Module 4

TRIGONOMETRY

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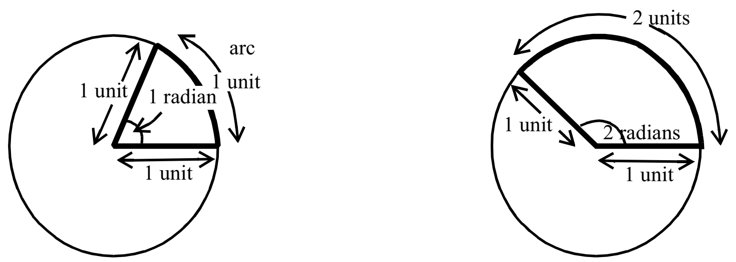
Introduction

In the Revision Module we briefly mentioned the six trigonometric functions sine, cosine, tangent, cosecant, secant and cotangent and we defined these functions for some angle θ in a right angled triangle. The trigonometric functions can also be defined in terms of the unit circle and this enables us to describe processes that fluctuate in some 'regular oscillatory' or periodic manner e.g. heartbeat, rise and fall of tides, vibration of a violin string.

4.1 Radian Measure

When we describe such processes mathematically the angles are measured in **radians**. You met radian measure in Level C so here is just a brief revision. A radian is defined as the angle subtended at the centre of a circle of radius 1 unit, by an arc (i.e. a section of the circumference) of length 1 unit.



Often we want to convert angles from degrees to radians. If we travel completely around the circle we have an arc length equal to the circumference of our circle, C.

i.e. $C = 2\pi r$

But $r = 1$

$\therefore$ $C = 2\pi$

Now the arc length is 2π and we know that for 1 full rotation the angle traversed is 360°

$$\therefore \quad 2\pi = 360°$$

$$\boxed{2\pi \text{ radians} = 360°}$$

$$\therefore \quad 1 \text{ radian} = \frac{360°}{2\pi} = \frac{180°}{\pi}$$

$\therefore$ 1 radian is 57.29°

Note that usually the word radian is not included in the measurement of an angle in radians. **So you always assume an angle is in radians unless the degree symbol is attached to the number.**

Exercise Set 4.1

Complete the following (Round answers to two decimal places if necessary).

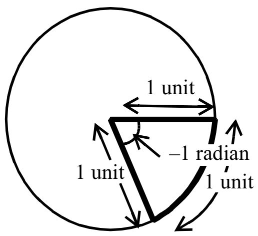
(a) $360° = \boxed{}$ radians (b) $\pi = \boxed{}°$ (c) $45° = \boxed{}$ radians

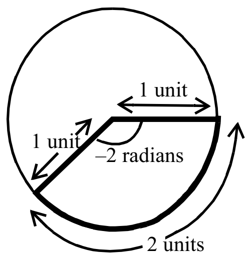
(d) $9.2 = \boxed{}°$ (e) $3\pi = \boxed{}°$ (f) $(3\pi)° = \boxed{}$ radians

Just as we do for an angle in degrees, an angle in radians is measured from the positive *x* axis, in the anticlockwise direction if the angle is positive and in the clockwise direction if the angle is negative.

We can imagine rotating around the circle more than once e.g. travelling through an angle of 720° means we have rotated twice around the circle

i.e. 4π radians $= 720°$
Similarly 8.4π radians $= 1\ 512°$





Check on your calculator for the key for the radian mode of operation. On the Casio fx- 82 super you need to press $\boxed{\text{MODE}}$ *then* $\boxed{5}$. *A small* **RAD** *message then appears in the upper section of the display. To get from radian mode to degree mode press* $\boxed{\text{MODE}}$ $\boxed{4}$.

Exercise Set 4.2

Use your calculator to find the following. (Round to three decimal places where necessary)

(a) $\sin\frac{\pi}{2}$ (b) $\sin 2.15$ (c) $\sin 20.4°$ (d) $\tan\frac{3\pi}{2}$ (e) $\tan\frac{1}{8}$

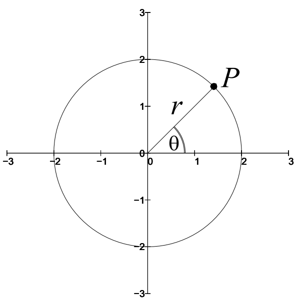
(f) $\cos -\frac{2\pi}{3}$ (g) $\sec -40°$ (h) $\text{cosec} -\frac{3\pi}{2}$ (i) $\cot\frac{8\pi}{3}$

4.2 Polar Co Ordinates

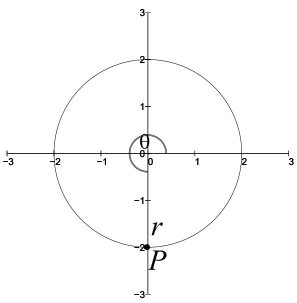
Using the unit circle we can extend our definition of sine, cosine, tangent etc. to incorporate angles bigger than 90°.

We are very familiar with specifying any point P in the XY plane by its x co-ordinate and y co-ordinate i.e. $P\,(x, y)$

We can also specify P as a point on a circle. To do this we need to know the radius of the circle and the angle made by a line from the centre of the circle (i.e. the ray) to point P and the x-axis in the +ve direction. i.e. we want r and θ .



Here $P(r, \theta)$ is $P(2, \frac{\pi}{4})$



Here $P(r, \theta)$ is $P(2, \frac{3\pi}{2})$

(alternatively $P(r, \theta)$ is $P(2, -\frac{\pi}{2})$

Note that the radius (or ray length) is **always** positive.

To convert from XY co-ordinates to polar co-ordinates and vice versa these relationships are used

Polar Co Ords $\rightarrow$ Cartesian Co Ords

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Cartesian Co Ords $\rightarrow$ Polar Co Ords

$$r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$

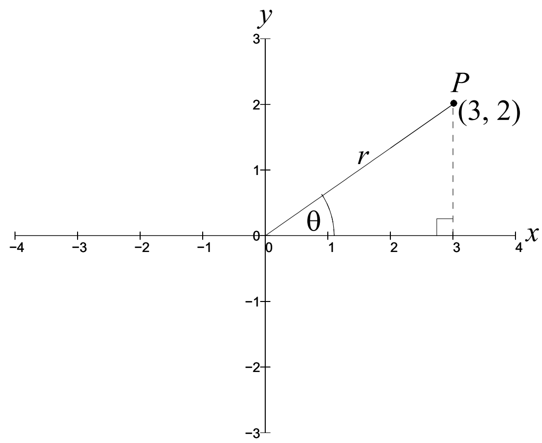
where r is the ray length from the origin to the point P and θ is the angle measured anticlockwise from the x axis to the ray.

Example 4.1:

Convert the Cartesian co-ordinates of $P(3, 2)$ into polar co-ordinates

Solution:

A diagram is always useful



We note that P is in the first quadrant so θ will be between 0 and 90°

i.e. $0 \le \theta < \frac{\pi}{2}$ (radians)

Length of $r = \sqrt{3^2 + 2^2} = \sqrt{13}$

$\tan\theta = \frac{2}{3} \quad \therefore \theta = \tan^{-1}\frac{2}{3} = 0.588$ radians or $33.69°$

$\therefore \; P(x, y) = P(3, 2)$ is $P(r, \theta) = P(\sqrt{13}, 0.588 \text{ rad})$ or $P(\sqrt{13}, 33.69°)$

Example 4.2:

Convert the polar co-ordinates $Q(8, 3.665 \text{ rad})$ into Cartesian co-ordinates.

Solution:

We note that 3.665 rad is an angle in the third quadrant so we expect both x and y to be negative

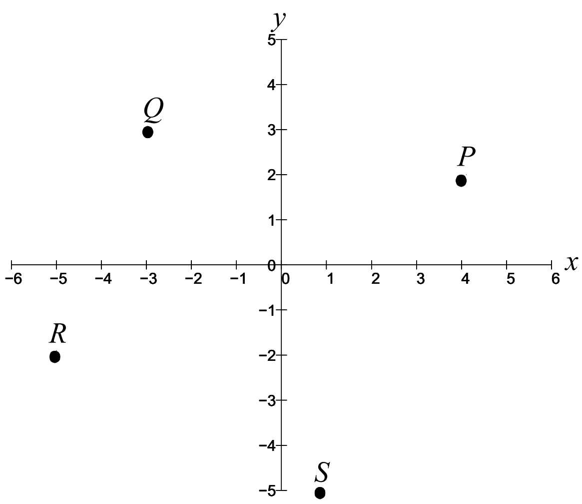
$$x = r\cos\theta \qquad \therefore\ x = 8 \times \cos 3.665 = -6.929$$
$$y = r\sin\theta \qquad \therefore\ y = 8 \times \sin 3.665 = -3.999$$

$\therefore \quad Q(r, \theta) = Q(8, 3.665 \text{ rad})$ is $Q(x, y) = Q(-6.929, -3.999)$

Exercise Set 4.3

This is an important set of exercises because in it you will revise some of the applications of radian measure.

