Algebra: Solving quadratic equations using the quadratic formula

Overview

This presentation will cover:

- quadratic equations
- solving quadratic equations using the quadratic formula
- finding the number of solutions the quadratic equation will have.

Quadratic equations

Quadratic equations have a general form of

$$ax^2 + bx + c = 0$$

where a, b and c are constant terms.

Quadratic equations are used in many disciplines and can be solved by a number of methods.

This presentation will focus on using the quadratic formula.

For a different method, please see the previous presentation on solving quadratic equations using factorisation.

Quadratic formula

If the quadratic equation of the form $ax^2 + bx + c = 0$ cannot be factorised (or if you cannot readily determine its factors), the solutions are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The $\pm$ sign means you have to do both the calculations twice (once with the $+$ and again for the $-$). Thus,

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The quadratic formula will obtain the solutions of any quadratic equation.

Example

Solve $3x^2 - 15x + 17 = 9$.

Solution: Note that the equation does not match the general form, therefore, we must rearrange the equation.

$$\begin{aligned} 3x^2 - 15x + 17 - 9 &= 0\,, \\ 3x^2 - 15x + 8 &= 0\,. \end{aligned}$$

Comparing this with the standard form $ax^2 + bx + c$ we note that $a = 3$, $b = -15$, and $c = 8$.

Solution (continued)

Substituting the values $a = 3$, $b = -15$, and $c = 8$ into the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-15) \pm \sqrt{(-15)^2 - 4 \times 3 \times 8}}{2 \times 3} \\ &= \frac{15 \pm \sqrt{225 - 96}}{6} \\ &= \frac{15 \pm \sqrt{129}}{6}\,. \end{aligned}$$

Solution (continued)

Now the we have simplified the quadratic equation $\left(x = \frac{15 \pm \sqrt{129}}{6}\right)$ we can do the both the $+$ and $-$ then evaluate each of them. Thus,

$$x = \frac{15 + \sqrt{129}}{6} \quad \text{or} \quad x = \frac{15 - \sqrt{129}}{6}\,,$$

$$x \approx 4.39 \quad \text{or} \quad x \approx 0.61\,.$$

(Check that each value of x satisfies the original equation.)

Exercise

Solve $3 = x + 4x^2$.

Solution

Firstly, you need to rearrange the equation so that is fits the general form of a quadratic:

$$4x^2 + x - 3 = 0$$

Comparing this with the standard form $ax^2 + bx + c$ we note that $a = 4$, $b = 1$, and $c = -3$.

(Note: this equation could be solved using factorisation. Have a go at factorising and solving this equation and check your solutions with using the quadratic formula.)

Solution (continued)

Substituting $a = 4$, $b = 1$, and $c = -3$ into the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-1 \pm \sqrt{1^2 - 4 \times 4 \times (-3)}}{2 \times 4} \\ &= \frac{-1 \pm \sqrt{49}}{8} \\ &= \frac{-1 \pm 7}{8}. \end{aligned}$$

Solution (continued)

Evaluating gives:

$$\begin{aligned} x &= \frac{6}{8} \quad \text{or} \quad \frac{-8}{8} \\ &= 0.75 \quad \text{or} -1. \end{aligned}$$

(Check that each value of x satisfies the original equation.)

The discriminant — types of solutions

We have seen that the solutions of any quadratic equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Quadratic equations do not always have two real solutions. The number of possible real solutions depends on the quantity under the square root sign, $b^2 - 4ac$.

The quantity, $b^2 - 4ac$, is called the **discriminant**.

Number of Solutions:

- *Two different solutions.* When $b^2 - 4ac$ is a positive number, there are two possible solutions.
- *No solution.* If $b^2 - 4ac$ is a negative number, there are no solutions because the square root of a negative number is not defined.
- *One solution.* If $b^2 - 4ac$ is zero, the formula gives us

$$x = \frac{-b \pm \sqrt{0}}{2a} = \frac{-b}{2a}.$$

Thus there is only one solution.

Exercise

Solve the following quadratic equations where possible.

1. $x^2 - 5x + 6 = 0$
2. $3x^2 - 2x + 4 = 0$
3. $2x^2 + 2x - 30 = 0$
4. $m^2 = 14m - 49$

Solution: Question 1

$x^2 - 5x + 6 = 0$ is already in general form, therefore $a = 1$, $b = -5$ and $c = 6$.

Firstly check the discriminant:

$$b^2 - 4ac = (-5)^2 - 4 \times 1 \times 6 = 1$$

therefore, there are two solutions.

Solution: Question 1 (continued)

Substituting into the quadratic formula gives:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 1 \times 6}}{2 \times 1} \\ &= \frac{5 \pm \sqrt{1}}{2} \\ &= \frac{5-1}{2} \quad \text{or} \quad \frac{5+1}{2} \\ &= 2 \quad \text{or} \quad 3\,. \end{aligned}$$

Solutions: Question 2

$3x^2 - 2x + 4 = 0$ is already in general form, therefore $a = 3$, $b = -2$ and $c = 4$.

Firstly check the discriminant:

$$b^2 - 4ac = (-2)^2 - 4 \times 3 \times 4 = -44$$

therefore, there are no real solutions.

Solution: Question 3

$2x^2 + 2x - 30 = 0$ is already in general form, therefore $a = 2$, $b = 2$ and $c = -30$.

Firstly check the discriminant:

$$b^2 - 4ac = 2^2 - 4 \times 2 \times -30 = 244$$

therefore, there are two solutions.

Solution: Question 3 (continued)

Substituting into the quadratic formula gives:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4 \times 2 \times (-30)}}{2 \times 2} \\ &= \frac{-2 \pm \sqrt{244}}{4} \\ &= \frac{-2 - \sqrt{244}}{4} \quad \text{or} \quad \frac{-2 + \sqrt{244}}{4} \\ &\approx -4.41 \quad \text{or} \quad 3.41\,. \end{aligned}$$

Solution: Question 4

$m^2 = 14m - 49$ requires rearranging to be in the general form.

Rearranging gives:

$$\begin{aligned} m^2 &= 14m - 49 \\ m^2 - 14m + 49 &= 0\,, \end{aligned}$$

therefore $a = 1$, $b = -14$ and $c = 49$.

Solution: Question 4 (continued)

Checking the discriminant gives:

$$b^2 - 4ac = (-14)^2 - 4 \times 1 \times 49 = 0$$

therefore, there is one solution.

Solution: Question 4 (continued)

$$\begin{aligned} m &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-14) \pm \sqrt{(-14)^2 - 4 \times 1 \times 49}}{2 \times 1} \\ &= \frac{14 \pm \sqrt{0}}{2} \\ &= \frac{14}{2} \\ &= 7\,. \end{aligned}$$

Conclusion

This presentation covered:

- the quadratic formula
- using the quadratic formula to solve quadratic equations
  - when the equation matches the general form
  - when the equation requires rearranging before it is in general form
- finding the number of solutions.