7.3891	log 7.3891 to the base e, is 2
10	2.3026	$\ln 10 \approx 2.3026$	What power do we raise e to, to get 10	log 10 to the base e, is 2.3026
100	4.6052	$\ln 100 \approx 4.6052$	What power do we raise e to, to get 100	log 100 to the base e, is 4.6052
1 000	6.9078	$\ln 1\,000 \approx 6.9078$	What power do we raise e to, to get 1000	log 1 000 to the base e, is 6.9078
2	0.6931	$\ln 2 \approx 0.6931$	What power do we raise e to, to get 2	log 2 to the base e, is 0.6931
7 567	8.9316	$\ln 7\,567 \approx 8.9316$	What power do we raise e to, to get 7567	log 7 567 to the base e, is 8.9316

So far we have only looked at two types of logarithms (base 10 and base e), but we can calculate logarithms for any base as long as we remember that **logarithms are just the power of a number**.

In fact, we can write all numbers in either an exponential form or a logarithmic form as in the following table.

Exponential form	**Logarithmic form**
$10^2 = 100$	$\log_{10} 100 = 2$
$3^2 = 9$	$\log_3 9 = 2$
$2^3 = 8$	$\log_2 8 = 3$
$9^{\frac{3}{2}} = 27$	$\log_9 27 = \frac{3}{2}$
$7^{-2} = \frac{1}{49}$	$\log_7 \frac{1}{49} = -2$

Can you recognize a pattern in the relationship between the exponential form and the logarithmic form?

It will be something like this and gives us the definition of a logarithm,

$$\boxed{\text{If } a^x = n \text{ then } x = \log_a n.}$$

Example

Use your knowledge of the definition of a logarithm to evaluate $\log_4 64$ and $\log_{\frac{1}{2}} 32$.

To evaluate $\log_4 64$, let's make the expression equal to an unknown, say *p*.

So $p = \log_4 64$. Using the definition of a logarithm we can rewrite this expression in an exponential form

$$p = \log_4 64 \quad \Rightarrow \quad 4^p = 64$$

By trial and error or our knowledge of arithmetic, you will notice that $64 = 4^3$, so

$$\begin{aligned} 4^p &= 64 \\ 4^p &= 4^3 \\ p &= 3 \end{aligned}$$

So that means $\log_4 64 = 3$, or we could say what number do we have to raise 4 to, to get 64.

To evaluate $\log_{\frac{1}{2}} 32$, let's make the expression equal to an unknown, say *p*.

So $p = \log_{\frac{1}{2}} 32$, using the definition of a logarithm we can rewrite this expression in an exponential form

$$p = \log_{\frac{1}{2}} 32 \quad \Rightarrow \quad (\frac{1}{2})^p = 32$$

By trial and error or your knowledge of arithmetic, you will notice that $32 = 2^5$, so

$$\begin{aligned} (\frac{1}{2})^p &= 32 && \\ (\frac{1}{2})^p &= 2^5 && \text{Write 32 as a power of 2.} \\ \frac{1^p}{2^p} &= 2^5 && \text{Use index law to break fraction as shown.} \\ 1 &= 2^5 \times 2^p && \text{Multiply both sides by 2 to the power } p. \\ \frac{1}{2^5} &= 2^p && \text{Divide both sides by 2 to the power 5.} \\ 2^{-5} &= 2^p && \text{Write LHS with a negative power.} \\ p &= -5 && \end{aligned}$$

So that means $\log_{\frac{1}{2}} 32 = -5$ or we could say what number do we raise $\frac{1}{2}$ to, to get 32.

Example

Solve the logarithmic equation $\log_5 x = 3$ for x.

To solve this equation first change from the logarithmic form to an exponential form

$\log_5 x = 3 \quad \Rightarrow \quad 5^3 = x$

By trail and error or our knowledge of arithmetic we know that $5^3 = 125$, so $x = 125$.

Something to talk about…

What happens when you try to calculate the logarithm of a negative number? Discuss with your colleagues or the discussion group why there is a problem calculating logarithms of negative numbers.

Activity 5.7

1. Change each of the following from exponential form to logarithmic form.

Exponential form	**Logarithmic form**
$5^2 = 25$	
$9^{\frac{1}{2}} = 3$	
$10^{1.8} \approx 63$	
$2^{-2} = \frac{1}{4}$	
$3^x = 10$	

2. Write the following equations in index form:

   (a) $\log 1000 = 3$

   (b) $\log_4 16 = 2$

   (c) $4 = \log_2 16$

   (d) $\ln 20 \approx 3$

3. Use your knowledge of logarithms to evaluate:

   (a) $\log(1\ 000\ 000)$

   (b) $\log_2 \frac{1}{8}$

4. Evaluate $\log_{25} 5$ and $\log_2 0.25$

5. Solve the following equations for x:

   (a) $x = \log 10$

   (b) $\log_2 x = 5$

   (c) $\log_4 x = 0$

   (d) $\log_x 27 = -3$

   (e) $\ln x = 3$

5.2.2 Properties of logarithms

From our work above it appears that logarithms are really powers in a different form. So it might follow that some of the properties that apply to powers might be used to help develop some similar properties for logarithms. If you have forgotten the index laws from module 3 now would be a good time to revise them.

Before examining the properties let's think about the relationship between indices and logarithms in more detail.

When we multiply two numbers which are in index form we add the indices, so
$10^3 \times 10^4 = 10^7$

But logarithms are really indices, so

$10^3 = 1000$ and $10^4 = 10000$ and $10^7 = 10000000$ can be written as
$\log_{10} 1000 = 3$ and $\log_{10} 10000 = 4$ and $\log_{10} 10000000 = 7$

Now

$\log_{10} 1000 + \log_{10} 10000 = 3 + 4$	Evaluate logarithms.
$= 7$	Write 7 in its logarithmic form.
$= \log_{10} 10000000$	Write 10000000 as a product.
$= \log_{10}(1000 \times 10000)$	

We can now see that $\log_{10} 1000 + \log_{10} 10000 = \log_{10} 10000000$

This is really a special case of our first property below. Many of the other properties are based on this relationship.

The logarithm of a product is the sum of the logarithms $\Rightarrow$ $\log(mn) = \log m + \log n$

This only works when the bases of the logarithms are the same.

For example,

$$\begin{aligned}\log_{10} 200 + \log_{10} 5 &= \log_{10}(200 \times 5) && \text{The bases are the same so can use log properties.}\\ &= \log_{10} 1000 \\ &= 3 && \text{Calculate either from the calculator or using the definition of a logarithm.}\end{aligned}$$

When we divide two numbers which are in index form we subtract the indices so in logarithms the following will result.

The logarithm of a quotient is the logarithm of the numerator minus the logarithm of the denominator $\Rightarrow$ $\log(\frac{m}{n}) = \log m - \log n$

This only works when the bases of the logarithms are the same.

For example,

$$\begin{aligned}\log_2 32 - \log_2 8 &= \log_2(\frac{32}{8}) && \text{The bases are the same so can use log properties.}\\ &= \log_2 4 \\ &= 2 && \text{Recall using the definition of a logarithm } 2^2 = 4 \text{ so } \log_2 4 = 2\end{aligned}$$

The logarithm of a number raised to a power is the logarithm of the number multiplied by the power $\Rightarrow$ $\log m^n = n \log m$

For example,

$$\begin{aligned}\log_7 64 &= \log_7 4^3 \\ &= 3\log_7 4\end{aligned}$$

This last property is particularly useful in solving exponential equations…we will return to this later.

The logarithm of one to any base will always be 0 $\Rightarrow$ $\log_a 1 = 0$

This is a direct result of the index rules. Remember that $a^0 = 1$, so when this is written in logarithmic form $\log_a 1 = 0$ (applies only when a ≠ 0).

Examples of this property are:

$$\log_{10} 1 = 0$$
$$\log_e 1 = \ln 1 = 0$$
$$\log_2 1 = 0$$

Note because any number to a power will never be equal to zero, then the logarithm of zero is undefined.

The logarithm of any number to the base of that number will be 1 $\Rightarrow$ $\log_a a = 1$

This is a direct result of the index rules. Remember for example that $a^1 = a$, so when this is written in logarithmic form $\log_a a = 1$

Examples of this property are:

$\log_{10} 10 = 1$
$\log_e e = 1$
$\log_2 2 = 1$

For interest only …

Modern calculators can only be used to directly evaluate logarithms to base 10 or base *e*, using the log and the *ln* button on the calculator respectively.

However the values of logarithms to other bases can be approximated by using a simple change of base rule. For example:

$$\log_2 3 = \frac{\log_{10} 3}{\log_{10} 2} \approx 1.584 \text{ or } \log_2 3 = \frac{\log_e 3}{\log_e 2} \approx 1.584$$

You can change the base to either base 10 or to base *e* the final answer will be the same.

In general this rule is called the change of base rule and is expressed as:

$$\log_a b = \frac{\log_{10} b}{\log_{10} a} = \frac{\log_e b}{\log_e a}$$

Do not learn this.

Example

Without using a calculator simplify the following expression $2\log_{10} 3 + \log_{10} 16 - 2\log_{10} \frac{6}{5}$

There are numerous ways to simplify this expression, here is one alternative.

$2\log_{10} 3 + \log_{10} 16 - 2\log_{10} \frac{6}{5}$	The bases are the same so can use log properties.
$= \log_{10} 3^2 + \log_{10} 16 - \log_{10} (\frac{6}{5})^2$	Use logarithm of a power property to write each as a single logarithm.
$= \log_{10} (3^2 \times 16 \div (\frac{6}{5})^2)$	Use multiplication of a log to write as one logarithm.
$= \log_{10} (100)$	
$= 2$	Recall that, $10_2 = 100$, $\log_{10} 100 = 2$

Example

Without using a calculator write the following expression as a single logarithm or number
$\log_5 25 + \log_5 125 - \log_5 0.04$

There are numerous ways to simplify this expression, here is one alternative.

$\log_5 25 + \log_5 125 - \log_5 0.04$ The bases are the same so can use log properties.

$= \log_5 5^2 + \log_5 5^3 - \log_5 \frac{1}{25}$

$= 2\log_5 5 + 3\log_5 5 - \log_5 5^{-2}$

$= 2\log_5 5 + 3\log_5 5 - -2\log_5 5$

$= 2\log_5 5 + 3\log_5 5 + 2\log_5 5$

$= 7\log_5 5$

$= 7$ Recall that because $5_1 = 5$, $\log_5 5 = 1$

Example

Use the definition of the logarithm and its properties to solve the following equation
$\log_{10}(x-2) + \log_{10} 3 = 1$

$\log_{10}(x-2) + \log_{10} 3 = 1$ The bases are the same so can use log properties.

$\log_{10}[(x-2)\times 3] = 1$

$3(x-2) = 10^1$ Use log property to write as a single logarithm.

$3(x-2) = 10$ Use logarithm in an exponential form.

$(x-2) = \frac{10}{3}$

$x = \frac{10}{3} + 2$

$x = \frac{16}{3}$ or $5\frac{1}{3}$

Check: When

$x = \frac{16}{3}$, LHS $= \log_{10}(\frac{16}{3} - 2) + \log_{10} 3 = \log_{10}\frac{10}{3} + \log_{10} 3 = \log_{10}(\frac{10}{3}\times 3) = \log_{10} 10 = 1 =$ RHS

Solution is $x = \frac{16}{3}$

Activity 5.8

1. Without using a calculator simplify the following expressions:
   (a) $\log_3 27 + \log_3 \frac{1}{3}$
   (b) $\log_2 16 - \log_2 8$
   (c) $\log 125 + \log 32 - \log 4$
2. Find the value of each of the following in simplest form.
   (a) $2\log_{12} 3 + 4\log_{12} 2$
   (b) $\ln 25 + 2\ln 0.2$
   (c) $4\log_5 10 - 3\log_5 2 - \log_5 10$
3. Use the logarithmic properties to solve the equations below.
   (a) $\log_3 x = \log_3 4 + \log_3 2$
   (b) $\log 5m = 2\log m$
   (c) $2\log p + 3 - \log p^5 = 0$
   (d) $\log 2 + \log 5 + \log y - \log 3 = 2$

A bit of history… **For interest only**

Today to many of us logarithms appear to be the most perverse and artificial of mathematical functions. But when they were invented they were thought to be the washing machine of the 17th century in that they saved many professionals from the drudgery of long multiplication and division, especially in the field of astronomy. The idea was first developed by John Napier in about 1594 and perfected by Henry Briggs in 1614. The idea is simple:

If you want to multiply two numbers, say 2376 and 34678 first write them as a power to the base of ten (this is logarithm), then instead of multiplying the numbers we can add the indices.

$$2376 = 10^{3.3759....}, 34678 = 10^{4.5400....}$$

$$\begin{aligned} 2376 \times 34678 &= 10^{3.3759....} \times 10^{4.5401....} \\ &= 10^{3.3759....+4.5401....} \\ &= 10^{7.9160.....} \\ &= 82394928 \end{aligned}$$

(Actual answer on a calculator is 82394928)

If you went to secondary school in Australia before the mid 1970s you might remember doing this with logarithm tables or a slide rule. Log Tables were books in which every number was written as a power of 10 or power of *e*.

Today we have calculators to do this job but logarithms are now seen to be very useful in their own right, see case studies below. **Do not learn this.**

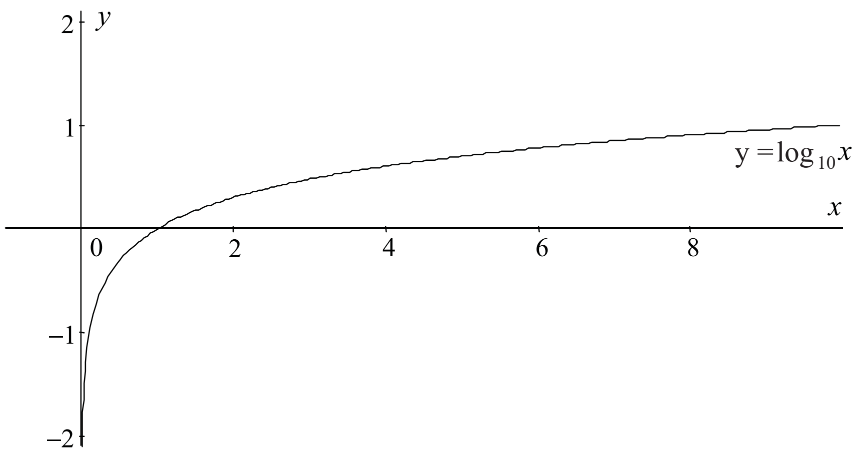
5.2.3 The function and its graph

Let's now examine the logarithmic function, $y = \log_{10} x$, in its own right.

To help us understand the shape of this function in more detail, use your calculator to complete the table of values below and use it to sketch the function $y = \log_{10} x$.

x	$\log_{10} x$
10	
9	
8	
7	
6	
5	
4	
3	
2	
1	
0.1	
0.001	
0.0001	
0.00001	

How did you go? Did you get a graph something like this?



Think about the shape, the domain and range of the curve and describe in your own words the characteristics of this logarithmic function.

Important characteristics of the graph you might have noticed are:

- it is a function because there is only one value of the dependent variable for each independent variable
- its domain is restricted and includes only real numbers greater than zero
- the range of each function is unrestricted and includes all real numbers
- the horizontal intercept is one
- as the independent variable decreases (approaches zero) the dependent variable approaches negative infinity
- as the independent variable increases (approaches infinity) the dependent variable increases slowly.

You should notice two things:

1. Compare this function with the inverse of the exponential function drawn at 5.1.5. They are identical. The logarithmic function is an inverse of the exponential growth function so all of its properties are mirror images of the properties of the growth function e.g. domain, range etc.