1. Convert (i) P (ii) Q (iii) R (iv) S into polar co-ordinates. (Express θ in degrees and radians) Round answers to 2 decimal places if necessary.



2. Convert P, Q, R, S into Cartesian co-ordinates. Round answers to two decimal places if necessary.

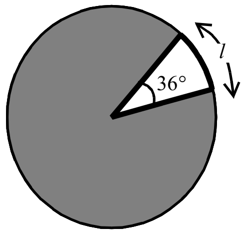
   (i) $P(r, \theta) = P(2, -\pi)$

   (ii) $Q(r, \theta) = Q(4.1, 233°)$

   (iii) $R(r, \theta) = R(5, \frac{7\pi}{2})$

   (iv) $S(r, \theta) = S(10, -\frac{3\pi}{8})$

3. (a) Find the length of the arc, l shown below, if the circumference of the circle is 45π cm.



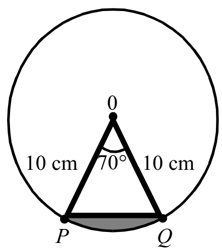
(b) Find the area of the sector of the circle shaded above.

4. (a) Find the angle θ, subtended by of a sector of a circle if the area of the sector is 6π m^2 and the area of the circle is 9π m^2. Give θ in degrees and radians.

(b) Find the arc length of the sector in (a) above.

5. A chord PQ of a circle (i.e. the line joining two points on the circumference of the circle) of radius 10 cm subtends an angle of 70° at the centre. Find

(i) Length of the chord PQ
(ii) Length of the arc PQ
(iii) The area of the sector subtended by 70°
(iv) The area shaded below.



4.3 Trigonometric Identities and Multiple Angle Formulae

From your study of Level C or similar work you will be familiar with the basic trigonometric identities and relationships for angles in degrees. These identities and relationships are given below as a reminder but note that here the angles are given in radians.

Identities:

$\sin^2\theta + \cos^2\theta = 1$
$1 + \tan^2\theta = \sec^2\theta$
$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Rotations:

$\sin(2n\pi + \theta) = \sin\theta$
$\cos(2n\pi + \theta) = \cos\theta$
$\tan(2n\pi + \theta) = \tan\theta$

2nd quadrant angles:

$\sin(n\pi - \theta) = \sin\theta$
$\cos(n\pi - \theta) = -\cos\theta$
$\tan(n\pi - \theta) = -\tan\theta$

3rd quadrant angles:

$\sin(n\pi + \theta) = -\sin\theta$
$\cos(n\pi + \theta) = -\cos\theta$
$\tan(n\pi + \theta) = \tan\theta$

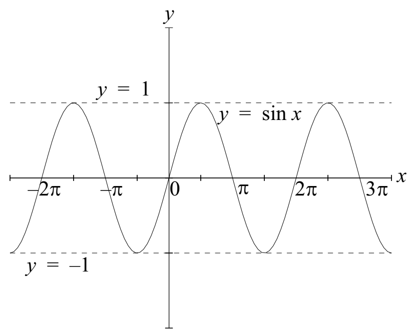
4th quadrant angles:

$\sin(2n\pi - \theta) = -\sin\theta$
$\cos(2n\pi - \theta) = \cos\theta$
$\tan(2n\pi - \theta) = -\tan\theta$

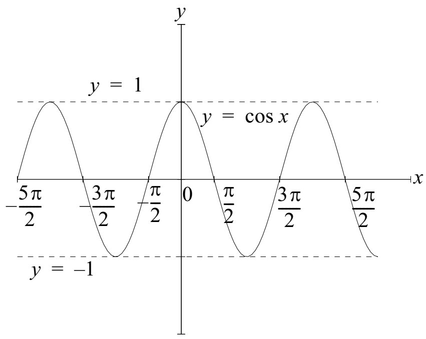
If you feel unsure of these identities choose some values of n *and* θ *and use your calculator to demonstrate the identities and relationships.*

Inverse Trigonometric Functions

The graphs of $y = \sin x$ and $y = \cos x$ given below show that these trigonometric functions are not one-to-one functions. So they will not have an inverse unless the domains are restricted. (The same applies for $\tan x$, $\operatorname{cosec} x$, $\sec x$ and $\cot x$.)



See Note 1



Examine the graphs of $y = \sin x$ and $y = \cos x$ and complete the tables below which show that these functions are not one-to-one.

x	$-\frac{5\pi}{2}$	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$
$y = \sin x$											

x	$-\frac{5\pi}{2}$	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$
$y = \cos x$											

Notes

1. You must be able to sketch the graphs of $\sin x$, $\cos x$ and $\tan x$, without the need to draw up a table of values etc.

✪ Look at the tables and decide:

- for what interval of x is the graph of $y = \cos x$ not repeated?
- for what interval of x is the graph of $y = \sin x$ not repeated?

Answer:

The answer to both of these is 2π. (Recall that this is called the period of the function.)

So if we restrict the domain of x so that the interval covered by x is less than 2π we can find the inverse of $\sin x$ and the inverse of $\cos x$.

The inverse of $\sin x$ is written as $\sin^{-1}x$.

$$\left[\text{Note that } \sin^{-1}x \text{ does not mean } \frac{\mathbf{1}}{\mathbf{sin}\ \boldsymbol{x}}\right]$$

$\sin^{-1}x$ is read as the angle (or number) whose sine is x (or arcsine x)

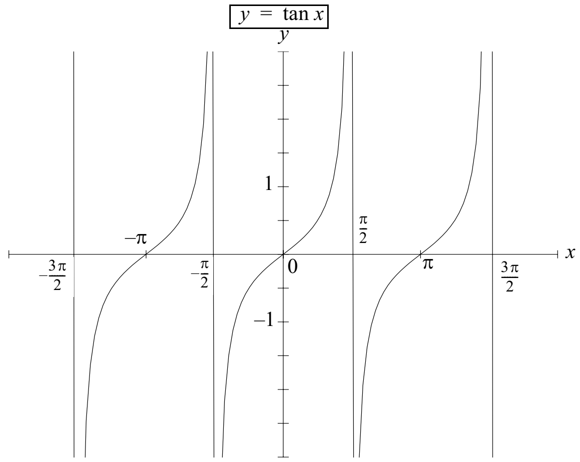
Similarly $\cos^{-1}x$ is read as the angle (or number) whose cosine is x (or arccos x)

The range specified for $\sin^{-1}x$ and $\cos^{-1}x$ must be less than 2π

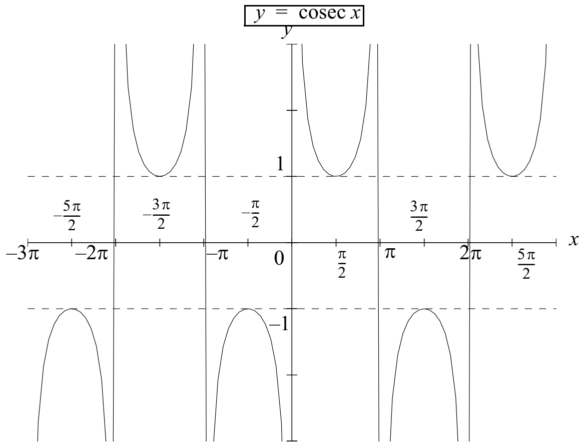
e.g. if $y = \sin^{-1}x$ then $x = \sin y$ for the range $0 \le y < 2\pi$ or $-\frac{\pi}{2} < y \le \frac{3\pi}{2}$

or $2\pi \le y < 4\pi$

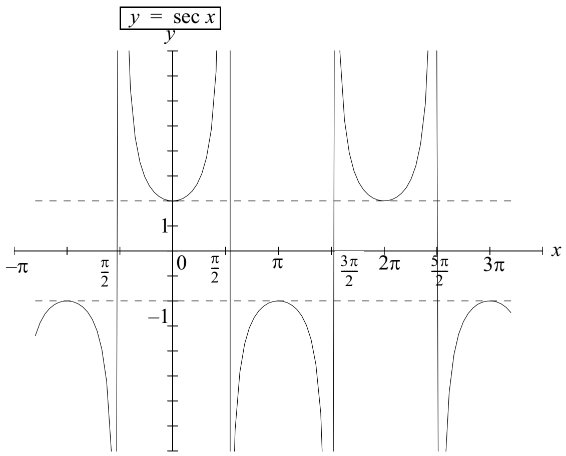
✪ Examine the graphs of $\tan x$, $\operatorname{cosec} x$, $\sec x$ and $\cot x$ given below. (If you are unfamiliar with these graphs make sure you sketch them by hand also.) Note that these functions are undefined for certain values of x hence the asymptotes on the graphs. Write down the period of each function and give (i) a suitable interval centred on the origin and (ii) another suitable interval, if you want to define the inverse functions.



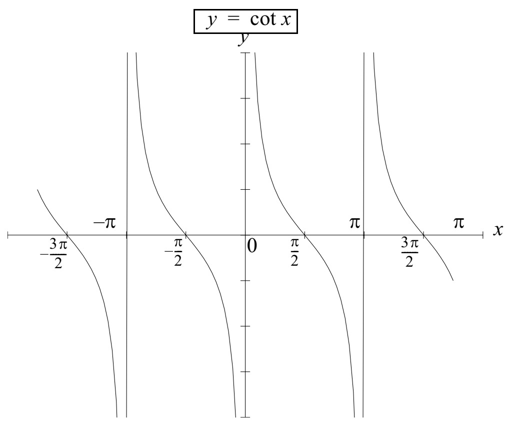
Period of $\tan x =$
If $y = \tan x$
then $x = \tan^{-1} y$ for (i) $\le x <$
or (ii) $\le x <$
provided $\cos x \neq 0$



Period of $\operatorname{cosec} x =$
If $y = \operatorname{cosec} x$
then $x = \operatorname{cosec}^{-1} y$ for (i) $\le x <$
or (ii) $\le x <$
provided $\cos x \neq 0$



Period of $\sec x =$
If $y = \sec x$
then $x = \sec^{-1} y$ for (i) $\le x <$
or (ii) $\le x <$
provided $\sin x \neq 0$



Period of $\cot x =$
If $y = \cot x$
then $x = \cot^{-1} y$ for (i) $\le x <$
or (ii) $\le x <$
provided $\sin x \neq 0$

Answer:

Period of $\tan x = \pi$
For the inverse to exist

(i) $-\frac{\pi}{2} \le x < \frac{\pi}{2}$

(ii) e.g. $\frac{\pi}{2} \le x < \frac{3\pi}{2}$

Period of $\operatorname{cosec} x = 2\pi$
For the inverse to exist
(i) $-\pi \le x < \pi$
(ii) e.g. $\pi \le x < 3\pi$

Period of $\sec x = 2\pi$
For the inverse to exist
(i) $-\pi \le x < \pi$
(ii) e.g. $-\frac{\pi}{2} \le x < \frac{3\pi}{2}$

Period of $\cot x = \pi$
For the inverse to exist

(i) $-\frac{\pi}{2} \le x < \frac{\pi}{2}$

(ii) e.g. $-\frac{3\pi}{2} \le x < -\frac{\pi}{2}$

4.4 Solving Trigonometric Equation

Trigonometric equations are solved using the algebraic techniques you are already familiar with e.g. the quadratic formula, completing the square etc. Because of the periodicity of the trigonometric functions, and the relationships and identities often there can be many solutions. The particular solutions you require are selected based on either the physical interpretation or the domain stipulated.

Do not always assume that the solution to a trigonometric equation is an angle between 0 and $\frac{\pi}{2}$ radians (i.e. in the first quadrant).

Example 4.3:

Find all angles between 0° and 360° which satisfy the equation

$2\cos^2\theta - \sin\theta - 1 = 0$

Solution:

$2\cos^2\theta - \sin\theta - 1 = 0$
$2(1 - \sin^2\theta) - \sin\theta - 1 = 0$ {Using Identity $\sin^2\theta + \cos^2\theta = 1$}
$2 - 2\sin^2\theta - \sin\theta - 1 = 0$
$-2\sin^2\theta - \sin\theta + 1 = 0$
$\therefore \quad 2\sin^2\theta + \sin\theta - 1 = 0$

This is a quadratic which can be solved for $\sin\theta$. **See Note** 1

$$\sin\theta = \frac{-1 \pm \sqrt{1^2 - 4\times 2\times(-1)}}{2\times 2}$$

$$= \frac{-1 \pm \sqrt{9}}{4}$$

$$\therefore \sin\theta = \frac{1}{2} \quad \text{or} \quad \sin\theta = -1$$

If $\sin\theta = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\frac{1}{2}$

$\therefore\ \theta = 30° \text{ or } (180° - 30°)$

because $\sin(\pi - \theta) = \sin\theta$ {or in degrees $\sin(180° - \theta°) = \sin\theta°$} **See Note** 2

$\therefore\ \theta = 30° \text{ or } 150°$

If $\sin\theta = -1 \Rightarrow \theta = \sin^{-1}(-1)$

$\therefore\ \theta = 270°$ **See Note** 3

$\therefore$ There are three solutions to the equation.

$\theta = 30°,\ 150° \text{ or } 270°$

Checking:

When $\theta = 30°$

$$\begin{aligned} \text{LHS} = 2\cos^2\theta - \sin\theta - 1 &= 2\cos^2 30° - \sin 30° - 1 \\ &= 2\times 0.866^2 - 0.5 - 1 \\ &= 2\times 0.75 - 0.5 - 1 = 0 = \text{RHS} \ ✓ \end{aligned}$$

When $\theta = 150°$

$$\begin{aligned} \text{LHS} = 2\cos^2\theta - \sin\theta - 1 &= 2\cos^2 150° - \sin 150° - 1 \\ &= 2\times(-0.866)^2 - 0.5 - 1 \\ &= 2\times 0.75 - 0.5 - 1 = 0 = \text{RHS} \ ✓ \end{aligned}$$

When $\theta = 270°$

$$\begin{aligned} \text{LHS} = 2\cos^2\theta - \sin\theta - 1 &= 2\cos^2 270° - \sin 270° - 1 \\ &= 2\times 0 - (-1) - 1 = 0 = \text{RHS} \ ✓ \end{aligned}$$

Note that there is no need to consider multiple rotations of θ as we are given that the solution(s) lies between $0°$ and $360°$

Notes

1. If you cannot see this, substitute x for $\sin\theta$ to get $2x^2 + x - 1 = 0$ and solve.
2. Remember All Stations To Central. You should expect another angle in the second quadrant to have the same sine as 30°.
3. θ is the third quadrant. You would not expect to get another angle between 0° and 360° with the same sine as 270°.

Example 4.4:

The average daily income, R of a tourist shop for any month of the year (in hundreds of dollars) is given by the function

$$R(t) = 18 + 12 \sin \frac{\pi}{6} t \quad \text{where } t \text{ is the number of months since 1st January}$$

(i) Find the average daily income expected in the month commencing 1st July
(ii) In what month(s) of the year would you expect the daily revenue to be \$2 400?

Solution:

(i) 1st July is 6 months after 1st January $\therefore$ we want to find $R(6)$

$$R(6) = 18 + 12 \sin \frac{\pi}{6} \times 6$$ **See Note** 1

$$= 18 + 12 \times 0$$
$$= 18$$

$\therefore$ In July the average daily income should be \$1 800.

(ii) We are given $R = 24$ $\therefore$ we want to solve for t

$$24 = 18 + 12 \sin \frac{\pi}{6} t$$

$$\therefore \quad 6 = 12 \sin \frac{\pi}{6} t$$

$$\therefore \quad \frac{1}{2} = \sin \frac{\pi}{6} t$$

$$\therefore \quad \frac{\pi}{6} t = \sin^{-1} 0.5$$
$$= 0.5236 \quad \text{or} \quad (\pi - 0.5236)$$

$$\therefore \text{If } \frac{\pi}{6} t = 0.5236 \qquad \therefore \text{If } \frac{\pi}{6} t = (\pi - 0.5236)$$

$$t = 0.5236 \times \frac{6}{\pi} \qquad t = (\pi - 0.5236) \times \frac{6}{\pi}$$
$$= 1 \qquad = 5$$

$\therefore$ The average daily income of \$2 400 occurs in the month, 1 month after the 1st January and in the month 5 months after the 1st January i.e. in February and in June each year.

Notes

1. You must use radians in this problem. (Degrees have no meaning here)

Exercise Set 4.4

1. Simplify (a) $\cos A \quad \tan A \quad \sec A$

   (b) $\dfrac{\sec A \quad \tan A}{\operatorname{cosec} A}$

   (c) $\dfrac{1}{\cos^2\theta} - \tan^2\theta$

   (d) $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$

2. Find all solutions between 0 and 2π which satisfy

   (a) $\sec^2\theta = 3 + \tan\theta$

   (b) $\cos^2\theta + 3\sin^2\theta = 2$

   (c) $2\sin^2\theta - 9\cos\theta + 3 = 0$

   Prove your solutions are correct.

3. The number of rabbits N, (in thousands) in a certain area, is given by the formula $N(t) = 10 + \sin\dfrac{\pi}{24}t$ where t is the number of months after 1 January 1994.

   (i) How many rabbits would you expect to be present in July 1995?

   (ii) What is the minimum possible rabbit population? What is the maximum possible?

   (iii) Draw the graph of N against t for four years from 1st January 1994 (assume 30 days in each month if necessary).

   (iv) Solve the trigonometric equation to determine when the population is 10 500 and 9 500 during the period 1st January 1994 – 30 June 1996.

   (v) Verify your solution(s) in (iv) by comparing the results with the graph drawn in (iii)

4. The number of saplings N, (in thousands) in a forest, t months after a given date is given by $N(t) = 5 + 5\cos\dfrac{\pi}{18}t$

   (i) Draw the graph of N against t, for $0 < t < 72$

   (ii) Find when the number of saplings is first 8 000 after the given date (use the inverse cosine function).