2. The vertical axis is an asymptote.

Activity 5.9

1. Sketch the graph of $y = \ln x$, in the domain $0 < x < 3$
2. (a) On your graph of $y = \ln x$ sketch the graphs
   (i) $y = 2\ln x$
   (ii) $y = 2 + \ln x$
   (b) What effect does multiplying by 2 or adding 2 have on the shape and position of the logarithmic function?

5.2.4 Case studies

Logarithms are used to model a range of situations that occur in science, economics and engineering. They are used in isolation or in combination with other functions. For example,

In mechanical technology, belt friction in a pulley system is modelled by $\ln(\frac{T_L}{T_S}) = \mu\theta$, where T are the large and small tensions in the rope on the pulley, θ is the angle of wrap of the rope around the pulley and μ is the coefficient of friction.

In chemistry, time of reaction (t) and concentration of a substance (x) are related by the equation, $t = k_1 + \ln(\frac{k_2 - x}{k_3 - x})$, where the k values are all constants.

In economics the growth of an economy could be represented by the formula, $t = A\ln(\frac{x}{a})$, where t is time and x the value in dollars.

However, by far the most common use of the logarithmic function is in the development of measurement scales. This application of the function makes use of the fact that as the independent variable increases there are only small changes to the dependent variable. This characteristic gives us the ability to work with very large and very small numbers more manageably within the one function. The following case studies emphasize this characteristic.

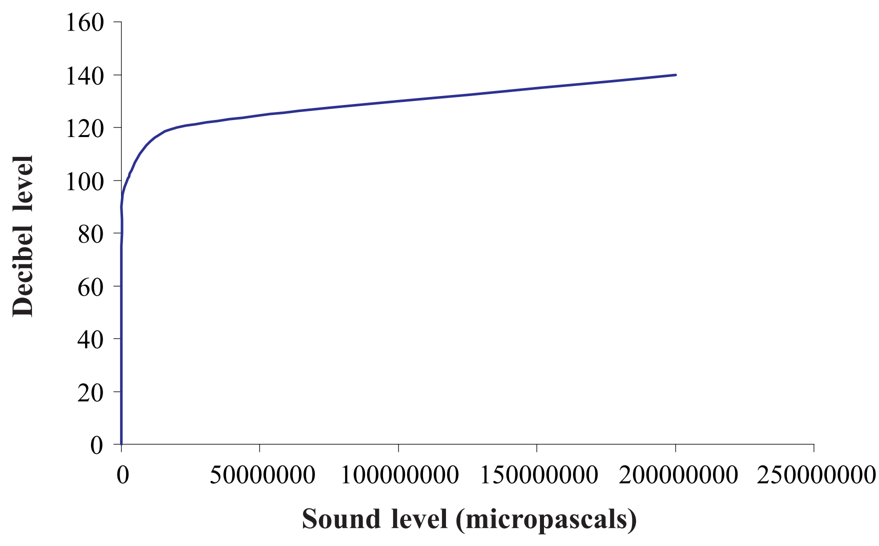
Measuring loudness

Sounds can vary in intensity from the lowest level of hearing (a ticking watch 7 metres away) to the pain threshold (the roar of a jumbo jet). Sound is detected by the ear as changes in air pressure measured in micropascals (μP). The ticking watch is about 20 μP, conversational speech about 20 000 μP, a jet engine close up about 200 000 000 μP…an enormous range of values. A scale was required to compress the range of 20 to 200 000 000 into a more manageable and useful form from 0 to 140. The decibel scale was invented for this purpose.

If P is the level of sound intensity to be measured and P_0 is a reference level, then $n = 20\log_{10}(\frac{P}{P_0})$, where n is the decibel scale level. If we assume 20 μP to be the threshold level, then the equation would be:

$n = 20\log_{10}(\frac{P}{20})$ and the graph of the relationship would resemble the one below.

Decibel scale for loudness of sound



Note that because of the nature of the scale many of the very small values are crowded along the vertical axis and are not clearly discernible.

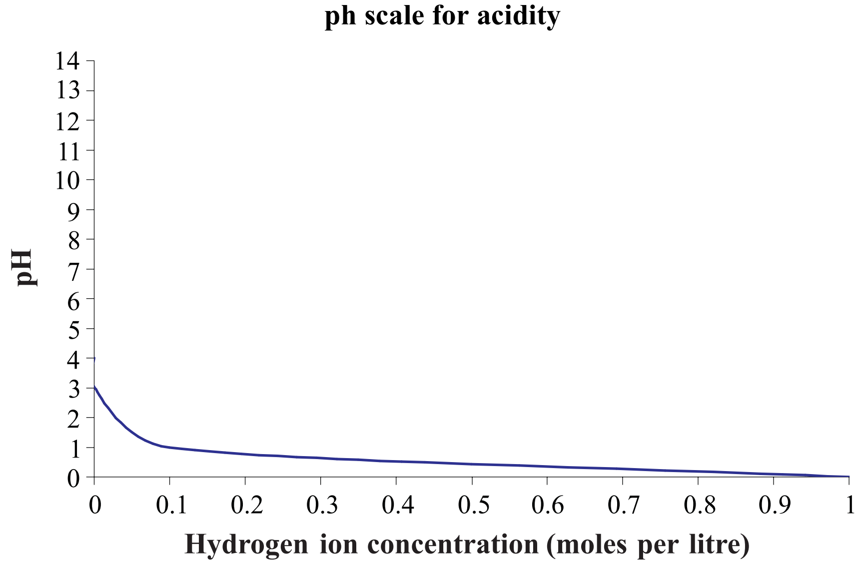
Measuring acidity

Chemists measure the acidity of a substance by determining the concentration of hydrogen ions in moles per litre. However, these concentrations can range from 1 mole per litre in very concentrated acids to 0.001 in lemon juice, 0.00000007 in milk and 0.000000000001 in washing soda. As before a very wide range of values can be reduced to a more convenient scale of measurement by applying a logarithmic scale. In this case the concentrations are all negative powers of 10 (see table below) so the relationship will be $\text{pH} = -\log_{10} x$, where pH is the level of acidity and x is the concentration of hydrogen ions in moles per litre. The values are calculated below. So a pH of 0 is very acid, 7 is neutral and 14 is alkaline (the opposite to acid).

Concentration of hydrogen ions (moles per litre)	**pH**	**Acidity**
$1 = 10^0$	0	High – hydrochloric acid
$0.1 = 10^{-1}$	1	
$0.01 = 10^{-2}$	2	
$0.001 = 10^{-3}$	3	Acid – lemon juice
$0.0001 = 10^{-4}$	4	
$0.00001 = 10^{-5}$	5	Acid – soft drinks
$0.000001 = 10^{-6}$	6	
$0.0000001 = 10^{-7}$	7	Neutral – milk
$0.00000001 = 10^{-8}$	8	
$0.000000001 = 10^{-9}$	9	
$0.0000000001 = 10^{-10}$	10	
$0.00000000001 = 10^{-11}$	11	
$0.000000000001 = 10^{-12}$	12	Alkaline – washing soda
$0.0000000000001 = 10^{-13}$	13	
$0.00000000000001 = 10^{-14}$	14	Very alkaline – sodium hydroxide

If we graphed the function we would get the following. Notice two things about this curve:

- it is shaped differently from the usual logarithmic curve because of the negative sign at the front of the formula
- because of the nature of the scale many of the very small values are crowded close to the vertical axis and are not clearly discernible.



Activity 5.10

1. In general a relationship exists between annual sugar consumption (S in kg) and income (w in \$) per head of population which follows the model $S = 12\log\frac{w}{3}$. Sketch a graph to show sugar consumption as a function of income up to \$45 000.

2. A video franchise has determined that in its stores the number of new releases (N, in hundreds) hired out each week depends on the hiring charge (C, in \$ per video). The number of hires is given by the equation $N = -10\log\frac{C}{10}$. Graph this function for charges between \$0 and \$10.

   (a) From your graph (or otherwise) calculate how many new releases they could expect to hire out at \$3.50 per video.

   (b) What would you advise this video franchise charge for its movies (to the nearest \$)? Give reasons for your answer.

3. The stellar magnitude of a star is another negative logarithmic scale like pH, but the quantity measured is the brightness of the star. If $SM = -\log B$, where SM is stellar magnitude and B is brightness, sketch the graph and answer the following questions.

   (a) What is the stellar magnitude of a star which has a brightness of 0.7943?

   (b) Another star has a magnitude of 2.1, what is its brightness?

   (c) Compare the brightness of the two stars.

5.2.5 Average rate of change

We can get an estimate of the rate of change of the logarithmic function just the same as we would measure the average rate of change of any curved function. Let's have a look at some of the examples above.

Example

Determine the average rate of change in the function $f(x) = \ln x$, between the values of $x = 1$ and $x = 5$.

This question requires us to calculate the average rate of change of the function between the values of 1 and 5. To do this we need to find the gradient of the straight line joining the points where $x = 1$ and $x = 5$.

Recall that gradient is change in the height divided by the change in the horizontal distance,

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\begin{aligned} m &= \frac{f(5) - f(1)}{5 - 1} \\ &= \frac{\ln 5 - \ln 1}{4} \\ &\approx 0.4 \end{aligned}$$

The rate of change between 1 and 5 is 0.4. This means that for every 1 unit change in the x value the y value changes by 0.4 units.

Activity 5.11

1. Complete the table of values for the function $f(x) = \ln x$

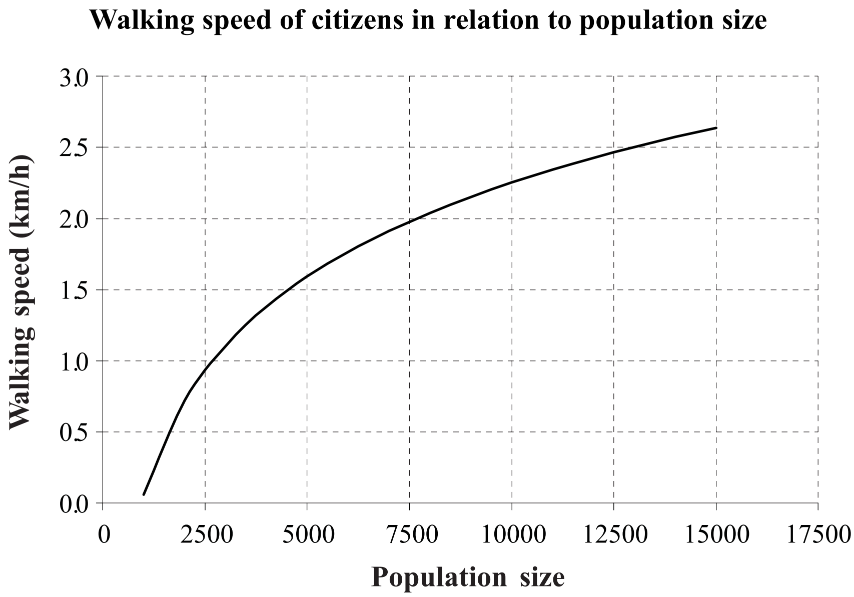
x	$f(x) = \ln x$
0.01	
0.5	
1	
1.5	
2	
3	
3.5	
4	
4.5	
5	
5.5	

(a) Find the average rate of change of y with respect to x as x changes from 0.5 to 1

(b) Find the average rate of change of y with respect to x as x changes between 5 and 5.5

(c) Compare the two rates of change.

2. The relationship between the population size of a city and the walking speed of its inhabitants can be modelled by a logarithmic function. This function is graphed below.



(a) Use the graph to determine the average rate of change of walking speed with respect to population size of cities of populations between 2 500 and 5 000.

(b) Repeat part (a) for population centres between 12 500 and 15 000.

(c) Use this information to help describe the functional relationship between walking speed and population size.

(d) Do you think this is a reasonable relationship to predict the pace of life in a city?

5.3 Putting it all together – solving equations and real world applications

On their own the exponential and logarithmic functions are powerful tools to help describe and measure natural phenomena. However, the inverse relationship between the functions allows us to extend their applications even further.

Solving exponential equations

Previously, we solved exponential equations by trial and error.

If we had to solve $3^x = 81$, we had to recall that $3^4 = 81$ and thus say $x = 4$.

However, we can solve these types of equations using our knowledge of the relationship between logarithms and exponentials, and the logarithmic properties.

To solve $3^x = 81$ we would do the following:

$$3^x = 81$$

As we have two expressions or numbers that are equal then their logarithms will also be equal. Our next step is to take logarithms of both sides (we could pick any logarithm but usually we would choose either base 10 or *e*).

$$\log_{10} 3^x = \log_{10} 81$$

Using the log property that $\log m^n = n \log m$ we have

$$x \log_{10} 3 = \log_{10} 81$$

Dividing both sides by $\log_{10} 3$

$$\frac{x \log_{10} 3}{\log_{10} 3} = \frac{\log_{10} 81}{\log_{10} 3}$$

$$x = \frac{\log_{10} 81}{\log_{10} 3}$$

Use your calculator to evaluate the logarithms of these numbers and then divide, giving

$$x = 4$$

Alternatively you might recall that,

$$\begin{aligned} x &= \frac{\log_{10} 81}{\log_{10} 3} \\ &= \frac{\log_{10} 3^4}{\log_{10} 3} \\ &= \frac{4 \log_{10} 3}{\log_{10} 3} \\ &= 4 \end{aligned}$$

Check: When $x = 4$, $LHS = 3^4 = 81 = RHS$

Example

Solve the following equation for *a*, $7^{2a} = 2.73$

$7^{2a} = 2.73$	
$\ln 7^{2a} = \ln 2.73$	Take log to the base *e* of both sides.
$2a \ln 7 = \ln 2.73$	Use power property to bring 2*a* to the front of ln7.
$2a = \dfrac{\ln 2.73}{\ln 7}$	Divide both sides by ln7.
$a = \dfrac{\ln 2.73}{2\ln 7}$	Divide both sides by 2.
$a \approx 0.2581$	

Check: When $a \approx 0.2581, LHS = 7^{2\times 0.2581} = 2.73 = RHS$

Example

What values of *p* satisfy this exponential equation, $2\times e^{-0.1p} + 3 = 4$?

$2\times e^{-0.1p} + 3 = 4$	
$2\times e^{-0.1p} = 1$	Subtract 3 from both sides.
$e^{-0.1p} = \dfrac{1}{2}$	Divide both sides by 2.
$\ln e^{-0.1p} = \ln\dfrac{1}{2}$	Take log to base *e* of both sides.
$-0.1p\ln e = \ln\dfrac{1}{2}$	Use power property to bring –0.1*p* to the front of ln*e*.
$-0.1p = \dfrac{\ln\frac{1}{2}}{\ln e}$	Divide both sides by ln*e*.
$-0.1p = \ln\dfrac{1}{2}$	Recall ln*e*=1.
$p = \dfrac{\ln\frac{1}{2}}{-0.1}$	Divide both sides by –0.1.
$p \approx 6.9315$	

Check: When $p \approx 6.9315, LHS = 2\times e^{-0.1\times 6.9315} + 3 \approx 4 = RHS$

Example

Rearrange the following formula to make x the subject, $L = 2e^{-2x} + 7$

$L = 2 \times e^{-2x} + 7$	
$L - 7 = 2 \times e^{-2x}$	Subtract 7 from both sides.
$\dfrac{L-7}{2} = e^{-2x}$	Divide both sides by 2.
$\ln(\dfrac{L-7}{2}) = \ln e^{-2x}$	Take logs to the base e of both sides.
$\ln(\dfrac{L-7}{2}) = -2x \ln e$	Use power property to bring $-2x$ to the front of $\ln e$.
$-2x = \ln(\dfrac{L-7}{2})$	Recall the $\ln e$=1.
$x = -\dfrac{1}{2}\ln(\dfrac{L-7}{2})$	Divide both sides by -2.