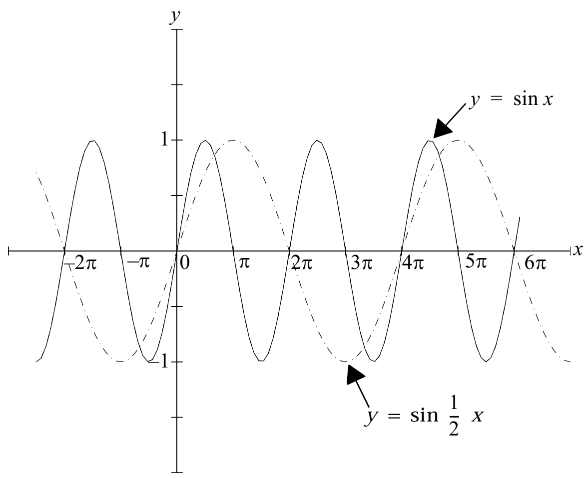
   (iii) Check your answer to (ii) using your graph and by substituting in the original equation.

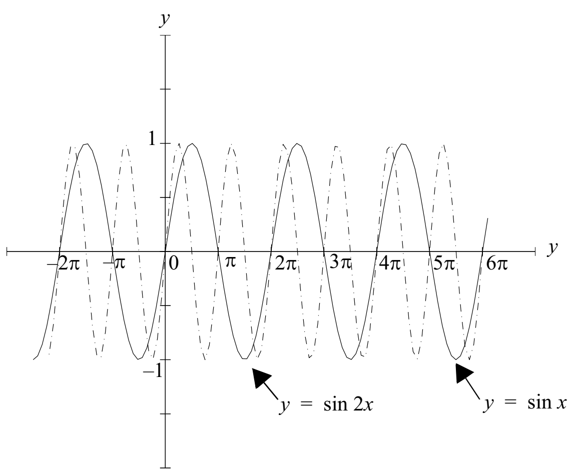
4.5 Periodicity

In Level C you saw the difference in periodicity of trigonometric functions such as $\sin x$, $\sin \frac{1}{2}x$, $\sin 2x$, and $\sin kx$.

If k, the coefficient of x, is less than one, the sine curve is spread out along the x axis. It has a period of $\frac{2\pi}{k}$. e.g. $\sin \frac{1}{2}x$ has period $\frac{2\pi}{1/2} = 4\pi$.

If k is greater than one, the sine curve is compressed along the x axis. It still has period $\frac{2\pi}{k}$. e.g. $\sin 2x$ has period $\frac{2\pi}{2} = \pi$.





4.6 Amplitude

Another feature of trigonometric functions that is important in physics, engineering, or ecology is the amplitude.

✪ Look at the graphs of $\sin x$, $\sin \frac{1}{2}x$ and $\sin 2x$. Note that the maximum value of each of these functions is 1 and the minimum value of each is –1. It is this value which gives the measurement of the feature known as amplitude.

Consider $y = k \sin x$ where k is a constant. This constant is the amplitude.

e.g. for $y = \sin x$ we can write $y = k \sin x = 1. \sin x$

$y = \sin \frac{1}{2}x$ we can write $y = k \sin x = 1. \sin \frac{1}{2}x$

$y = \sin 2x$ we can write $y = k \sin 2x = 1. \sin 2x$

The amplitude is the magnitude of the vertical displacement from the base position. The base position is the x axis i.e. at $y = 0$.

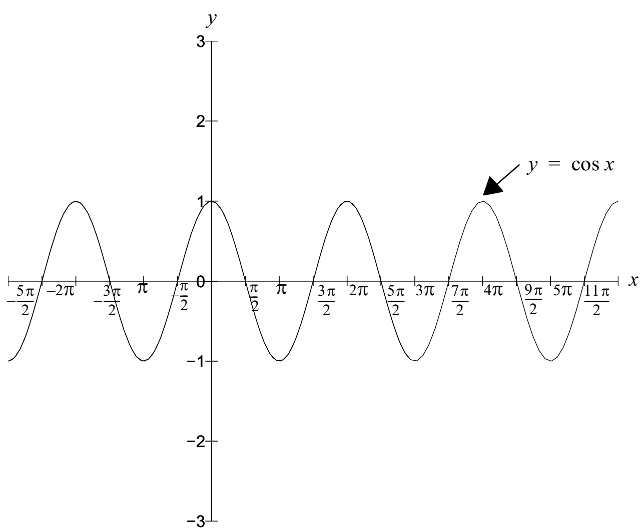
✪ What do you think is the amplitude of the function $y = 3 \cos x$? . . . .

Answer: Its amplitude is 3

✪ Sketch the curve $y = 3 \cos x$ on the axes below where I have already drawn $y = \cos x$. State the base position, the maximum and minimum values and the period of $y = 3 \cos x$?

Base position = ... ; Maximum = ... ; Minimum = ...

Period = ...



Answer:

Base position is the x axis i.e. $y = 0$; Maximum $= 3$; Minimum $= -3$; Period $= 2\pi$

✪ Note that a function such as $y = 20 + 3 \sin x$ still has an amplitude of 3 because the amplitude is measured from the base position whether this be from the x axis or some other line parallel to it. In this case the base position is given by $y = 20$.

✪ What is the effect of k in a function such as:

(i) $k \sin x$ or $k \cos x$

(ii) $\sin kx$ or $\cos kx$

(i) . . . .

. . . .

(ii) . . . .

. . . .

Answer:

(i) A constant k, in a function such as $k \cos x$, affects the amplitude of the function in that the amplitude $= 1 \times k$ e.g. $y = 5 \cos x$ has amplitude of 5.

(ii) A constant k in a function such as $\cos kx$ affects the periodicity of the function in that the period $= \dfrac{2\pi}{k}$ e.g. $y = \cos 5x$ has period $\dfrac{2\pi}{5}$

Exercise Set 4.5

1. Give (i) the period and (ii) the amplitude of the functions

   (a) $N = 10 + \sin \dfrac{\pi}{24} t$

   (b) $N = 5 + 5 \cos \dfrac{\pi}{18} t$

   (You met these functions in Exercise Set 4.4)

2. If the frequency of a sine or cosine function is the reciprocal of the period, find the frequency of (a) and (b) above. [The frequency is the number of waves in an interval of unit length along the time axis t]

3. Consider the function $y = -2.5 \cos 2x$

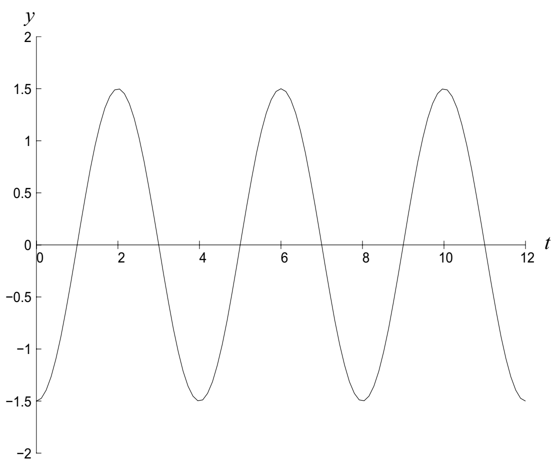
   (i) Sketch the graph for $0 < x < 8$

   (ii) What is the period, amplitude and frequency?

   (iii) If the general cosine function is $y = A \cos kx$ and A is negative what does this tell you about the graph of $y = A \cos kx$?

4. (a) Give the equation of the sine function $y(x)$ with amplitude of 3; 'base' position of $y = 4$, and frequency of $\dfrac{1}{2}$

   (b) Give the equation of this cosine function.

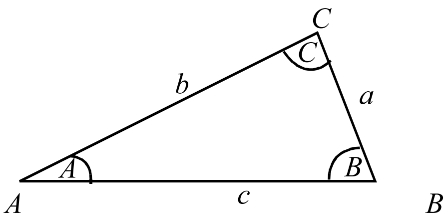


4.7 Triangle Solution

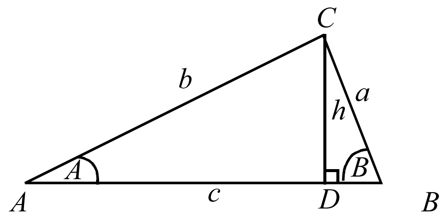
Some of the most common applications of trigonometry occur in surveying where heights, angles, lengths etc. must be measured or calculated from known measurements. You've already solved problems where right angled triangles were involved and an unknown angle or sidelength had to be determined. But obviously, all real world problems will not be able to be modelled with right angled triangles. In this section of work we will determine how to find unknown angles or sidelengths in any triangle.

The Sine Rule

Consider any triangle ABC. (By convention we label the length of the side **opposite** each angle with the corresponding small letter)



Drop a perpendicular from C to D on AB and let the length of this line be h



Consider triangle ACD

Complete the following

$$\sin A = \frac{\boxed{}}{b}$$

$$\therefore \; h = b\,\boxed{}$$

Consider triangle BCD

$$\sin B = \frac{\boxed{}}{\boxed{}}$$

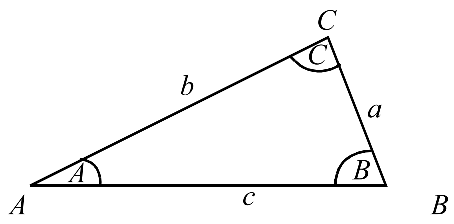
$$\therefore \; h = a \sin B$$

The two expression for h must be equal

$$\therefore \; b \sin A = a \sin B$$

$$\therefore \; \frac{a}{\sin A} = \frac{b}{\sin B}$$

Here is triangle ABC again



✪ Draw a perpendicular from A onto BC and using the same reasoning as above show that
$\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

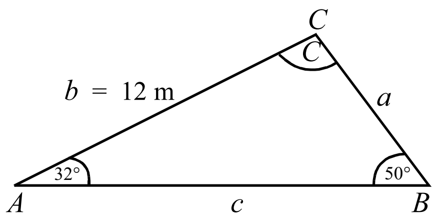
So for any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Thus if we know any two sidelengths and one of the opposite angles or any two angles and one of the opposite sidelengths we can find all the angles and sidelengths of a triangle.

Example 4.5:

Find the angle C and sidelengths a and c of triangle ABC below.



Solution:

$A = 32°$; $B = 50°$; $b = 12\text{m}$ and we need to find C, a and c.

There are 180° in a triangle $\therefore$ $C = 180° - (32° + 50°) = 98°$

We know three angles and one sidelength $\therefore$ the sine rule can be used to find a and b when any two angles and an opposite sidelength are known, we can use the sine rule to find the other opposite sidelength.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\therefore \quad \frac{a}{\sin 32°} = \frac{12}{\sin 50°}$$

$$\therefore \quad a = \frac{12 \times 0.5299}{0.7660}$$

$$= 8.301 \text{ m}$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \quad \frac{12}{\sin 50°} = \frac{c}{\sin 98°}$$

$$\therefore \quad C = \frac{12 \times 0.9903}{0.7660}$$

$$= 15.512 \text{ m}$$

$\therefore$ a = 8.301 m; c = 15.512 m and angle C = 98°

Don't forget the correct units of measurement.

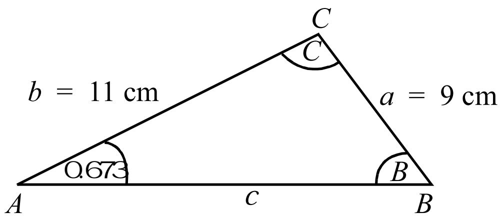
Check that the solution makes sense. We would expect the largest angle to have the longest opposite sidelength and the smallest angle to have the shortest opposite sidelength. Is this the case here? Yes! So the solution looks sensible.

Example 4.6:

Find all the unknown angles of triangle ABC in which the length of AC is 11 cm, the length of BC is 9 cm and angle A is 0.673 radians.

Solution:

It's always a good idea to draw a rough diagram. (Note that here we don't really know the shape of the triangle.)



$A = 0.673$ radians; $a = 9$ cm; $b = 11$ cm and we need to find B and C.

We know two sidelengths and one opposite angle $\therefore$ the sine rule can be used.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{9}{\sin 0.673} = \frac{11}{\sin B}$$ **See Note** 1

$$\therefore \quad \sin B = \frac{11 \times 0.6233}{9}$$

$$= 0.7619$$

$$\therefore \quad B = \sin^{-1} 0.7619$$ **See Note** 2

$$\therefore \quad B = 0.866 \text{ radians} \quad \text{or} \quad B = (\pi - 0.866) \text{ radians}$$
$$= 2.276 \text{ radians}$$

So there are two possible triangles.

If $A = 0.673$ radians and $B = 0.866$ radians **See Note** 3
then $C = \pi - (0.673 + 0.866)$ radians
$= 1.603$ radians

If $A = 0.673$ radians and $B = 2.276$ radians
then $C = \pi - (0.673 + 2.276)$ radians
$= 0.193$ radians

I suggest you use your protractor to draw both these triangles. You will need to first convert the angles into degrees by using your calculator.

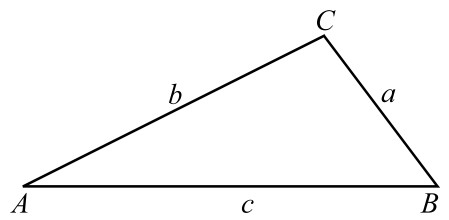
Notes

1. B will be in radians. Make sure calculator is in the radians mode before finding $\sin 0.673$
2. Remember sine is positive in the 1st and 2nd quadrant.
3. There are π radians (i.e. 180°) in a triangle.

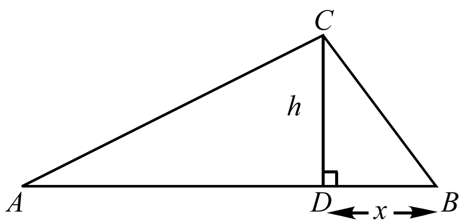
The Cosine Rule

If we only know two sidelengths and the included angle or only three sidelengths of a triangle we cannot use the sine rule to find the other unknown angles and sidelengths. However we can develop another rule, called the cosine rule for such situations. You can develop this rule yourself by completing the following.

Consider any triangle ABC



Drop a perpendicular from C to D on AB and let the length of this line be h and the length of BD be x



If AB has length c and BD has length x.
What is the length of AD? . . . .

Answer:

Label all sides of the triangles ABC, ADC, and BCD.

Length of $AD = c - x$

Consider the triangle ACD.

Use Pythagoras' Rule to write an expression for the length of the hypotenuse squared.

$$\begin{aligned} b^2 &= (\qquad)^2 + h^2 \\ &= c^2 - 2cx + x^2 + h^2 \end{aligned}$$

Now from triangle BDC, $x^2 + h^2 = \square$
$\therefore \quad b^2 = c^2 - 2cx + a^2$

Also in triangle BDC, $\cos B = \dfrac{\square}{\square}$

$\therefore \quad x = a \cos B$
$\therefore \quad b^2 = c^2 - 2c(a \cos B) + a^2$

i.e.

$$\boxed{b^2 = a^2 + c^2 - 2ac \cos B}$$

Can you see now that by knowing any two sidelengths e.g. a and c and the included angle, (in this case B) you can find the other sidelength and the other angles?

Also if 3 sidelengths are known, we can use the cosine rule to find any unknown angle in the triangle.

Note: lengths are always positive so when taking the square root of b^2 to get b only the positive root is used.

Complete the other cosine formulae.

$$a^2 = b^2 + c^2 - 2\,\boxed{}\,\boxed{}\cos\boxed{}$$

$$c^2 = \boxed{} + \boxed{} - 2\,\boxed{}\,\boxed{}\cos C$$

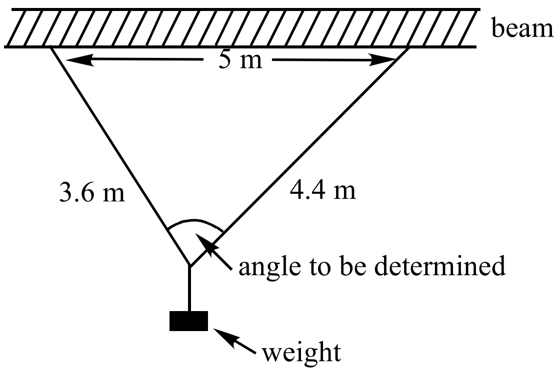
Of course you only need to remember ONE cosine rule.

Example 4.7:

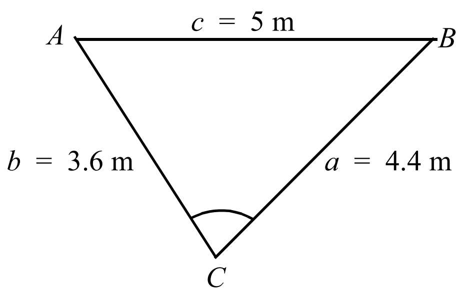
A weight hangs from the junction of two wires 4.4 m and 3.6 m long. The other ends of the wires are attached to a horizontal beam 5 m apart. Find the angle between the wires.

Solution:

The first step is try to draw a diagram of the situation



and then model it, in this case with a triangle.



Now we know three sidelengths, so we can use the cosine rule to find angle ACB (in radians or degrees).

The formula we want is

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\begin{aligned} 5^2 &= 4.4^2 + 3.6^2 - 2 \times 4.4 \times 3.6 \times \cos C \\ 31.68\cos C &= 19.36 + 12.96 - 25 \\ &= 7.32 \end{aligned}$$

$$\therefore \quad \cos C = \frac{7.32}{31.68} = 0.2311$$

$$\therefore \quad C = \cos^{-1} 0.2311$$

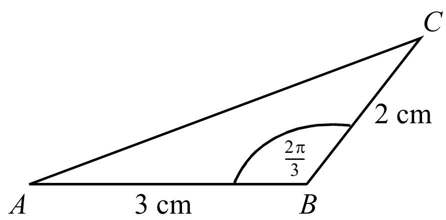
Now we can choose radians or degrees for angle C. Choosing degrees, I need first ensure that the calculator is in the degree mode.

$$\begin{aligned} \therefore \quad \text{angle } ACB &= \cos^{-1} 0.2311 \\ &= 76.64° \end{aligned}$$

The angle between the two wires is 76.64° (in radians, angle ACB is 1.3376)

Example 4.8:

Find all the unknown angles and sidelength of the triangle below.