Activity 5.12

1. Solve the following exponential equations.
   (a) $2^a = 1024$
   (b) $9^b = 25$
   (c) $7^{c+2} = 11$
   (d) $5^{1-d} = 7$
   (e) $3^{2x} = 40$
2. Find the value of the unknown in the following equations.
   (a) $4^{a+1} = 8$
   (b) $4^{b-1} = \dfrac{1}{8}$
   (c) $2 \times 3^{c-1} = 15$
   (d) $10^{d-2} - 1 = 2.444$
   (e) $3^{n+1} = 5^n$
3. Rearrange the following to make n the subject.
   (a) $a^{2n} = b^2$
   (b) $y = ae^{4n}$
   (c) $A = P(1 + \dfrac{r}{100})^n$
   (d) $D = P(1 - \dfrac{r}{100})^n$
   (e) $C = 79.345e^{-0.0166n}$
   (f) $L = 2(3e)^{-n}$

Real world applications

This section will examine some real world applications which require solution of exponential equations.

Example

The Earth's population is growing according to the function $P = P_0 e^{kt}$, where P is the population in billions, P_0 is the initial population, t is time in years and k is a constant. If the population in 1980 was 4.478 billion and in 1994 was 5.462 billion, what will this model predict the population to be in 2010?

To solve this problem you first have to determine the values of the constants P_0 and k in the formula. As 1980 is the first date we have we can let this be the initial year. This means that when $t = 0$ the population is 4.478 billion. We substitute this into the formula:

$$\begin{aligned} P &= P_0 e^{kt} \\ 4.478 &= P_0 e^{k\times 0} \\ P_0 &= 4.478 \end{aligned}$$

So the formula is now $P = 4.478e^{kt}$

To find the value of k substitute the other two values into the formula, $t = 14$ and $P = 5.462$.

$P = 4.478e^{kt}$	
$5.462 = 4.478e^{k\times 14}$	Substitute in values.
$\dfrac{5.462}{4.478} = e^{14k}$	Divide both sides by 4.478.
$\ln\dfrac{5.462}{4.478} = \ln e^{14k}$	Take logs to the base e of both sides.
$\ln\dfrac{5.462}{4.478} = 14k\ln e$	Use power property to bring $14k$ to front of $\ln e$.
$14k = \ln\dfrac{5.462}{4.478}$	Recall $\ln e=1$.
$k = \dfrac{1}{14}\ln\dfrac{5.462}{4.478}$	Divide both sides by 14.
$k \approx 0.01419$	Evaluate the expression.

So the formula is now $P = 4.478e^{0.01419t}$

To find the population in the year 2010, find the value of P when $t = 30$

$$\begin{aligned} P &= 4.478e^{0.01419t} \\ P &= 4.478e^{0.01419\times 30} \\ P &\approx 6.854 \end{aligned}$$

Population is predicted to be 6.854 billion.

Example

Radioactive decay is modelled by the equation $N = N_0 e^{-kt}$, where N represents the mass of the substance, N_0 the initial mass of the substance and t the time. If a certain radioactive substance has a half-life of 5 years and 20 grams of it was initially secured, how much of the substance would be left after 10 years? If the substance could only be safely moved in batches of 0.1 g, when would the original 20 g be safe to move?

This first step in the solution of this question is to determine the value of k. We know that we started with 20 g so $N_0 = 20$, we know that in 5 years we only have half of this (10 g) so $N = 10$ when $t = 5$.

So using our model we get,

$$\begin{aligned}
N &= N_0 e^{-kt} \\
10 &= 20e^{-5k} \\
0.5 &= e^{-5k} \\
\ln(0.5) &= \ln(e^{-5k}) \\
\ln(0.5) &= -5k \ln e \\
\ln(0.5) &= -5k \\
k &= \frac{\ln 0.5}{-5} \\
k &\approx 0.138629
\end{aligned}$$

Now that we know the value of k we can determine the amount of the substance left after 10 years.

$$\begin{aligned}
N &= 20e^{-0.138629\times 10} \\
&= 20e^{-1.38629} \\
&\approx 5
\end{aligned}$$

There would be approximately 5 g left after 10 years.

We can also determine how long it would take to have decayed to 0.1 g of the substance.

$$
\begin{aligned}
0.1 &= 20e^{-0.138629t} \\
\frac{0.1}{20} &= e^{-0.138629t} \\
\ln(\frac{0.1}{20}) &= \ln e^{-0.138629t} \\
\ln(\frac{0.1}{20}) &= -0.138629t \ln e \\
\ln(\frac{0.1}{20}) &= -0.138629t \\
t &= \frac{\ln(\frac{0.1}{20})}{-0.138629} \\
t &\approx 38.22
\end{aligned}
$$

It would be safe to transport the substance after 38.22 years.

Something to talk about...

This module contains some new and different concepts. Share with the discussion group your technique for coming to terms with the concept of logarithms.

Activity 5.13

1. The growth function $A = Pe^{kt}$ models the amount to which \$$P$ grows in t years at an interest rate of k% compounded continuously. At the current rate \$100 will grow to \$185 in 10 years. What would it amount to in 20 years?
2. A radioactive material is decaying exponentially. If N_0 is the initial mass of the material, N the mass at any time t and k the rate of decay, then the function is modelled by the equation $N = N_0e^{-kt}$. It takes 2 years for 100 g of the material to reduce to 60 g. How long would it take the same mass to reduce to 30 g?
3. The present temperature of a star is 10 000°C, and it is losing heat continuously in such a way that its temperature T in t million years may be obtained from the decay function $T = T_0e^{-kt}$. If the temperature has halved after 5 million years what will its temperature be in 10 million years?
4. The number of bacteria in a culture was initially 2 000. The number increases exponentially at the rate k% per hour for t hours, i.e. $N = N_0e^{kt}$. After 2 hours the number present is 2 420. How many will be present after 24 hours?
5. The population of a city grew exponentially ($P = P_0e^{kt}$) from 120 000 to 180 000 over a period of 10 years. How long before it reaches 200 000?

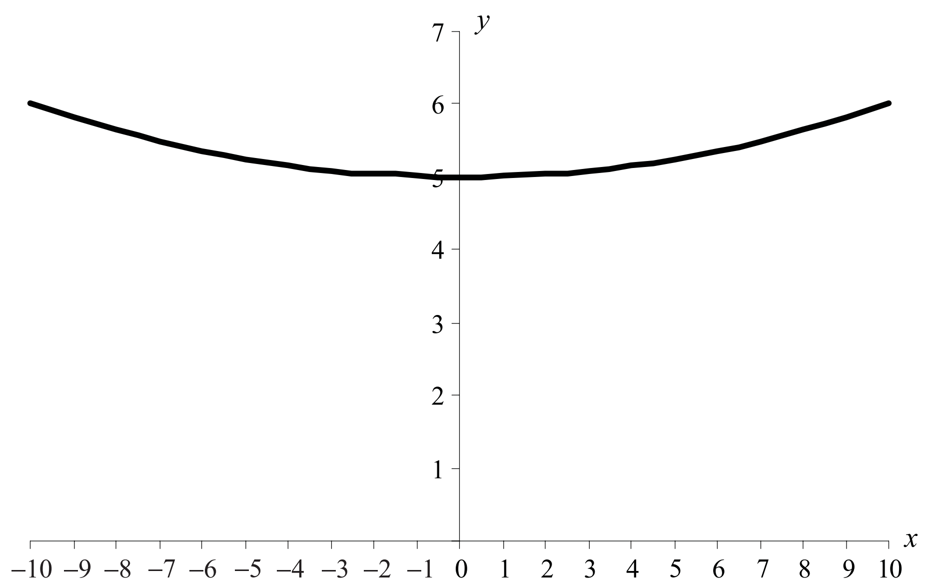
That's the end of this module. You will have experienced a lot of new concepts and algebraic techniques to arm you for your future studies in this unit and later.

But before you are really finished you should do a number of things:

1. Have a close look at your action plan for study. Are you still on schedule? Or do you need to restructure your action plan or contact your tutor to discuss any delays or concerns?
2. Make a summary of the important points in this module noting your strengths and weaknesses. Add any new words to your personal glossary. This will help with future revision.
3. Practise some real world problems by having a go at 'A taste of things to come'.
4. Check your skill level by attempting the Post-test.
5. When you are ready, complete and submit your assignment.

5.4 A taste of things to come

1. An electrical power cable hangs between two supports 20 metres apart. Engineers model the shape of its suspension by the equation $y = 25(e^{\frac{x}{50}} + e^{-\frac{x}{50}}) - 45$, where x is the distance in metres from the supports and y is the height in metres above the ground. The graph of the function is depicted below.



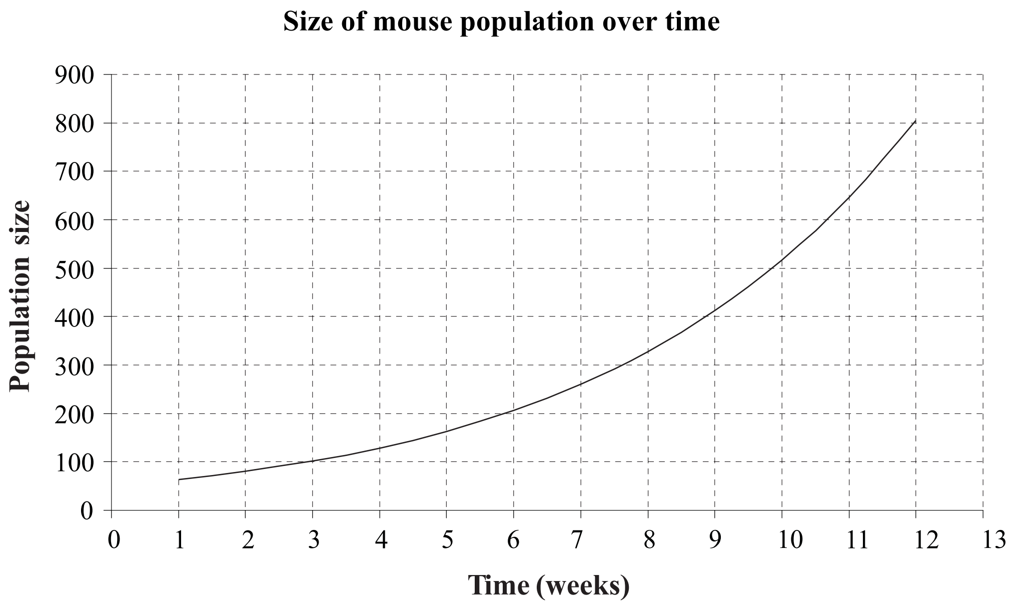
(a) How much does the cable sag?

(b) Engineers need to know the slope of cable where it attaches to each pylon. Find the approximate value of this by finding the average slope of the cable 1 metre from the support?

(c) What is the average slope of the cable 1 metre from the lowest point?

2. In environmental studies researchers are interested in the concept of carrying capacity and population growth. Carrying capacity is the maximum population that can be supported in a particular environment. Growth rates are often modelled to include this variable.

In practice mice populations do not grow exponentially because of the limits of food and other resources. It could follow the model $N = \dfrac{50e^{0.24t}}{1+0.006e^{0.24t}}$, where t is time in weeks and N the number of mice. The function looks like this.



(a) On the axes above sketch a graph of the traditional exponential model $N = N_0 e^{at}$, where N_0 is 50 mice and a is 0.24.

(b) Comment on the graphs of the two models.

(c) Compare the populations predicted by the two models after 30 weeks.

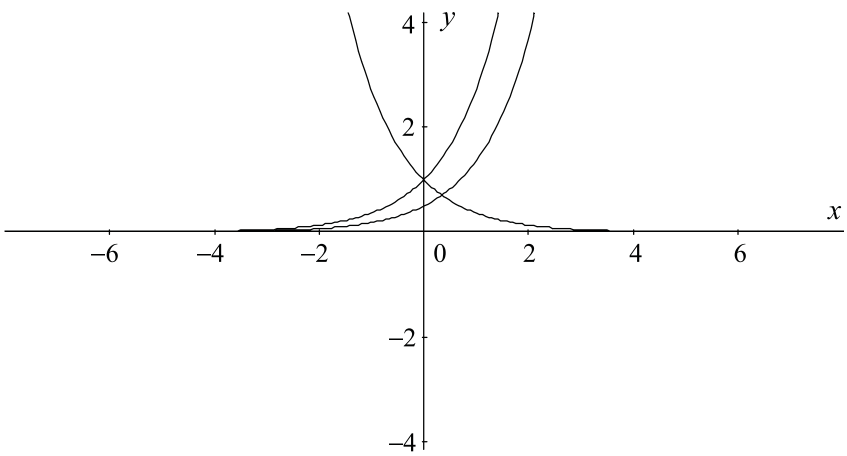
3. Following an advertising campaign at a recycling plant more goods are supplied at each point. The table below contains data for two supply schedules. The first for supply one month prior to the campaign and the second for supply in the month following the advertising. For example, the table shows a shift in the quantity supplied from 4 million to 8 million items per month for goods priced at \$3 each.

Before campaign		**After campaign**	
Price (\$/item)	**Quantity (millions)**	**Price (\$/item)**	**Quantity (millions)**
1	0	1	3
2	3	2	6
3	4	3	8
4	5	4	10
5	6	5	11

(a) Taking quantity (x) as the independent variable and price (P) as the dependent variable, $y = P(x)$, these curves can be **approximated** by $y = P_1 = e^{0.3x}$ and $y = P_2 = e^{0.2x}$ respectively. Use the actual data from the table to sketch the two curves. Find the shift in supply of a \$4 item and mark this on your graph.

5.5 Post-test

1. Match the equations to the graphs of exponential functions: $y = e^x$, $y = e^{-x}$, $y = 0.5e^x$



2. Determine the y-intercepts for each of the following:

   (a) $y = 2 \times (\frac{1}{2})^x$

   (b) $y = 2^x - \frac{1}{2}$

   (c) $y = 2^{-x} + \frac{1}{2}$

   (d) $y = \frac{1}{2} \times 2^x$

3. Find the value of y when $x = 2$ in $y = 1.5 \times (1.5)^{-x}$

4. Write the following equations in logarithmic form.

   (a) $10^{x+1} = 5$

   (b) $2 = e^y$

5. Use your knowledge of the definition of a logarithm to evaluate:

   (a) $\log_3 81$

   (b) $\log_{\frac{1}{5}} 125$

6. Write the following equations in exponential form.

   (a) $a = \log_{10} 0.1$

   (b) $\log_2 8 = b$

7. Determine p and q in the following equations.

   (a) $\log_2 (2p) = 5$

   (b) $\ln \frac{10}{q} = -0.405$

8. Find t in the following equations.

   (a) $2^t = 7.5$

   (b) $1 = 8(1 - e^{2t})$

9. Without using a calculator write the following expressions as a single logarithm or number.

   (a) $\log_3 27 + \log_3 \frac{1}{9} - \log_3 9$

   (b) $\ln 0.4e + \ln 10e - 2\ln 2$

10. Use logarithmic properties to solve:

    (a) $\log_4 3 = \log_4 x - \log_4 2$

    (b) $\log_2 \frac{1}{8} + \log_2 y = \log_2 \frac{1}{4}$

11. What value of r satisfies this exponential equation, $4 \times 10^{\frac{r}{100}} - 1 = 4$

12. Rearrange the following formula to make t the subject, $y = 1.4e^{-0.6t} - 3$

13. If $g = 3 \times (ae)^n$, show that $n = \dfrac{\ln g - \ln 3}{\ln a + 1}$

14. If $A = P(1+i)^n$, find n in terms of A, P and i.

15. Use the formula found in Q14 for n to find how long it would take \$50 (P) to amount to \$75 (A) if the interest is compounded at 8.5% (i = 0.085).

16. Machinery originally costing \$250 000 depreciates at 20% pa on the diminishing value.

    (a) Construct a depreciation schedule (nearest \$) for the first 6 years after purchase, presenting the schedule as a table and as a graph.

    (b) Use the above information to write the exponential function for the depreciation value (D) as a function of time (n)

    (c) Use the function to estimate its value in 10 years time.