Solution:

$a = 2$ cm; $c = 3$ cm; $ABC = \dfrac{2\pi}{3}$ **(We note the angle is in radians)**
$b = ?$; $CAB = ?$; $BCA = ?$

We have two sidelengths, a and c and the included angle ABC
$\therefore$ We can find b using the cosine rule.

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos B \\ b^2 &= 2^2 + 3^2 - 2 \times 2 \times 3\cos\frac{2\pi}{3} \\ &= 4 + 9 - 12 \times -0.5 \qquad \{\text{Check that calculator is in the radians mode}\} \\ &= 19 \end{aligned}$$

$\therefore \quad b = \sqrt{19}$ (Positive root only as b is the length of a side of a triangle)

Now b is known, we can use the sine rule or the cosine rule to find another angle.

Using the sine rule.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\therefore \quad \frac{2}{\sin A} = \frac{\sqrt{19}}{\sin \frac{2\pi}{3}}$$

$$\therefore \quad \sin A = \frac{2 \times \sin \frac{2\pi}{3}}{\sqrt{19}}$$

See Note 1

$$= \frac{2 \times \frac{\sqrt{3}}{2}}{\sqrt{19}}$$

$$= 0.3974$$

$$\therefore \quad A = \sin^{-1} 0.3974$$
$$= 0.4086$$

If $A = 0.4086$ and $B = \frac{2\pi}{3}$,

$$C = \pi - (0.4086 + \frac{2\pi}{3})$$
$$= 0.6386$$

$\therefore \; b = \sqrt{19}$ cm ; angle $BAC = 0.4086$ radians and angle $ACB = 0.6386$ radians

✪ Why didn't we have to worry about there being two solutions to $A = \sin^{-1} 0.3974$, one in the first quadrant (i.e. lying between 0 and $\frac{\pi}{2}$) and the other in the second quadrant (i.e. lying between $\frac{\pi}{2}$ and π)?

. . . .

. . . .

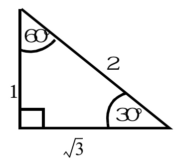
. . . .

Answer:

Because we already know one angle in the triangle is $\frac{2\pi}{3}$ (i.e. greater than $\frac{\pi}{2}$) so the other two angles **together** must be less than $\frac{\pi}{2}$ (as there are only π radians in a triangle). So there is no way that A could be a second quadrant angle.

Notes

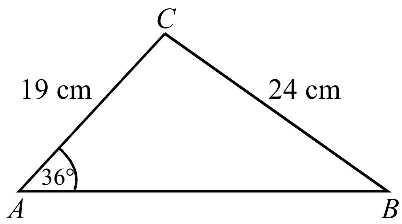
1. $\frac{2\pi}{3}$ (or $120°$) is a common angle whose trig ratios you can get from this triangle and the relationships between angles in various quadrants. (See page 4.7)



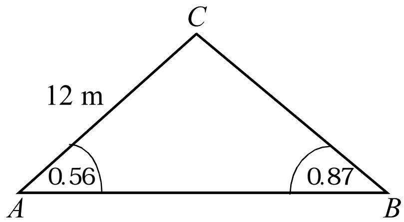
Exercise Set 4.6

1. In triangle ABC below, find all unknown angles and sidelengths (i.e. solve the triangle)

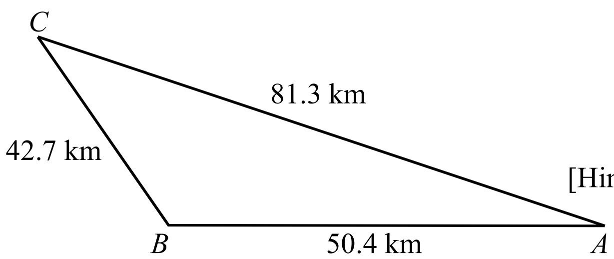
(a)



(b)



(c)



[Hint: Find the largest angle first]

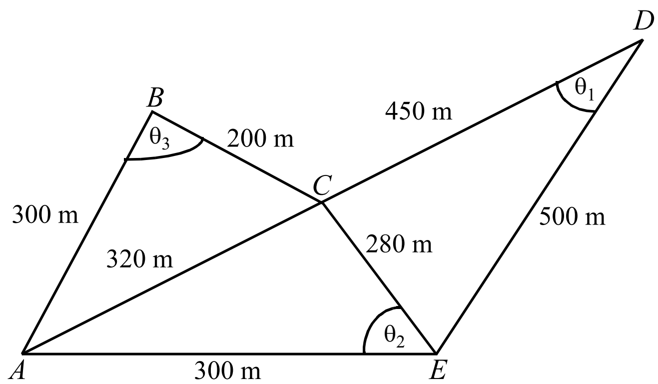
2. Solve the triangle ABC in which $a = 9$ cm, $b = 11$ cm and $A = 50°$

3. The area of **any** triangle can be found if two sidelengths and the included angle are known e.g. if b, c and A are known.

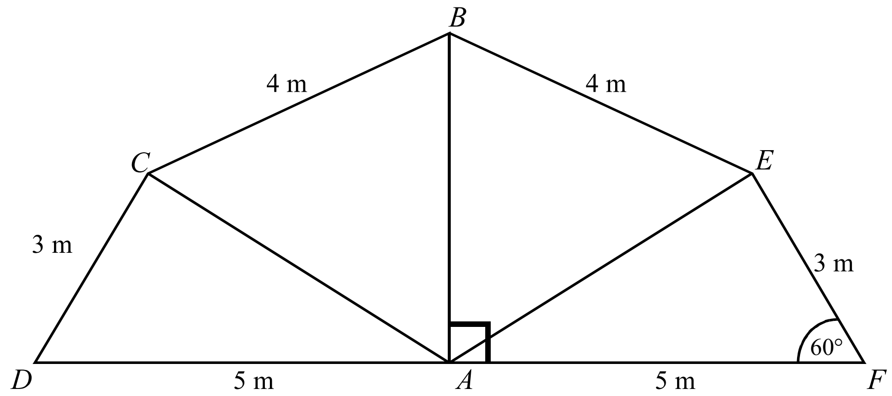
   Area $= \frac{1}{2}(bc \sin A)$ *Make sure you write this rule in your rule book.*

   Find the area of each triangle in Question 1 above. (Round answers to nearest whole number)

4. Find the area of this mining lease. Note that θ_1, θ_2 and θ_3 are all acute angles.



5. Find the length of AC and AB below. Note that the diagram is symmetrical about AB.



4.8 Compound Angles

Earlier we met the transformations for converting polar coordinates to Cartesian coordinates

i.e. $x = r\cos\theta$
$y = r\sin\theta$

where r is the ray length from the origin to the point $P(x, y)$ or $P(r, \theta)$ and θ is the angle measured anticlockwise from the x axis to the ray.

We can use these formulae and the cosine rule to derive expressions for the trigonometric ratios of sums and differences of angles. We'll do this shortly but first let's get an idea of how trigonometric ratios of compound angles (i.e. angles that are the sum or difference of two other angles) can be expressed.

✪ Use your calculator to complete the following (Round to 4 decimal places where necessary).

$\cos 173° = \boxed{}$; $\cos 28° = \boxed{}$; $\sin 173° = \boxed{}$; $\sin 28° = \boxed{}$

$\cos 173° \times \cos 28° = \boxed{}$

$\sin 173° \times \sin 28° = \boxed{}$

$\cos 173° \times \cos 28° + \sin 173° \times \sin 28° = \boxed{}$

Now $\cos(173° - 28°) = \cos 145°$
$= \boxed{}$

Answer:

You should have found that

$\cos(173° - 28°) = \cos 173° \times \cos 28° + \sin 173° \times \sin 28°$ (with a small rounding error)

Exercise Set 4.7

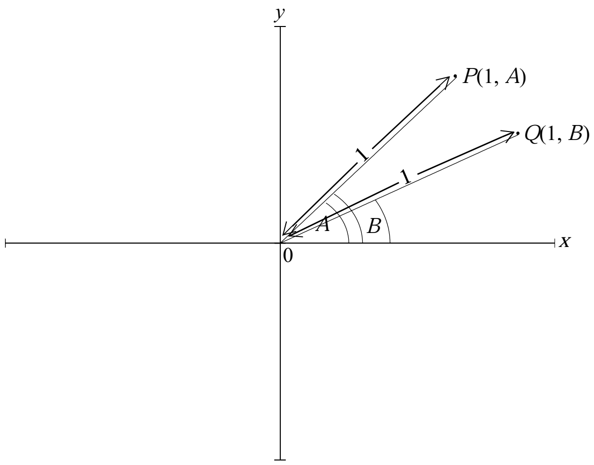
Use the angles 173° and 28° and the same procedure used before to show that

(a) $\cos(173° + 28°) = \cos 173° \times \cos 28° - \sin 173° \times \sin 28°$
(b) $\sin(173° + 28°) = \sin 173° \times \cos 28° + \cos 173° \times \sin 28°$
(c) $\sin(173° - 28°) = \sin 173° \times \cos 28° - \cos 173° \times \sin 28°$

Now you should have some idea that we can express the trigonometric ratios of the sum or difference of two angles as the sum or difference of the products of various trigonometric ratios. This is very useful for simplifying expressions and in calculus.