17. The voltage (V measured in volts) across a capacitor is modelled by the equation, $V = 10e^{\frac{-t}{3}}$, where t is measured in seconds. Find V when t = 5.

18. The function, $R = 12e^{-0.075t}$ (t measured in years, R in grams) approximates the decay of a certain element.

    (a) What is its weight when t = 0?

    (b) How long will it take to reduce to half its weight?

19. The function $P = 4.5e^{0.0142t}$ (P in billions of people and t in years) modelled Earth's population growth in 1981. Compare the rate of growth over the years 1981 to 1986 with the expected rate of growth 20 years later, i.e. 2001 to 2006.

20. The equation $P = 20 \times 10^{0.1n}$ can be used to convert any number of decibels (n) to the corresponding number of micropascals (μP) used to measure loudness. Show that a 60 decibel sound is 10 times as loud as a 50 decibel sound, and 100 times as loud as a 40 decibel sound.

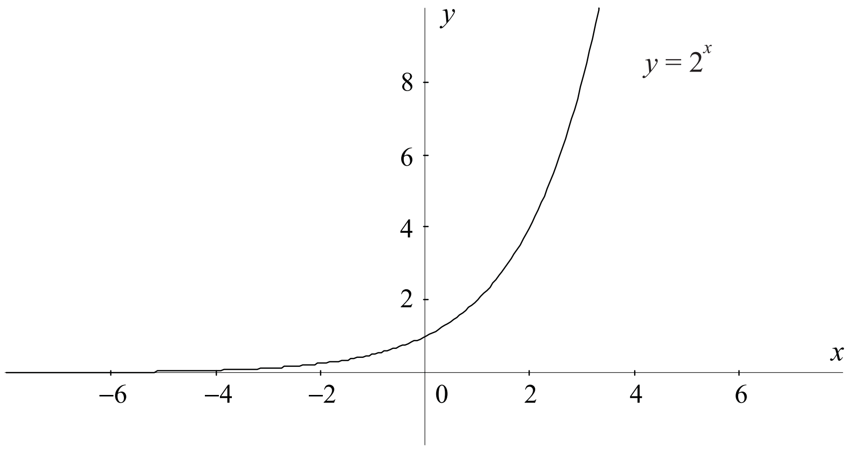
5.6 Solutions

Solutions to activities

Activity 5.1

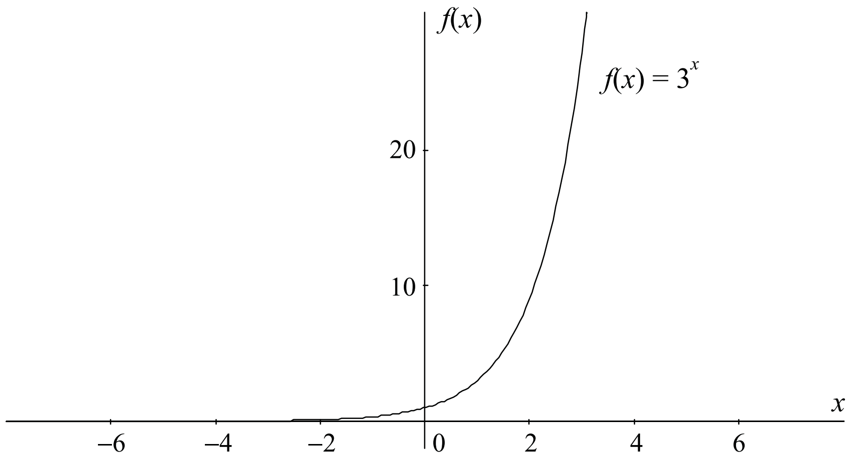
1. To sketch graph first calculate table of values

x	$y = 2^x$
–3	0.125
–2	0.25
–1	0.5
0	1
1	2
2	4
3	8



2. To sketch graph first calculate table of values

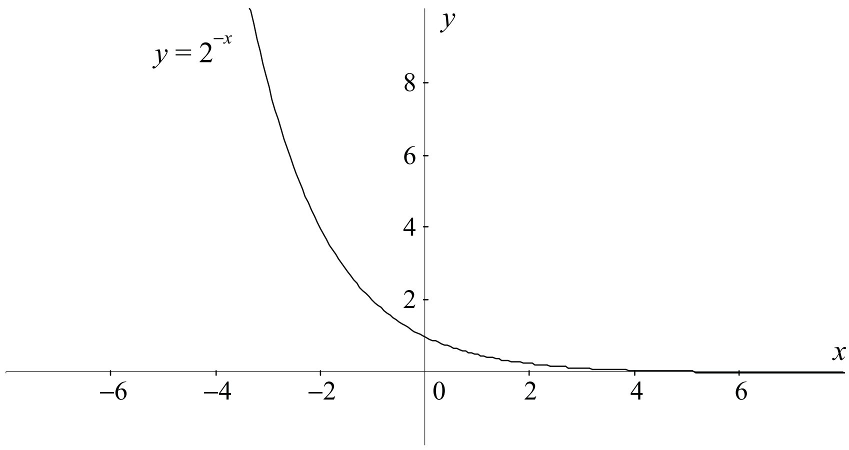
x	$f(x)=3^x$
–3	0.04
–2	0.11
–1	0.33
0	1.00
1	3.00
2	9.00
3	27.00



Activity 5.2

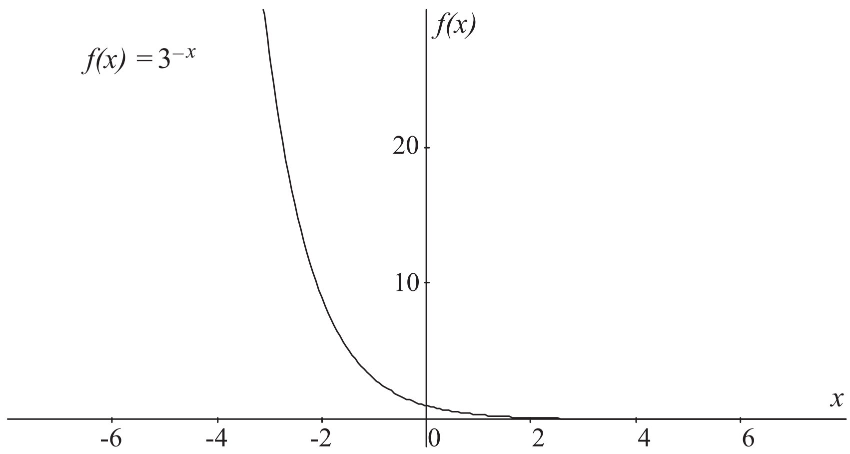
1.

x	$y = 2^{-x}$
–3	8
–2	4
–1	2
0	1
1	0.5
2	0.25
3	0.125



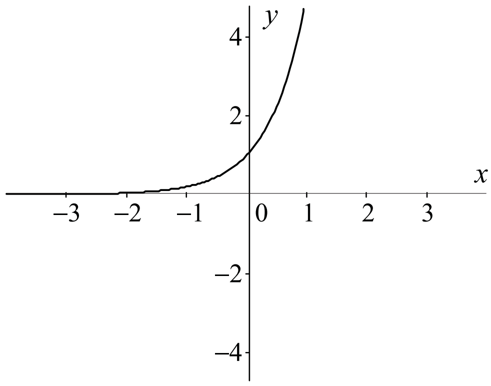
2.

x	$f(x)=3^{-x}$
-3	27.00
-2	9.00
-1	3.00
0	1.00
1	0.33
2	0.11
3	0.04

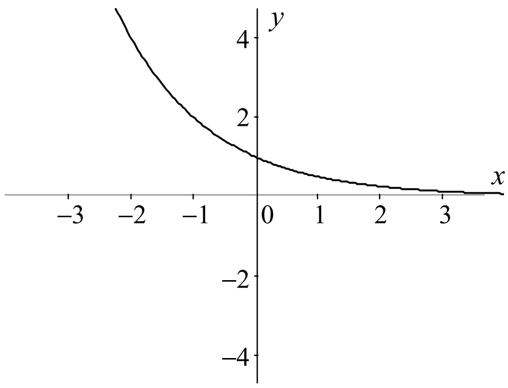


Activity 5.3

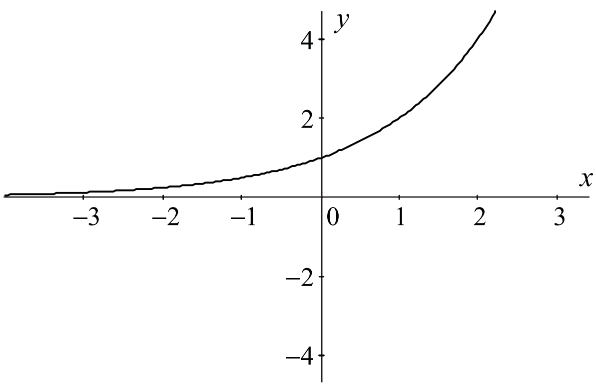
1. (a)



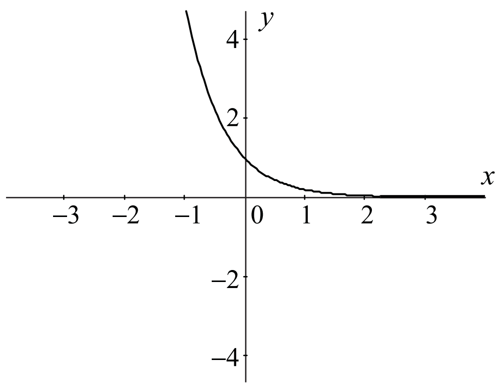
A $\Rightarrow y = 5^x$



B $\Rightarrow y = (0.5)^x = (\frac{1}{2})^x = 2^{-x}$

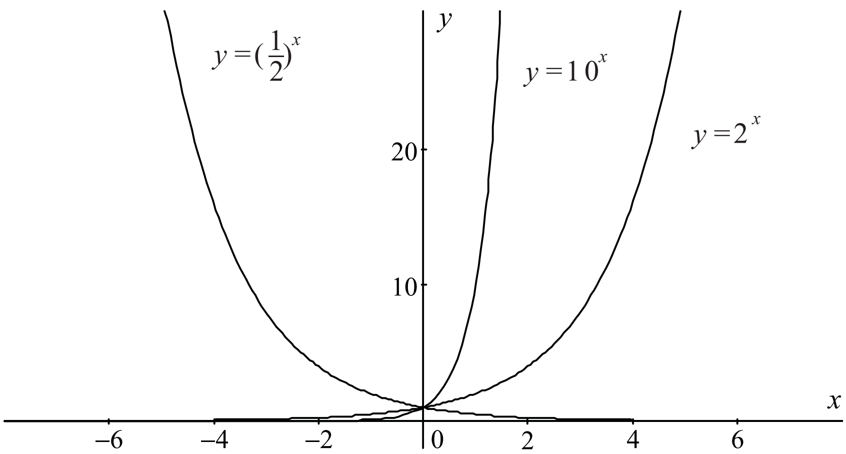


C $\Rightarrow y = (0.5)^{-x} = (\frac{1}{2})^{-x} = 2^x$



D $\Rightarrow y = 5^{-x}$

2. (a)



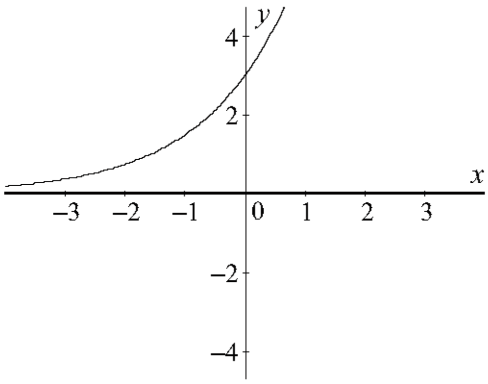
(b) Domain for each graph is all real values of x.
Range is all real values of y greater than zero, $y > 0$

(c) Point of intersection is (0, 1)

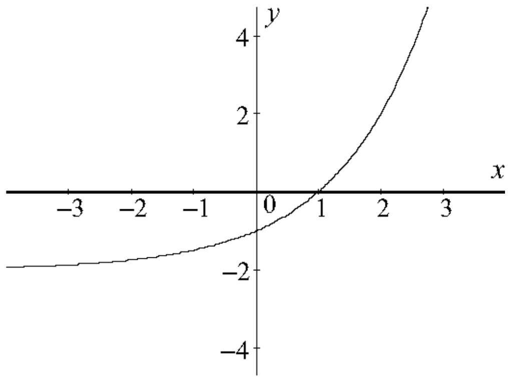
(d) Horizontal asymptote is the x axis, $y = 0$

3. Note the easiest way to distinguish between each of these graphs is to first determine the y intercepts by evaluating y when $x = 0$, then to plot some more points.

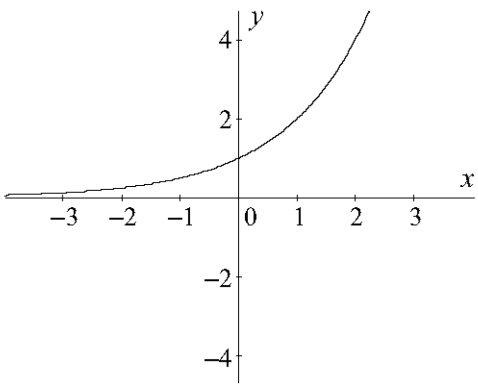
$y = 2^x$	$y = 3 \times 2^x$	$y = 2^x + 2$	$y = 2^x - 2$
$y = 2^0$	$y = 3 \times 2^0$	$y = 2^0 + 2$	$y = 2^0 - 2$
$y = 1$	$y = 3 \times 1$	$y = 1 + 2$	$y = 1 - 2$
	$y = 3$	$y = 3$	$y = -1$



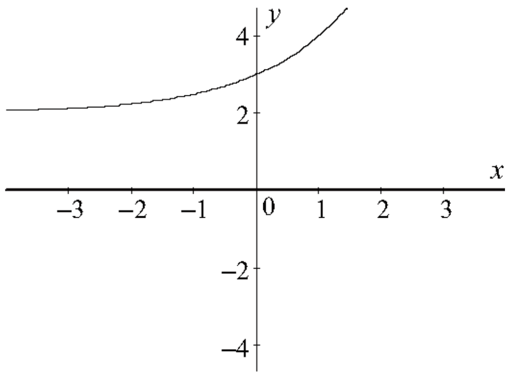
A $\Rightarrow y = 3 \times 2^x$



B $\Rightarrow y = 2^x - 2$



C $\Rightarrow y = 2^x$



D $\Rightarrow y = 2^x + 2$

4. (a) To find the value of a substitute the point (5, a) into the equation given.

$$y = \frac{1}{3} \times 3^x$$
$$a = \frac{1}{3} \times 3^5$$
$$a = 3^4$$
$$a = 81$$

(b) To find the value of b substitute the point (b, $\frac{1}{27}$) into the equation given.

$$y = \frac{1}{3} \times 3^x$$
$$\frac{1}{27} = \frac{1}{3} \times 3^b$$
$$\frac{3}{27} = 3^b$$
$$\frac{1}{9} = 3^b$$
$$3^{-2} = 3^b$$
$$b = -2$$

(c) To find the vertical intercept put $x = 0$ in the equation of the function

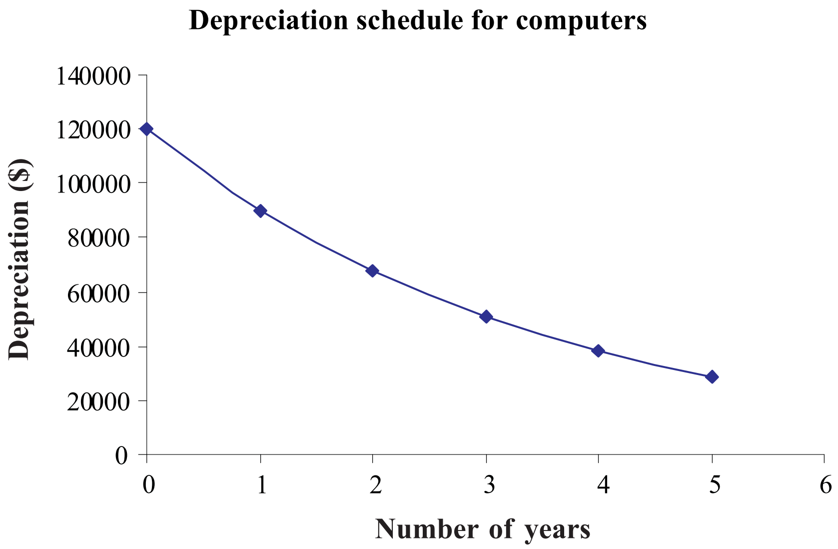
$$y = \frac{1}{3} \times 3^x$$
$$y = \frac{1}{3} \times 3^0$$
$$y = \frac{1}{3} \times 1$$
$$y = \frac{1}{3}$$

Vertical intercept is one third.