Let's derive a general formula for the **cosine** of the **difference** between two angles using the polar coordinate form for points.

Let P and Q be two points such that in polar coordinates they have raylength of 1 and angles A and B respectively as shown below.



Now we know that $x = r\cos\theta$; $y = r\sin\theta$ for $P(x, y) \equiv P(r, \theta)$

In this case $r = 1$ for both P and Q,

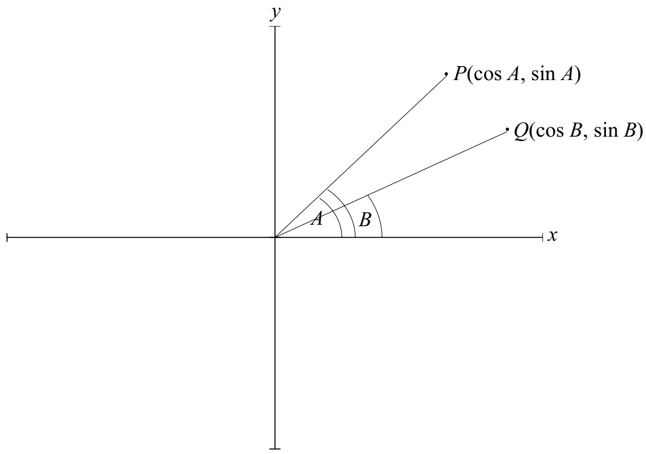
$\therefore$ For P (1, A)

$x_P = 1 \times \cos A$ and $y_P = 1 \times \sin A$
$\therefore \ P(x, y) = P(\cos A, \sin A)$

and for Q (1, B)

$x_Q = 1 \times \cos B$ and $y_Q = 1 \times \sin B$
$\therefore \ Q(x, y) = Q(\cos B, \sin B)$

So let's redraw the figure using these alternative Cartesian coordinates to label P and Q.



Recall that the distance, d between any two points in the XY plane is

$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

$\therefore$ The distance between P and Q is given by

$$d = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}$$

$$\therefore \quad d^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$$

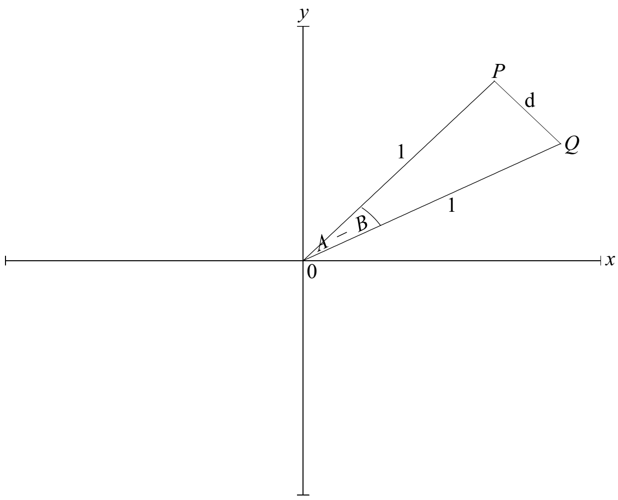
Expanding RHS yields

$$d^2 = \cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A - 2\sin A\cos B + \sin^2 B$$

$$\left.\begin{array}{ll} \text{Now} & \cos^2 A + \sin^2 A = 1 \\ \text{and} & \cos^2 B + \sin^2 B = 1 \end{array}\right\}\text{Pythagoras' Identity}$$

$$\therefore \quad d^2 = 2 - 2\cos A\cos B - 2\sin A\cos B$$

Let's get another expression for d^2 by using the cosine rule on triangle POQ



Applying the cosine rule for triangle POQ we get **See Note** 1

$$d^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \cos(A - B)$$
$$\therefore \quad d^2 = 2 - 2\cos(A - B)$$

Now we have two expressions for d^2 which must be equal

$$\therefore \quad 2 - 2\cos(A - B) = 2 - 2\cos A \cos B - 2\sin A \sin B$$
$$\therefore \quad \cos(A - B) = \cos A \cos B + \sin A \sin B$$

and this is the rule for the **cosine** of the **difference** of two angles.

$$\boxed{\cos(A - B) = \cos A \cos B + \sin A \sin B}$$

Note: This is the rule which we showed an example of earlier with $\cos(173° - 28°)$

The second rule we want is for the **cosine** of the **sum** of two angles. If we substitute $(-B)$ for B everywhere in the above formula we get

$$\cos(A - (-B)) = \cos A \cos(-B) + \sin A \sin(-B)$$

i.e.
$$\boxed{\cos(A + B) = \cos A \cos B - \sin A \sin B}$$
See Note 2

To get the rules for the **sine** of the sum and difference we need to recognise that every cosine function can be written as a sine function and every sine function can be written as a cosine function. If you are not aware of this do the following activity.

Choose one of the following methods to show that

✪

- $\cos x = \sin(x + \frac{\pi}{2})$
- $\sin x = \cos(x - \frac{\pi}{2})$

1. Use your graphing package to draw

   (i) $\cos x$ and $\sin(x + \frac{\pi}{2})$ on the same graph

   (ii) $\sin x$ and $\cos(x - \frac{\pi}{2})$ on the same graph.

   (iii) and note that the graphs in each case are coincidental.

Notes

1. **Remember:** cosine rule $b^2 = a^2 + c^2 - 2ac\cos B$

2. **Remember:** $\cos(-B) = \cos(2\pi - B) = \cos B$

   $\sin(-B) = \sin(2\pi - B) = -\sin B$

2. Verify the formulae $\cos x = \sin(x + \frac{\pi}{2})$

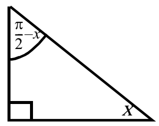
$\sin x = \cos(x - \frac{\pi}{2})$

by choosing some values for x and using your calculator to find the sines and cosines.

For a right angled triangle OPC of known side lengths (as below) and show that

$\sin x = \cos(\frac{\pi}{2} - x)$ _________ (i)

$\cos x = \sin(\frac{\pi}{2} - x)$ _________ (ii)



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Now if $\sin x = \cos(\frac{\pi}{2} - x)$ and $x = A - B$

We can write $\sin(A - B) = \cos\left\{\frac{\pi}{2} - (A - B)\right\}$

$= \cos\left\{\frac{\pi}{2} - A + B\right\}$

$= \cos\left\{\left(\frac{\pi}{2} - A\right) + B\right\}$ and we have on the RHS the cosine of a sum

∴ using the general formula for the cosine of a sum of two angles we can write the RHS as

$= \cos\left(\frac{\pi}{2} - A\right)\cos B - \sin\left(\frac{\pi}{2} - A\right)\sin B$

And using (i) and (ii) above

∴ $\boxed{\sin(A - B) = \sin A\cos B - \cos A\sin B}$

and this is the rule for the **sine** of the **difference** between two angles.

Now if we substitute $(-B)$ for B everywhere in the above formula we get

$$\sin(A - (-B)) = \sin A \cos(-B) - \cos A \sin(-B)$$

$$\therefore \quad \sin(A + B) = \sin A \underbrace{\cos B}_{\substack{\text{because} \\ \cos(-B) = \cos B}} - \cos A . \underbrace{+(-\sin B)}_{\substack{\text{because} \\ \sin(-B) = -\sin B}}$$

$$\therefore \quad \boxed{\sin(A + B) = \sin A \cos B + \cos A \sin B}$$

and this is the rule for the **sine** of the **sum** of two angles.

Two special cases of the compound angles sum and difference formulae that often are useful are the '**double angle formulae**'. In these cases A and B are the same so $A = B$ and

$\sin(A + B) = \sin A \cos B + \cos A \sin B$ becomes

$\sin(A + A) = \sin A \cos A + \cos A \sin A$

i.e. $\boxed{\sin 2A = 2 \sin A \cos A}$

In other books you may see a similar formula as one of the '**half-angle formulae**'. In this case the authors have just let $A = B = \frac{x}{2}$

Then $\sin(A + B) = \sin\left(\frac{x}{2} + \frac{x}{2}\right) = \sin\frac{x}{2}\cos\frac{x}{2} + \cos\frac{x}{2}\sin\frac{x}{2}$

i.e. $\sin x = 2 \sin\frac{x}{2} \cos\frac{x}{2}$

or if you prefer to use A's instead of x's

$$\boxed{\sin A = 2 \sin\frac{A}{2} \cos\frac{A}{2}}$$

There is no need for you to learn the double angle formulae (or the half angle formulae). They are easily derived from the compound angle sum and difference formulae. In the next exercise you will derive the double angle formulae for $\sin\theta$ *and* $\cos\theta$.