Activity 5.4

1. Using the formula we know that $D = P(1 - \frac{r}{100})^n$, where P = \$120 000, r = 25% determined each year, and n changes from 0 to 5.

Year (n)	Depreciated value of computers (D in \$)
0	120000
1	90000
2	67500
3	50625
4	37969
5	28477



2. Using the formula for compounded interest we have the following,

$A = P(1 + \frac{r}{100})^n$, where P = \$1.25, r is 4% taken each year and n is 8 years.

$$A = P(1 + \frac{r}{100})^n$$
$$A = 1.25(1 + \frac{4}{100})^8$$
$$A = 1.25 \times 1.04^8$$
$$A \approx 1.7107....$$
$$A = 1.71$$

New value of milk is \$1.71

3. We use the compound interest formula, $A = P(1+\frac{r}{100})^n$, to calculate returns from both packages.

   Package 1: P = \$90 000, r is 6.5% every 6 months so $r = \frac{6.5\%}{2} = 3.25\%$, n is for two six month periods so is 2.

$$A = P(1+\frac{r}{100})^n$$
$$A = 90000(1+\frac{3.25}{100})^2$$
$$A = 90000(1.0325)^2$$
$$A = 95945.0625$$
$$A \approx 95945.06$$

   Return is \$95 945.06

   Package 2: P = \$90 000, r is 6.4% for 365 days so $r = \frac{6.4\%}{365}$, n is 365 days.

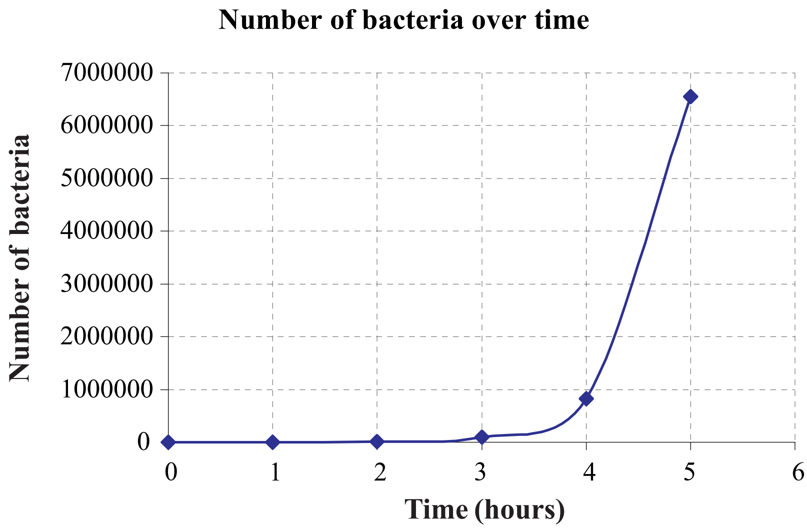
$$A = P(1+\frac{r}{100})^n$$
$$A = 90000(1+\frac{(\frac{6.4}{365})}{100})^{365}$$
$$A = 90000(1.000175...)^{365}$$
$$A = 95947.775$$
$$A \approx 95947.78$$

   Return is \$95 947.78

   Melvil should take the second package as his money will have earned an extra \$2.72.

4. Calculate a table of values.

Hours (*t*)	**Number of bacteria (*N(t)*)**
0	200
1	1600
2	12800
3	102400
4	819200
5	6553600
6	52428800
7	419430400
8	3355443200
9	26843545600



From the graph we see that it should take just over 4 hours for the number of bacteria to reach 1 000 000

5. Given the function $N(t) = 4e^{(-\frac{t}{5771})}$, to find the mass present initially put $t = 0$.

$$N(t) = 4e^{(-\frac{t}{5771})}$$
$$N(0) = 4e^{(-\frac{0}{5771})}$$
$$N(0) = 4 \times 1$$
$$N(0) = 4$$

Initial mass is 4 grams.

When $t = 4000$

$$N(t) = 4e^{(-\frac{t}{5771})}$$
$$N(4000) = 4e^{(-\frac{4000}{5771})}$$
$$N(4000) = 4 \times e^{-0.69312...}$$
$$N(4000) = 2.000052...$$
$$N(4000) \approx 2$$

After 4000 years the mass is 2 grams.

The mass has halved after four thousand years so the half-life of the substance must be 4 000 years.

6. Using the formula $W = W_0(1+\frac{r}{100})^n$ where W_0 is 50, r is 10% per month and n is 10 months, the growth period of a 10 month rodent,

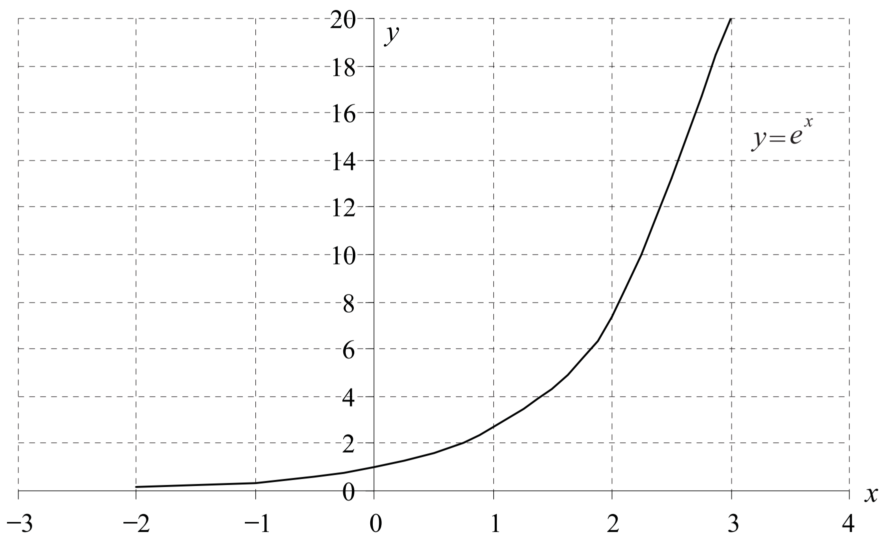
$$W = 50(1+\frac{10}{100})^{10}$$
$$W = 50(1.1)^{10}$$
$$W = 129.68...$$

The rodent should have a weight of approximately 130 g.

Activity 5.5

1. (a)

x	$y = e^x$
−2	0.14
−1	0.37
0	1.00
1	2.72
2	7.39
3	20.09



Rate of change, m, between $x = -2$ and $x = -1$

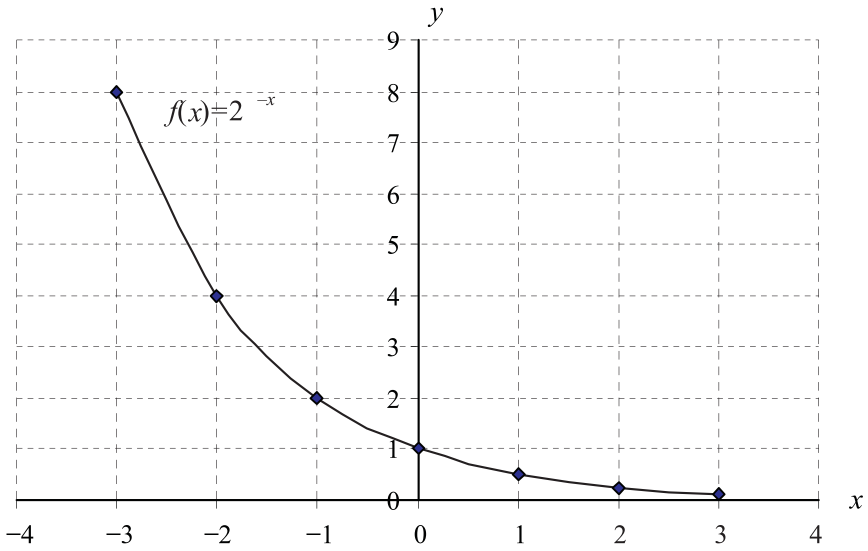
$$m = \frac{\text{change in height}}{\text{change in horizontal distance}}$$
$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
$$m = \frac{0.3679 - 0.1353}{-1 - -2}$$
$$m = \frac{0.2326}{1}$$
$$m = 0.2326$$

Rate of change, m, between $x = 2$ and $x = 3$

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}}$$
$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
$$m = \frac{20.0855 - 7.3891}{3 - 2}$$
$$m = \frac{12.6964}{1}$$
$$m = 12.6964$$

(a)

x	$f(x) = 2^{-x}$
–3	8
–2	4
–1	2
0	1
1	$\frac{1}{2} = 0.5$
2	$\frac{1}{4} = 0.25$
3	$\frac{1}{8} = 0.125$



Rate of change, m, between $x = -3$ and $x = -1$

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}}$$

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$m = \frac{2-8}{-1--3}$$

$$m = \frac{-6}{2}$$

$$m = -3$$

Rate of change, m, between $x = 1$ and $x = 3$

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}}$$

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$m = \frac{\frac{1}{8} - \frac{1}{2}}{3-1}$$

$$m = \frac{-\frac{3}{8}}{2}$$

$$m = -\frac{3}{16}$$

2. (a) From the graph the approximate coordinates are (9, 40) and (10, 60) to the nearest 10.

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}}$$

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$m = \frac{f(10) - f(9)}{10 - 9}$$

$$m = \frac{60 - 40}{1}$$

$$m = 20$$

Average gradient is 20.

(b) A range of alternatives are possible for this answer here are two.

(i) The values of y are increasing with respect to x at a rate of 20 for each unit of x.

(ii) When x increases by one unit the value of y increases 20 units.

3. The depreciation schedule for this car is as follows calculated from the formula, $D = f(n) = P(1 - \frac{r}{100})^n$ where P = \$32 000, r = 25% determined each year, and n changes from 0 to 6.

Year (*n*)	**Depreciated value (*D=f(n)*)**
0	32000
1	24000
2	18000
3	13500
4	10125
5	7594
6	5695

(a) Average rate of change of the value of the car over the first two years,

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}}$$
$$m = \frac{f(n_2) - f(n_1)}{n_2 - n_1}$$
$$m = \frac{f(2) - f(0)}{2 - 0}$$
$$m = \frac{18000 - 32000}{2}$$
$$m = \frac{-14000}{2}$$
$$m = -7000$$

Value of car is decreasing in the first two years by \$7000 per year.

(b) Average rate of change of the value of the car over the six years,

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}}$$
$$m = \frac{f(n_2) - f(n_1)}{n_2 - n_1}$$
$$m = \frac{f(6) - f(0)}{6 - 0}$$
$$m = \frac{5695 - 32000}{6 - 0}$$
$$m = \frac{-26305}{6}$$
$$m \approx -4384.17$$

Value of car is decreasing in the entire six years by about \$4384 per year.

4. (a) To find the rate at which the iceblock melts over the first five minutes calculate the rate of change between $t = 0$ and $t = 5$ for the function $V = f(t) = 30e^{-0.05672t}$

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}}$$
$$m = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$
$$m = \frac{f(5) - f(0)}{5 - 0}$$
$$m = \frac{22.592 - 30}{5}$$
$$m \approx -1.48$$

Block melts at a rate of –1.48 cm^3 per minute. The negative sign indicates it is decreasing in volume.

(b) To find rate at which iceblock melts over the five minutes from the 30th to 35th minute, calculate the rate of change between $t = 30$ and $t = 35$ for the function $V = f(t) = 30e^{-0.05672t}$

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}}$$

$$m = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

$$m = \frac{f(35) - f(30)}{35 - 30}$$

$$m \approx \frac{4.1206 - 5.4717}{5}$$

$$m \approx -0.27$$

Block melts at a rate of -0.27 cm^3 per minute. The negative sign indicates it is decreasing in volume.

(c) Since the melting rate in the 30 to 35 minute time span is much slower than the melting rate in the first 5 minutes, the ice melts much more slowly as time passes, decreasing in size more slowly.

5. To find the rate of change we need to use the equation $P = f(n) = e^{0.0198(n-1)}$ and evaluate it for $n = 4$ and $n = 5$, n is recorded in thousands unit lots.

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}}$$

$$m = \frac{f(n_2) - f(n_1)}{n_2 - n_1}$$

$$m = \frac{f(5) - f(4)}{5 - 4}$$

$$m \approx \frac{1.08 - 1.06}{1}$$

$$m \approx 0.02$$

Thus profit increased by 2 cents per unit for each lot of thousand units produced. This means that for a thousand units the profit increases by $20.

Activity 5.6

1. $A = P(1 + \frac{r}{100})^n$, P = 3000, r = 3.5, n is integral values 0 to 10.

(a)

Original function	
Years (n)	**Amount A(n) – $**
0	3000.00
1	3105.00
2	3213.68
3	3326.15
4	3442.57
5	3563.06
6	3687.77
7	3816.84
8	3950.43
9	4088.69
10	4231.80

(b)

Inverse function	
Amount (n) – $	**Years $A^{-1}(n)$**
3000.00	0
3105.00	1
3213.68	2
3326.15	3
3442.57	4
3563.06	5
3687.77	6
3816.84	7
3950.43	8
4088.69	9
4231.80	10

(c) The inverse function is used to determine the number of years required to earn a certain amount of money.

(d) From the second table, $3000 takes approximately 8.5 years to grow to $4000.

2. The function is $P(t) = 19.5e^{0.0163t}$

(a)

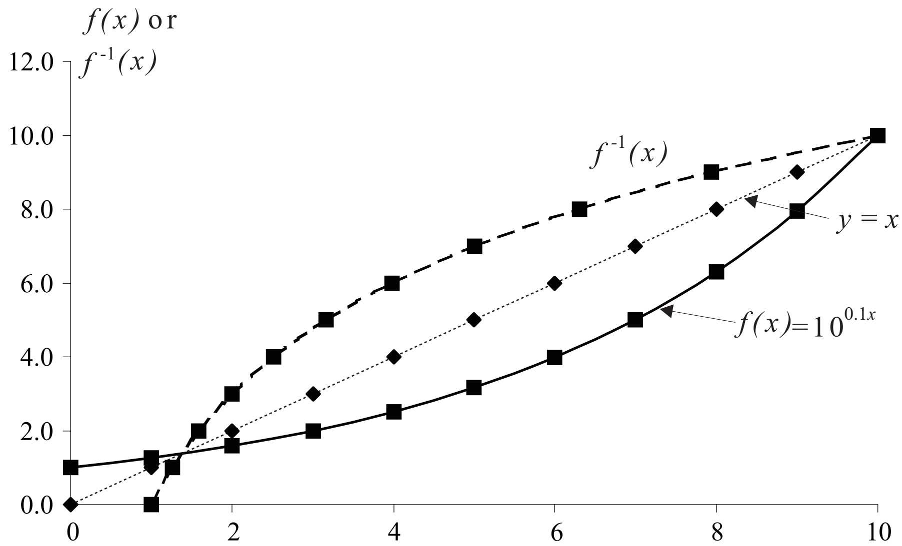
Original function	
Years (t)	**Population $P(t)$ – millions**
0	19.5
1	19.82045461
2	20.14617543
3	20.47724901
4	20.81376331
5	21.15580774
6	21.50347318
7	21.85685201
8	22.21603811
9	22.58112692

(b)

Inverse function	
Population t – millions	**Years $P^{-1}(t)$**
19.5	0
19.82045461	1
20.14617543	2
20.47724901	3
20.81376331	4
21.15580774	5
21.50347318	6
21.85685201	7
22.21603811	8
22.58112692	9

(c) Since it takes almost 9 years to reach 22.5 million people thus the target should be reached toward the end of the year 2008.

3.



4. You could use the original function or the inverse function to answer this question.

Original function	
Weeks (t)	**Radium $f(t)$ – (g)**
0	10.00
1	9.26
2	8.57
3	7.94
4	7.35
5	6.80
6	6.30
7	5.83
8	5.40
9	5.00

Inverse function	
Radium $f^{-1}(t)$ – (g)	**Weeks**
10.00	0
9.26	1
8.57	2
7.94	3
7.35	4
6.80	5
6.30	6
5.83	7
5.40	8
5.00	9

From either table we see that 10 grams of radium is reduced to 5 grams in 9 weeks.

5. We could use either the original function or the inverse to answer this question.

Original function	
thousands of years	**Carbon**
0	100
1	88
2	78
3	69
4	61
5	54

Inverse function	
Carbon	**thousands of years**
100	0
88	1
78	2
69	3
61	4
54	5

If we start with 100 g then 60% will occur when we have 60 g. From either table we can see that we have 60 g when the bone is 4 000 years old.

Activity 5.7

1.

Exponential form	**Logarithmic form**
$5^2 = 25$	$\log_5 25 = 2$
$9^{\frac{1}{2}} = 3$	$\log_9 3 = \frac{1}{2}$
$10^{1.8} \approx 63$	$\log_{10} 63 \approx 1.8$
$2^{-2} = \frac{1}{4}$	$\log_2 \frac{1}{4} = -2$
$3^x = 10$	$\log_3 10 = x$

2. (a) $10^3 = 1000$

   (b) $4^2 = 16$

   (c) $2^4 = 16$

   (d) $e^3 \approx 20$

3. (a) Since $1000000 = 10^6$, then $\log_{10} 1000000 = 6$

   (b) Since $\frac{1}{8} = \frac{1}{2^3} = 2^{-3}$, then $\log_2 \frac{1}{8} = -3$

4. Let

$$\log_{25} 5 = x$$
$$5 = 25^x$$
$$5^1 = (5^2)^x$$
$$5^1 = 5^{2x}$$
$$1 = 2x$$
$$x = \frac{1}{2}$$
$$\therefore \log_{25} 5 = \frac{1}{2}$$

Let

$$\log_2 0.25 = y$$
$$2^y = 0.25$$
$$2^y = \frac{1}{4}$$
$$2^y = 2^{-2}$$
$$y = -2$$
$$\therefore \log_2 0.25 = -2$$

5. (a) $x = \log 10$

$$10^x = 10^1$$
$$x = 1$$

(b) $\log_2 x = 5$

$$2^5 = x$$
$$x = 32$$

(c) $\log_4 x = 0$

$$4^0 = x$$
$$1 = x$$
$$x = 1$$

(d) $\log_x 27 = -3$

$$x^{-3} = 27$$
$$x^{-3} = 3^3$$
$$x^{-3} = (\frac{1}{3})^{-3}$$
$$x = \frac{1}{3}$$

(e) $\ln x = 3$

$$e^3 = x$$
$$x \approx 20$$

Activity 5.8

1. (a) $\log_3 27 + \log_3 \frac{1}{3} = \log_3(27 \times \frac{1}{3})$

$= \log_3 9$

$= \log_3 3^2$

$= 2\log_3 3$

$= 2 \times 1$

$= 2$

(b) $\log_2 16 - \log_2 8 = \log_2(\frac{16}{8})$

$= \log_2 2$

$= 1$

(c) $\log 125 + \log 32 - \log 4 = \log(\frac{125 \times 32}{4})$

$= \log 1000$

$= \log 10^3$

$= 3\log 10$

$= 3 \times 1$

$= 3$

2. (a) $2\log_{12} 3 + 4\log_{12} 2 = \log_{12} 3^2 + \log_{12} 2^4$

$= \log_{12}(3^2 \times 2^4)$

$= \log_{12} 144$

$= \log_{12} 12^2$

$= 2\log_{12} 12$

$= 2 \times 1$

$= 2$

(b) $\ln 25 + 2\ln\frac{1}{5} = \ln 25 + \ln(\frac{1}{5})^2$

$= \ln(25 \times (\frac{1}{5})^2)$

$= \ln 1$

$= 0$

(c) $$\begin{aligned} 4\log_5 10 - 3\log_5 2 - \log_5 10 &= \log_5 10^4 - \log_5 2^3 - \log_5 10 \\ &= \log_5\left(\frac{10^4}{2^3 \times 10}\right) \\ &= \log_5 125 \\ &= \log_5 5^3 \\ &= 3\log_5 5 \\ &= 3\times 1 \\ &= 3 \end{aligned}$$

3. (a) $$\begin{aligned} \log_3 x &= \log_3 4 + \log_3 2 \\ \log_3 x &= \log_3 (4\times 2) \\ \log_3 x &= \log_3 8 \\ x &= 8 \end{aligned}$$

(b) $$\begin{aligned} \log 5m &= 2\log m \\ \log 5m &= \log m^2 \\ 5m &= m^2 \\ m^2 - 5m &= 0 \\ m(m-5) &= 0 \\ m &= 0, m = 5 \end{aligned}$$

Note only m = 5 is possible because log0 is undefined (try it in the original equation).

Check solution by substituting into original equation.

(c) $$\begin{aligned} 2\log p + 3 - \log p^5 &= 0 \\ 2\log p - \log p^5 &= -3 \\ 2\log p - 5\log p &= -3 \\ -3\log p &= -3 \\ \log p &= 1 \\ p &= 10 \end{aligned}$$

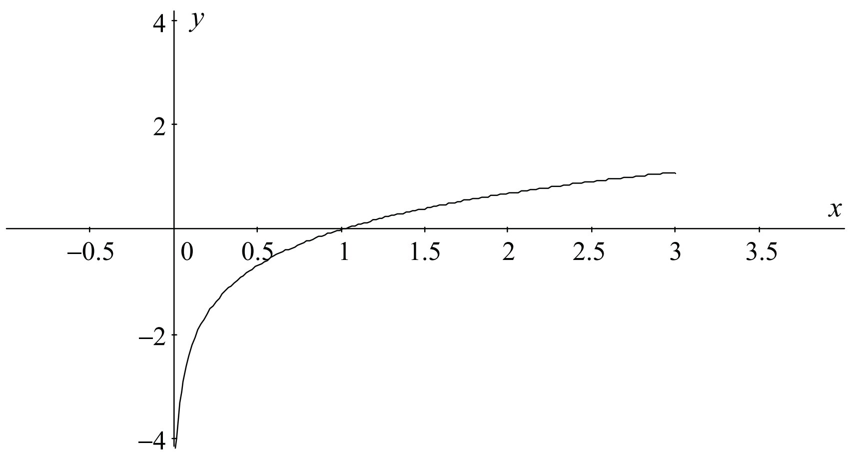
Check solution by substituting into original equation.

(d) $$\begin{aligned} \log 2 + \log 5 + \log y - \log 3 &= 2 \\ \log\left(\frac{2\times 5\times y}{3}\right) &= 2 \\ \log\left(\frac{10y}{3}\right) &= 2 \\ \frac{10y}{3} &= 10^2 \\ 10y &= 100\times 3 \\ y &= \frac{100\times 3}{10} \\ y &= 30 \end{aligned}$$

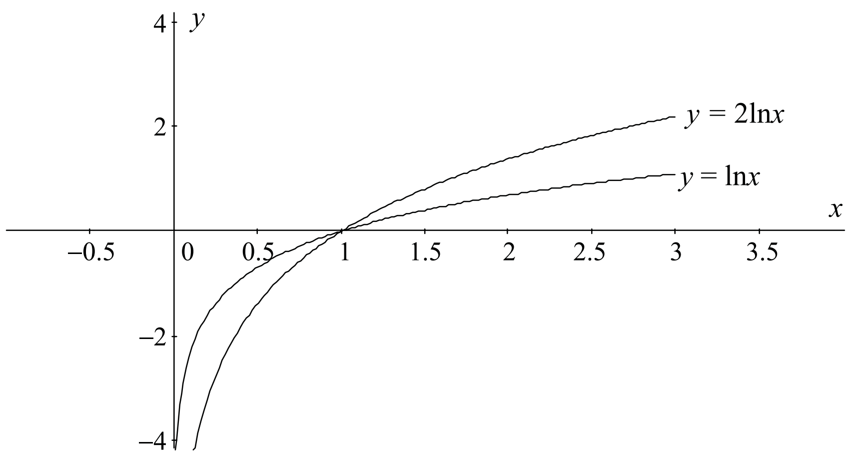
Check solution by substituting into original equation.

Activity 5.9

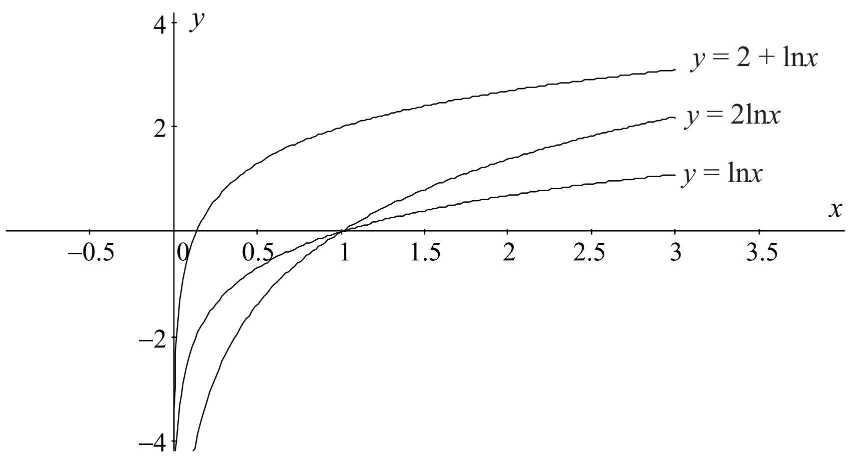
1. $y = \ln x$



2. (a) (i)



(ii)

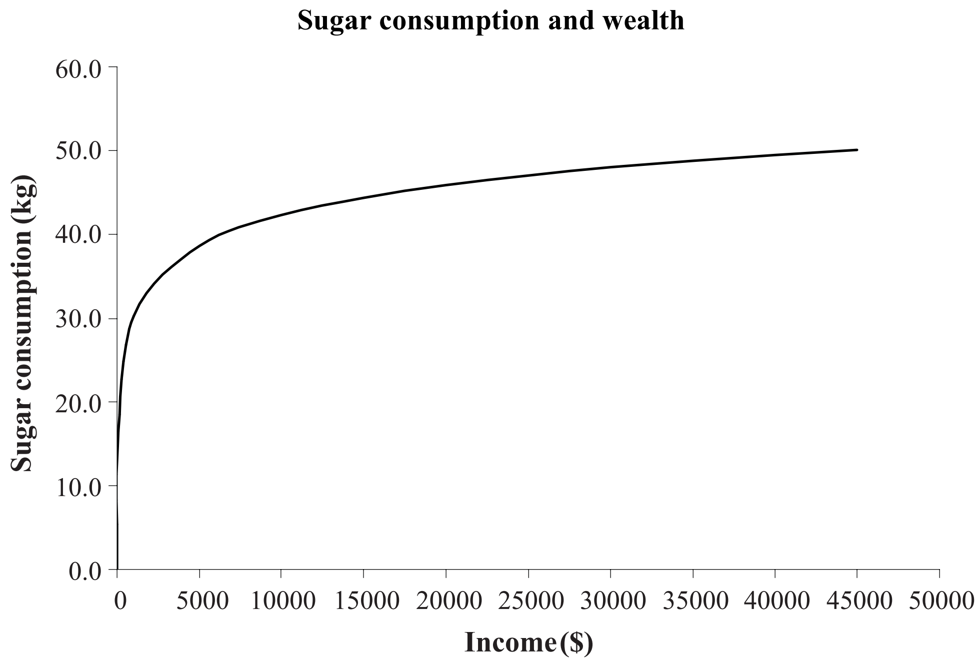


(b) Multiplying the function by 2 has the effect of increasing the rate of change of the logarithmic function. Notice it still intersects the x-axis at 1.

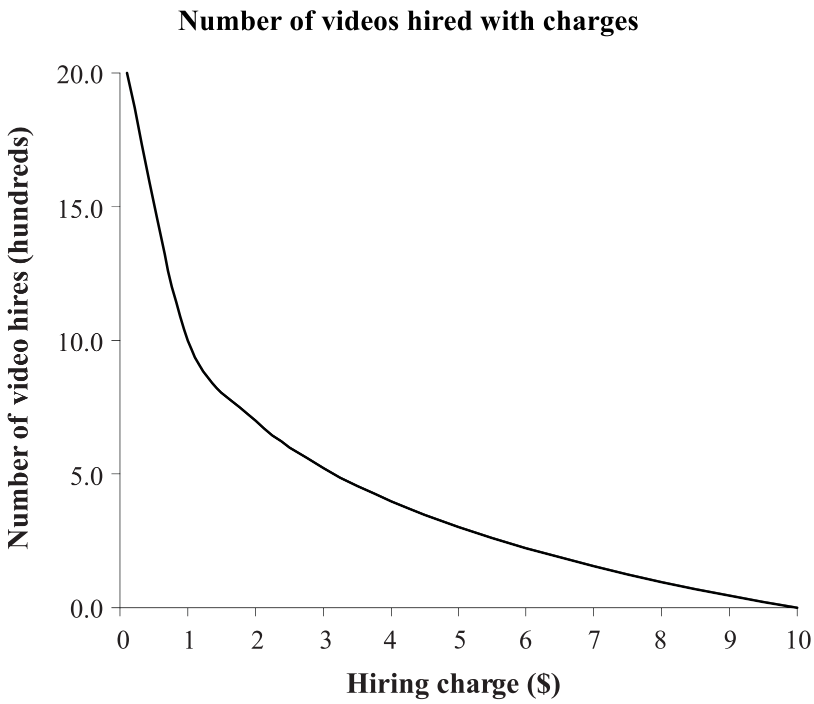
Adding 2 to the function has the effect of shifting the function up 2 units. It no longer passes through $x = 1$ but it has the same amount of curvature as the original function.

Activity 5.10

1.



2.



(a) From the graph when the hiring charge is $3.50 the number hired will be approximately 450. You could also substitute 3.5 into the original function

$$N = -10\log\frac{C}{10}$$

$$N = -10\log\frac{3.5}{10}$$

$$N \approx 4.56 \text{ hundreds of videos}$$

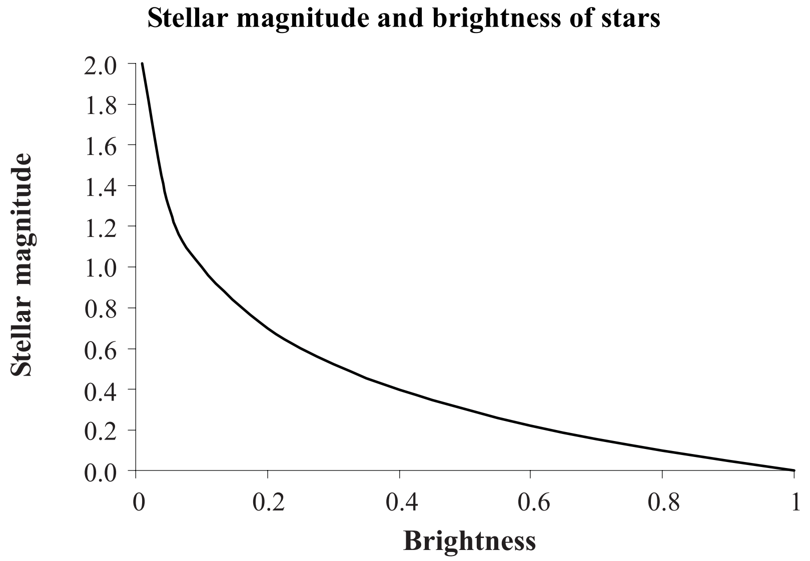
$$N \approx 456 \text{ videos}$$

(b) Gross income from hires is number of videos multiplied by charge per video. If you compare the gross income for different numbers of videos:

Charge per video (C)	**Number hired (N)**	**Gross income ($C \times N$)**
1	1000	1000
2	699	1398
3	523	1569
4	398	1592
5	301	1505
6	222	1332
7	155	1085
8	97	776
9	46	414

$4 is the most profitable charge.

3.



(a) If the star has a brightness of 0.7943 from the formula it will have a stellar magnitude of –log(0.7943) or 0.1. This can be confirmed from the graph.

(b) If the magnitude is 2.1 then

$$2.1 = -\log B$$
$$-2.1 = \log B$$
$$10^{-2.1} = B$$
$$B \approx 0.007943$$

The brightness is 0.007943 (you can get an estimate of this from the graph).

(c) Comparing the first star with the second star the first is about 100 times as bright as the second star.

Activity 5.11

1.

x	$f(x) = \ln x$
0.01	–4.61
0.5	–0.69
1	0.00
1.5	0.41
2	0.69
3	1.10
3.5	1.25
4	1.39
4.5	1.50
5	1.61
5.5	1.70

(a) Rate of change between $x = 0.5$ and $x = 1$

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$m \approx \frac{0 - -0.7}{1 - 0.5} \approx 1.4$$

(b) Rate of change between $x = 5$ and $x = 5.5$

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$m \approx \frac{1.7 - 1.6}{5.5 - 5} \approx 0.2$$

(c) Since 1.4 is 7 times greater than 0.2, the function must be increasing at a greater rate between $x = 0.5$ and $x = 1$ than after $x = 5$.

2. (a) Rate of change between $x = 2\ 500$ and $x = 5\ 000$

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$m \approx \frac{1.6 - 1}{5000 - 2500} \approx 0.00024$$

(b) Rate of change between $x = 12\ 500$ and $x = 15\ 000$

$$m = \frac{\text{change in height}}{\text{change in horizontal distance}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$m \approx \frac{2.6 - 2.5}{15000 - 12500} \approx 0.00004$$

(c) As the population size of a city increases the walking speed continues to increase but it is increasing at a reduced rate the large the population size by a factor of 6. (Note that this is just one way to describe the function…..you might have another way that is equally correct.)

(d) Although the function appears to work for the domain given I would imagine that there are finite limits to the speed of a person walking. The function could not increase indefinitely.

Activity 5.12

1. (a)

$$\begin{aligned} 2^a &= 1024 \\ \log 2^a &= \log 1024 \\ a \log 2 &= \log 1024 \\ a &= \frac{\log 1024}{\log 2} \\ a &= 10 \end{aligned}$$

Note: You could have used log base e it makes no difference to the solution. Check your answer by substituting it back into the original equation.

(b)

$$\begin{aligned} 9^b &= 25 \\ \log 9^b &= \log 25 \\ b \log 9 &= \log 25 \\ b &= \frac{\log 25}{\log 9} \\ b &\approx 1.465 \end{aligned}$$

Check your answer by substituting it back into the original equation.

(c)
$$
\begin{aligned}
7^{c+2} &= 11 \\
\log 7^{c+2} &= \log 11 \\
(c+2)\log 7 &= \log 11 \\
c+2 &= \frac{\log 11}{\log 7} \\
c &= \frac{\log 11}{\log 7} - 2 \\
c &\approx -0.768
\end{aligned}
$$

Check your answer by substituting it back into the original equation.

(d)
$$
\begin{aligned}
5^{1-d} &= 7 \\
\log 5^{1-d} &= \log 7 \\
(1-d)\log 5 &= \log 7 \\
1-d &= \frac{\log 7}{\log 5} \\
d &= 1 - \frac{\log 7}{\log 5} \\
d &\approx -0.209
\end{aligned}
$$

Check your answer by substituting it back into the original equation.

(e)
$$
\begin{aligned}
3^{2x} &= 40 \\
\log 3^{2x} &= \log 40 \\
2x\log 3 &= \log 40 \\
2x &= \frac{\log 40}{\log 3} \\
x &= \frac{\log 40}{2\log 3} \\
x &\approx 1.679
\end{aligned}
$$

Check your answer by substituting it back into the original equation.

2. (a)
$$
\begin{aligned}
4^{a+1} &= 8 \\
\log 4^{a+1} &= \log 8 \\
(a+1)\log 4 &= \log 8 \\
a+1 &= \frac{\log 8}{\log 4} \\
a &= \frac{\log 8}{\log 4} - 1 \\
a &= \frac{1}{2}
\end{aligned}
$$

Check your answer by substituting it back into the original equation.

(b)
$$
\begin{aligned}
4^{b-1} &= \frac{1}{8} \\
\log 4^{b-1} &= \log \frac{1}{8} \\
(b-1)\log 4 &= \log \frac{1}{8} \\
b-1 &= \frac{\log \frac{1}{8}}{\log 4} \\
b &= \frac{\log \frac{1}{8}}{\log 4} + 1 \\
b &= -\frac{1}{2}
\end{aligned}
$$

Check your answer by substituting it back into the original equation.
(Note that this question could also be solved using the power rules from module 3.)

(c)
$$\begin{aligned}
2\times 3^{c-1} &= 15\\
3^{c-1} &= \frac{15}{2}\\
\log 3^{c-1} &= \log\frac{15}{2}\\
(c-1)\log 3 &= \log\frac{15}{2}\\
c-1 &= \frac{\log\frac{15}{2}}{\log 3}\\
c &= \frac{\log\frac{15}{2}}{\log 3}+1\\
c &\approx 2.834
\end{aligned}$$

Check your answer by substituting it back into the original equation.

(d)
$$\begin{aligned}
10^{d-2}-1 &= 2.444\\
10^{d-2} &= 3.444\\
\log 10^{d-2} &= \log 3.444\\
(d-2)\log 10 &= \log 3.444\\
d-2 &= \log 3.444 \qquad \text{Recall that log 10 = 1}\\
d &= \log 3.444+2\\
d &\approx 2.537
\end{aligned}$$

Check your answer by substituting it back into the original equation.

(e)
$$\begin{aligned}
3^{n+1} &= 5^{n}\\
\log 3^{n+1} &= \log 5^{n}\\
(n+1)\log 3 &= n\log 5\\
n\log 3+\log 3 &= n\log 5\\
n(\log 5-\log 3) &= \log 3\\
n &= \frac{\log 3}{\log 5-\log 3}\\
n &\approx 2.151
\end{aligned}$$

Check your answer by substituting it back into the original equation.

Note all of the above questions could have just as easily been solved by taking log to the base e rather than log to the base 10 of both sides.

3. (a)

$$a^{2n} = b^2$$
$$\log a^{2n} = \log b^2$$
$$2n \log a = \log b^2$$
$$n = \frac{\log b^2}{2 \log a}$$

(b)

$$y = ae^{4n}$$
$$y = a \times e^{4n}$$
$$\frac{y}{a} = e^{4n}$$
$$\ln\frac{y}{a} = \ln e^{4n}$$

Since we have *e* in the question we should take log base *e*.

$$\ln\frac{y}{a} = 4n \ln e$$
$$n = \frac{\ln\frac{y}{a}}{4 \ln e}$$
$$n = \frac{\ln\frac{y}{a}}{4}$$

(c)

$$A = P(1 + \frac{r}{100})^n$$
$$\frac{A}{P} = (1 + \frac{r}{100})^n$$
$$\log\frac{A}{P} = \log(1 + \frac{r}{100})^n$$
$$\log\frac{A}{P} = n \log(1 + \frac{r}{100})$$
$$n = \frac{\log\frac{A}{P}}{\log(1 + \frac{r}{100})}$$

(d)
$$D = P(1 - \frac{r}{100})^n$$
$$\frac{D}{P} = (1 - \frac{r}{100})^n$$
$$\log\frac{D}{P} = \log(1 - \frac{r}{100})^n$$
$$\log\frac{D}{P} = n\log(1 - \frac{r}{100})$$
$$n = \frac{\log\frac{D}{P}}{\log(1 - \frac{r}{100})}$$

(e)
$$C = 79.345e^{-0.0166n}$$
$$C = 79.345 \times e^{-0.0166n}$$
$$\frac{C}{79.345} = e^{-0.0166n}$$
$$\ln(\frac{C}{79.345}) = \ln e^{-0.0166n}$$
$$\ln(\frac{C}{79.345}) = -0.0166n\ln e$$
$$\ln(\frac{C}{79.345}) = -0.0166n$$
$$n = \frac{\ln(\frac{C}{79.345})}{-0.0166}$$

(f)
$$L = 2(3e)^{-n}$$
$$\frac{L}{2} = (3e)^{-n}$$
$$\ln(\frac{L}{2}) = \ln(3e)^{-n}$$
$$\ln(\frac{L}{2}) = -n\ln(3e)$$
$$n = \frac{\ln(\frac{L}{2})}{-\ln 3e}$$
or
$$n = -\frac{\ln L - \ln 2}{\ln 3 + 1}$$

Activity 5.13

1. $A = Pe^{kt}$, $t = 0$, $A = 100$

$$A = Pe^{kt}$$
$$100 = Pe^{0\times k}$$
$$P = 100$$

To find k use the values of $t = 10$ and $A = 185$

$$A = Pe^{kt}$$
$$185 = 100 \times e^{10k}$$
$$1.85 = e^{10k}$$
$$\ln 1.85 = \ln e^{10k}$$
$$\ln 1.85 = 10k \ln e$$
$$10k = \ln 1.85$$
$$k = \frac{\ln 1.85}{10}$$
$$k \approx 0.0615$$

When $t = 20$

$$A = 100e^{0.0615\times 20}$$
$$A = 342.122$$

After 20 years the value is \$342.12. (\$342.25 if k is not rounded early in the process)

2. $N = N_0 e^{-kt}$, when $t = 0$ and $N = 100$,

$$N = N_0 e^{-kt}$$
$$100 = N_0 e^{-k0}$$
$$N_0 = 100$$

To find k use $t = 2$ and $N = 60$

$$60 = 100e^{-k2}$$
$$0.60 = e^{-k2}$$
$$\ln 0.60 = \ln e^{-k2}$$
$$\ln 0.60 = -k2 \ln e$$
$$k = \frac{\ln 0.60}{-2}$$
$$k \approx 0.2554$$

When $N = 30$

$$\begin{aligned}
30 &= 100e^{-0.2554t} \\
0.30 &= e^{-0.2554t} \\
\ln 0.30 &= \ln e^{-0.2554t} \\
\ln 0.30 &= -0.2554t \ln e \\
t &= \frac{\ln 0.30}{-0.2554} \\
t &\approx 4.714
\end{aligned}$$

It takes approximately 4.7 years to reduce to 30 g.

3. $T = T_0 e^{-kt}$, when $t = 0$ and $T = 10\ 000$, $T_0 = 10\ 000$.

   To find k, use $t = 5$ and $T = 5000$

$$\begin{aligned}
5000 &= 10000e^{-k5} \\
0.5 &= e^{-k5} \\
\ln 0.5 &= \ln e^{-k5} \\
\ln 0.5 &= -5k \ln e \\
k &= \frac{\ln 0.5}{-5} \\
k &\approx 0.1386
\end{aligned}$$

   When $t = 10$,

$$\begin{aligned}
T &= 10000e^{-0.1386\times 10} \\
T &= 2500
\end{aligned}$$

   Temperature will be 2500°C in 10 million years.

4. $N = N_0 e^{kt}$, when $t = 0$, $N = 2000$, $N_0 = 2000$

   To find k use $t = 2$ and $N = 2420$

$$\begin{aligned}
2420 &= 2000e^{k2} \\
1.21 &= e^{k2} \\
\ln 1.21 &= \ln e^{k2} \\
\ln 1.21 &= 2k \ln e \\
k &= \frac{\ln 1.21}{2} \\
k &\approx 0.0953
\end{aligned}$$

When $t = 24$,

$N = 2000e^{0.0953\times 24}$

$N \approx 19694.65$

After 24 hours the numbers present are 19695.

5. $P = P_0e^{kt}$, when $t = 0$ and $P = 120\,000$, then $P_0 = 120\,000$.

To find k use $t = 10$ and $P = 180\,000$

$$\begin{aligned}
180000 &= 120000e^{k10} \\
1.5 &= e^{k10} \\
\ln 1.5 &= \ln e^{k10} \\
\ln 1.5 &= 10k \ln e \\
k &= \frac{\ln 1.5}{10} \\
k &\approx 0.0405
\end{aligned}$$

When $P = 200\,000$

$$\begin{aligned}
200000 &= 120000e^{0.0405t} \\
1.6666 &\approx e^{0.0405t} \\
\ln 1.6666 &\approx \ln e^{0.0405t} \\
\ln 1.6666 &\approx 0.0405t \ln e \\
t &\approx \frac{\ln 1.6666}{0.0405} \\
t &\approx 12.6
\end{aligned}$$

It will take approximately 12.6 years to reach 200 000.

Solutions to a taste of things to come

1. (a) Sag is the difference in the values of the function at $x = 10$ and $x = 0$.

$$\begin{aligned}\text{Sag} &= y_{x=10} - y_{x=0}\\ &= [25(e^{\frac{10}{50}} + e^{\frac{-10}{50}}) - 45] - [25(e^{\frac{0}{50}} + e^{\frac{-0}{50}}) - 45]\\ &\approx 6.000... - 5\\ &\approx 1\end{aligned}$$

Therefore sag is approximately 1 m

(b) One metre from the support is $x = 9$ (or $x = –9$)

$$\begin{aligned}y_{x=9} &= 25(e^{\frac{9}{50}} + e^{\frac{-9}{50}}) - 45\\ &\approx 5.8\end{aligned}$$

$$\begin{aligned}y_{x=10} &= 25(e^{\frac{10}{50}} + e^{\frac{-10}{50}}) - 45\\ &\approx 6\end{aligned}$$

$$\text{Slope} = m = \frac{6.0 - 5.8}{10 - 9} \approx 0.2$$

Average rise/fall of 20 cm per horizontal metre.

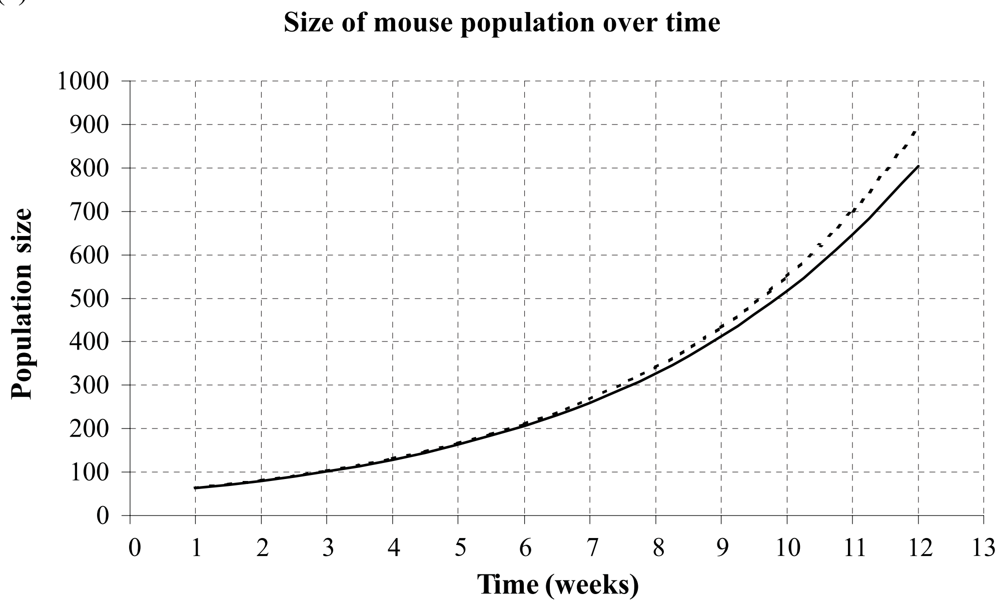
(c) Lowest point when $x = 0$ and $y = 5$. One metre from the lowest point is

$$\begin{aligned}y_{x=1} &= 25(e^{\frac{1}{50}} + e^{\frac{-1}{50}}) - 45\\ &\approx 5.01\end{aligned}$$

$$\text{Slope} = m \approx \frac{5.01 - 5}{1 - 0} \approx 0.01$$

Therefore average rise/fall of 1 cm per horizontal metre.

2. (a)

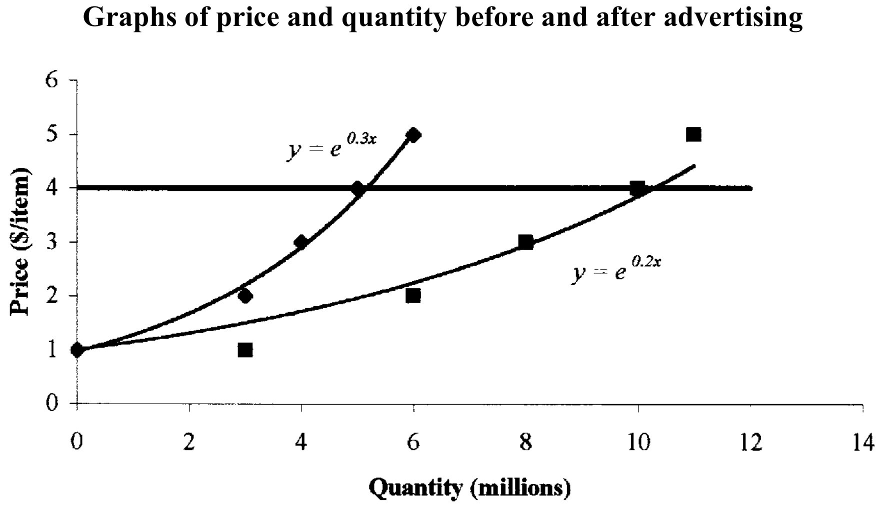


(b) The dotted graph is the traditional exponential model. For the first 12 weeks the graphs are very similar in shape. The traditional model increases slightly quicker than the corrected model.

(c) From the corrected model after 30 weeks the population is 7411, while with the traditional model the population would be 66972.

Therefore after 30 weeks the population predicted from traditional model is approximately 9 times as great as the population from the corrected model.

3. (a)

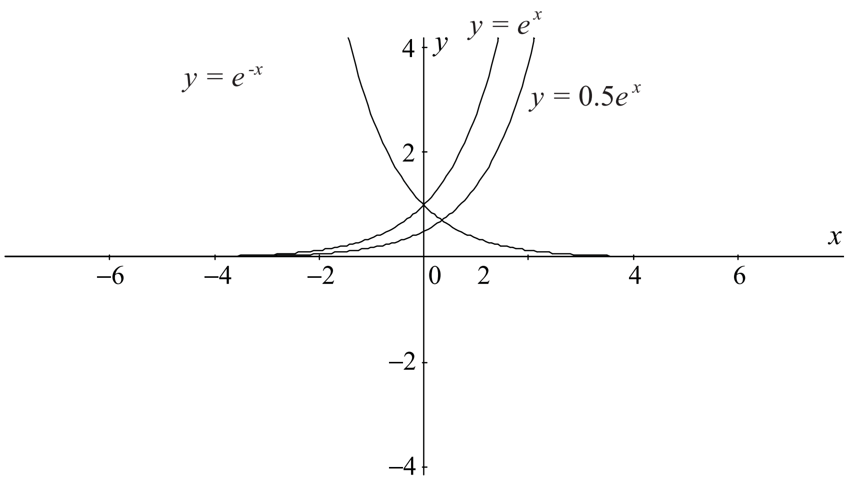


When $P = 4$ dollars the supply is 5 million before advertising and 10 million after advertising. It has been doubled.

Solutions to post-test

1. Match the equations to the graphs of exponential functions.

   $y = e^x$, $y = e^{-x}$, $y = 0.5e^x$



2. To find the y intercept put $x = 0$ and solve for y.

   (a) $y = 2 \times (\frac{1}{2})^x$

   $y = 2 \times (\frac{1}{2})^0$

   $y = 2 \times 1$

   $y = 2$

   (b) $y = 2^x - \frac{1}{2}$

   $y = 2^0 - \frac{1}{2}$

   $y = 1 - \frac{1}{2}$

   $y = \frac{1}{2}$

(c) $y = 2^{-x} + \frac{1}{2}$

$y = 2^{-0} + \frac{1}{2}$

$y = 1 + \frac{1}{2}$

$y = \frac{3}{2}$

(d) $y = \frac{1}{2} \times 2^{x}$

$y = \frac{1}{2} \times 2^{0}$

$y = \frac{1}{2} \times 1$

$y = \frac{1}{2}$

3. When $x = 2$,

$$y = 1.5 \times (1.5)^{-x}$$
$$y = 1.5 \times (1.5)^{-2}$$
$$y = \frac{1.5}{1.5^2}$$
$$y = \frac{1}{1.5}$$
$$y \approx 0.7$$

4. (a) $10^{x+1} = 5$

$\log_{10} 5 = x + 1$

(b) $2 = e^{y}$

$\ln 2 = y$

5. (a) $\log_3 81 = \log_3 3^4$

$= 4\log_3 3$

$= 4$

(b) $\log_{\frac{1}{5}} 125 = \log_{\frac{1}{5}} (\frac{1}{125})^{-1}$

$= \log_{\frac{1}{5}} (\frac{1}{5^3})^{-1}$

$= \log_{\frac{1}{5}} (\frac{1}{5})^{-3}$

$= -3\log_{\frac{1}{5}} (\frac{1}{5})$

$= -3$

6. (a) $a = \log_{10} 0.1$

$10^a = 0.1$

(b) $\log_2 8 = b$

$2^b = 8$

7. (a) $\log_2 (2p) = 5$

$2^5 = 2p$

$p = 2^4$

(b) $\ln\frac{10}{q} = -0.405$

$e^{-0.405} = \frac{10}{q}$

$q = \frac{10}{e^{-0.405}}$

$q \approx 15$

8. (a)
$$\begin{aligned}
2^t &= 7.5\\
\ln 2^t &= \ln 7.5\\
t\ln 2 &= \ln 7.5\\
t &= \frac{\ln 7.5}{\ln 2}\\
t &\approx 2.9069
\end{aligned}$$

(b)
$$\begin{aligned}
1 &= 8(1-e^{2t})\\
\frac{1}{8} &= 1-e^{2t}\\
e^{2t} &= 1-\frac{1}{8}\\
e^{2t} &= \frac{7}{8}\\
\ln e^{2t} &= \ln\frac{7}{8}\\
2t\ln e &= \ln\frac{7}{8}\\
2t &= \ln\frac{7}{8}\\
t &= \frac{1}{2}\ln\frac{7}{8}\\
t &\approx -0.0668
\end{aligned}$$

9. (a)
$$\begin{aligned}
\log_3 27 + \log_3\frac{1}{9} - \log_3 9 &= \log_3\frac{27\times\frac{1}{9}}{9}\\
&= \log_3\frac{1}{3}\\
&= \log_3 3^{-1}\\
&= -1\log_3 3\\
&= -1
\end{aligned}$$

(b)
$$\begin{aligned}
\ln 0.4e + \ln 10e - 2\ln 2 &= \ln(0.4e\times 10e) - \ln 2^2\\
&= \ln(\frac{0.4e\times 10e}{2^2})\\
&= \ln e^2\\
&= 2\ln e\\
&= 2
\end{aligned}$$

10. (a)
$$\begin{aligned}\log_4 3 &= \log_4 x - \log_4 2\\ \log_4 x &= \log_4 3 + \log_4 2\\ \log_4 x &= \log_4 3\times 2\\ \log_4 x &= \log_4 6\\ x &= 6\end{aligned}$$

(b)
$$\begin{aligned}\log_2 \frac{1}{8} + \log_2 y &= \log_2 \frac{1}{4}\\ \log_2 y &= \log_2 \frac{1}{4} - \log_2 \frac{1}{8}\\ \log_2 y &= \log_2 (\frac{1}{4} \div \frac{1}{8})\\ \log_2 y &= \log_2 2\\ y &= 2\end{aligned}$$

11.
$$\begin{aligned}4\times 10^{\frac{r}{100}} - 1 &= 4\\ 4\times 10^{\frac{r}{100}} &= 5\\ 10^{\frac{r}{100}} &= \frac{5}{4}\\ \log 10^{\frac{r}{100}} &= \log\frac{5}{4}\\ \frac{r}{100}\log 10 &= \log\frac{5}{4}\\ \frac{r}{100} &= \log\frac{5}{4}\\ r &= 100\log\frac{5}{4}\\ r &\approx 9.69\end{aligned}$$

12. $$y = 1.4e^{-0.6t} - 3$$
$$y + 3 = 1.4e^{-0.6t}$$
$$\frac{y+3}{1.4} = e^{-0.6t}$$
$$\ln\frac{y+3}{1.4} = \ln e^{-0.6t}$$
$$\ln\frac{y+3}{1.4} = -0.6t\ln e$$
$$\ln\frac{y+3}{1.4} = -0.6t$$
$$t = \frac{\ln(\frac{y+3}{1.4})}{-0.6}$$

13. Given the original equation we can rearrange it to become the second expression.

$$g = 3\times(ae)^n$$
$$\frac{g}{3} = (ae)^n$$
$$\ln\frac{g}{3} = \ln(ae)^n$$
$$\ln\frac{g}{3} = n\ln(ae)$$
$$n = \frac{\ln\frac{g}{3}}{\ln(ae)}$$
$$n = \frac{\ln g - \ln 3}{\ln a + \ln e}$$
$$n = \frac{\ln g - \ln 3}{\ln a + 1}$$

Thus the first equation is equivalent to the second equation.

14. $$A = P(1+i)^n$$
$$\frac{A}{P} = (1+i)^n$$
$$\ln\frac{A}{P} = \ln(1+i)^n$$
$$\ln\frac{A}{P} = n\ln(1+i)$$
$$n = \frac{\ln\frac{A}{P}}{\ln(1+i)}$$

15. When $P = 50$, $A = 75$ and $i = 0.085$

$$n = \frac{\ln \frac{A}{P}}{\ln(1+i)}$$

$$n = \frac{\ln \frac{75}{50}}{\ln(1+0.085)}$$

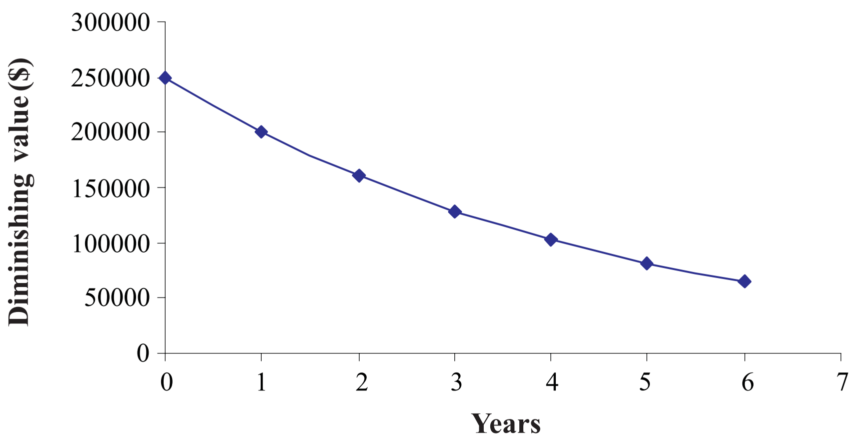
$$n \approx 4.97$$

This it takes approximately 5 years.

16. (a) Depreciation schedule for machinery is as follows:

Years (n)	**Diminishing value ($)**
0	250 000
1	200 000
2	160 000
3	128 000
4	102 400
5	81 920
6	65 536

Depreciation schedule for machinery



(b) $D(n) = P(1 - \frac{r}{100})^n$

$D(n) = 250000(1 - \frac{20}{100})^n$

$D(n) = 250000 \times (0.8)^n$

(c) If $n = 10$

$D(n) = 250000 \times (0.8)^{10}$

≈ 26843.55

Depreciated value is approximately \$26844.

17. When $t = 5$

$V = 10e^{\frac{-t}{3}}$

$V = 10e^{\frac{-5}{3}}$

$V = 1.88875....$

$V \approx 1.889$

Voltage is approximately 1.889 volts.

18. $R = 12e^{-0.075t}$

(a) Present value occurs when $t = 0$,

$R = 12e^{-0.075 \times 0}$

$R = 12$

Value is 12 g

(b) Half of the original mass is 6 g.

$$
\begin{aligned}
R &= 12e^{-0.075t} \\
6 &= 12e^{-0.075t} \\
0.5 &= e^{-0.075t} \\
\ln 0.5 &= \ln e^{-0.075t} \\
\ln 0.5 &= -0.075t \ln e \\
\ln 0.5 &= -0.075t \\
t &= \frac{\ln 0.5}{-0.075} \\
t &\approx 9.24
\end{aligned}
$$

It takes approximately nine and a quarter years to reach its half-life.

19. $P = 4.5e^{0.0142t}$

Find P for the years 1981, 1986, 2001 and 2006 using the above formula.

Year	**Time (years –*t*)**	**Population (billions –*P*)**
1981	0	4.5
1986	5	4.83
2001	20	5.98
2006	25	6.42

Rate of growth (1981 to 1986),

$$m_1 \approx \frac{4.83 - 4.5}{5 - 0}$$
$$\approx 0.066 \text{ billion per year}$$

Rate of growth (2001 to 2006),

$$m_2 \approx \frac{6.42 - 5.98}{25 - 20}$$
$$\approx 0.088 \text{ billion per year}$$

If we compare growth rates we find that 0.088 is one and one third times the value of 0.066. Thus population is predicted to grow faster in the next century.

20. $P = 20 \times 10^{0.1n}$

When $n = 60$

$P = 20 \times 10^{0.1n}$

$P = 20 \times 10^{0.1 \times 60}$

$P = 20 \times 10^{6}$ μP

When $n = 50$

$P = 20 \times 10^{0.1n}$

$P = 20 \times 10^{0.1 \times 50}$

$P = 20 \times 10^{5}$ μP

When $n = 40$

$P = 20 \times 10^{0.1n}$

$P = 20 \times 10^{0.1 \times 40}$

$P = 20 \times 10^{4}$ μP

Comparing 60 decibels with 50 decibels we find that the loudness 60 decibels is 10 times that of 50 decibels because $\dfrac{20\times10^6\ \mu\mathrm{P}}{20\times10^5\ \mu\mathrm{P}}=10$.

Comparing 60 decibels with 40 decibels we find that the loudness 60 decibels is 100 times that of 40 decibels because $\dfrac{20\times10^6\ \mu\mathrm{P}}{20\times10^4\ \mu\mathrm{P}}=100$.