Exercise Set 4.8

1. Complete the following

   $\sin(A + B) =$ $\qquad\qquad$ $+$ $\qquad$ **See Note** 1

   $\sin(A - B) =$ $\qquad\qquad$ $-$

   $\cos(A + B) =$ $\qquad\qquad$ $-$

   $\cos(A - B) =$ $\qquad\qquad$ $+$

2. Use the formulae in 1 above to find the **double angle formulae** for

   (i) $\sin 2\theta$
   (ii) $\cos 2\theta$

3. Use the double angle formulae for θ and Pythagoras' Identity ($\sin^2\theta + \cos^2\theta = 1$) to show that

   (i) $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

   (ii) $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

4. Rearrange the formulae in 3 above to get another pair of formulae for the cosine of double angles.

Being able to write trigonometric functions and express trigonometric ratios in different ways often enables us to simplify trigonometric expressions or express them in a way that allows us to differentiate or integrate them. You need to always be looking out for opportunities to simplify trigonometric expressions.

Follow through the next three examples and then do Exercise Set 4.9. You will have to use several of the formulae introduced in this module. There are many ways to show each relationship.

Notes

1. Take care with the signs between the expressions.

Example 4.9(a):

Show that $\cos A \tan A \operatorname{cosec} A = 1$

Solution:

$$\text{LHS} = \cos A \tan A \operatorname{cosec} A$$
$$= \cos A \times \frac{\sin A}{\cos A} \times \frac{1}{\sin A}$$
$$= 1 = \text{RHS}$$

Example 4.9(b):

Show that $\cot\theta = \sqrt{\dfrac{1+\cos 2\theta}{1-\cos 2\theta}}$

Solution:

$$\text{RHS} = \sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}$$
$$= \sqrt{\frac{2\cos^2\theta}{2\sin^2\theta}}$$

In Exercise 4.8 Question 4 (i) you showed (i) $\cos 2\theta = -1 + 2\cos^2\theta$
$\therefore\ 1 + \cos 2\theta = 2\cos^2\theta$ and (ii) $\cos 2\theta = 1 - 2\sin^2\theta$
$\therefore\ 1 - \cos 2\theta = 2\sin^2\theta$

$$= \sqrt{\cot^2\theta}$$
$$= \cot\theta$$
$$= \text{LHS}$$

Example 4.9(c):

Show that $\dfrac{\sin 4\theta - \cos 2\theta}{1-\cos 4\theta - \sin 2\theta} = \cot 2\theta$

Solution:

$$\text{LHS} = \frac{\sin 4\theta - \cos 2\theta}{1-\cos 4\theta - \sin 2\theta}$$
$$= \frac{2\sin 2\theta\cos 2\theta - \cos 2\theta}{1-(\cos^2 2\theta - \sin^2 2\theta) - \sin 2\theta}$$
$$= \frac{\cos 2\theta(2\sin 2\theta - 1)}{\sin^2 2\theta + \sin^2 2\theta - \sin 2\theta}$$

{Using Pythagoras, $\sin^2\theta + \cos^2\theta = 1$
$\therefore 1 - \cos^2 2\theta = \sin^2 2\theta$}

$$= \frac{\cos 2\theta(2\sin 2\theta - 1)}{2\sin^2 2\theta - \sin 2\theta}$$
$$= \frac{\cos 2\theta(2\sin 2\theta - 1)}{\sin 2\theta(2\sin 2\theta - 1)}$$
$$= \cot 2\theta = \text{RHS}$$

Exercise Set 4.9

To reinforce the formulae write the rule you use beside each line of working.

1. Show that

   (a) $\dfrac{\sec A}{\operatorname{cosec} A} = \tan A$

   (b) $\dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \dfrac{1}{\operatorname{cosec}\theta}$

2. Show that

   (a) $\tan A + \tan B = \dfrac{\sin(A+B)}{\cos A\cos B}$

   (b) $\tan\left(\dfrac{\pi}{4}+A\right) = \dfrac{1+\tan A}{1-\tan A}$

   (c) $\dfrac{\cos(A+B)}{\sin(A+B)} = \dfrac{\cot A - \tan B}{1+\cot A\tan B}$

   (d) $2\sin^2\theta - 2 = -2\cos^2\theta$

3. If $\cos x = \dfrac{4}{5}$ find, **without using the calculator**

   (i) $\cos 2x$ (ii) $\sin 2x$

Make a summary of the important formula from this module (Don't bother to remember formulae that you can easily derive).

4.9 Solving Equations Involving Trigonometric Functions

In module 2 we found that we could solve systems of equations by drawing their graphs and identifying the points of intersection. We also showed that we could solve quite complicated equations by splitting the equation into two parts y_1 and y_2 and finding the intersection points of the graphs of y_1 and y_2.

This is a powerful technique when the equations are a mixture of e.g. an algebraic function and a trigonometric function. The most important thing to remember when using the graphical technique is that numerical values must be substituted into the trigonometric function and the algebraic function (i.e. **you must use RADIANS and not DEGREES**).

Follow through this example.

Example 4.10:

Find all solutions of $x^2 - 1 - \sin x = 0$

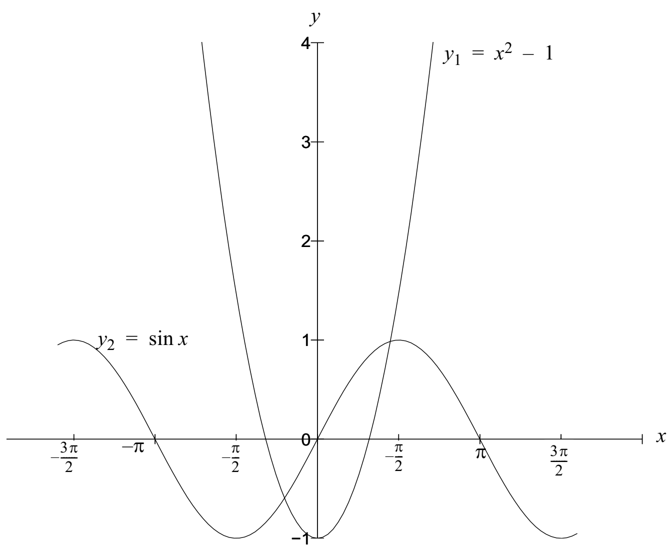
Solution:

We can write the equation as $x^2 - 1 = \sin x$

and let $y_1 = x^2 - 1$ (which we know will be a parabola)
and $y_2 = \sin x$ (a graph whose shape we also know)

Draw up a table of values – **remember to put the calculator in radian mode**.

x	$-\frac{\pi}{2}$	$-\frac{3\pi}{8}$	$-\frac{\pi}{4}$	$-\frac{\pi}{8}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
$y_1 = x^2 - 1$	1.47	0.39	–0.38	–0.85	–1	–0.85	–0.38	0.39	1.47
$y_2 = \sin x$	–1	–0.92	–0.71	–0.38	0	0.38	0.71	0.92	1



Examining the graph we see the points of intersection of y_1 and y_2 are approximately when $x = -\frac{\pi}{5}$ (i.e. approximately –0.63) and when $x = \frac{9\pi}{20}$ (i.e. approximately 1.41)

i.e. the **approximate** solutions of

$$x^2 - 1 - \sin x = 0$$

are $x \simeq -0.63$ and $x \simeq 1.41$

Checking:

When $x = -0.63$
LHS $= x^2 - 1 - \sin x = (-0.63)^2 - 1 - \sin(-0.63) = -0.01 \simeq$ RHS ✓

When $x = 1.41$
LHS $= x^2 - 1 - \sin x = 1.41^2 - 1 - \sin 1.41 \simeq 0 =$ RHS ✓

You can obtain more accurate solutions by drawing the graphs using your graphing package and zooming in on the points of intersection.

Complete this next example and then you should be ready to do the last set of Exercises in this module.

Example 4.11:

Find all solutions between –3 and 3 of the equation $2 + \sec x = 4x$

Solution:

We know $\sec x = \dfrac{1}{\cos x}$, so it will be easier to rearrange this equation with $\sec x$ on the LHS and $4x - 2$ on the RHS. ($4x - 2$ is just a straight line so this will be easy to draw).

Let $y_1 = \sec x$
and $y_2 = 4x - 2$

Draw up a table of values. We are looking for solutions between $x = -3$ and $x = 3$, so this guides us in the possible x values we will use as input into y_1 and y_2. Let's choose to draw y_1 and y_2 for x roughly between –3 and 3.

$\therefore$ in **radians** we will use $x = -\pi$ and $x = \pi$.

Complete the following table for $y_1 = \sec x$ and $y_2 = 4x - 2$. Round to two decimal places where necessary. Use the $\boxed{\cos x}$ key and then the $\boxed{1/x}$ key to get the y_1 values. REMEMBER: CALCULATOR IN RADIANS.

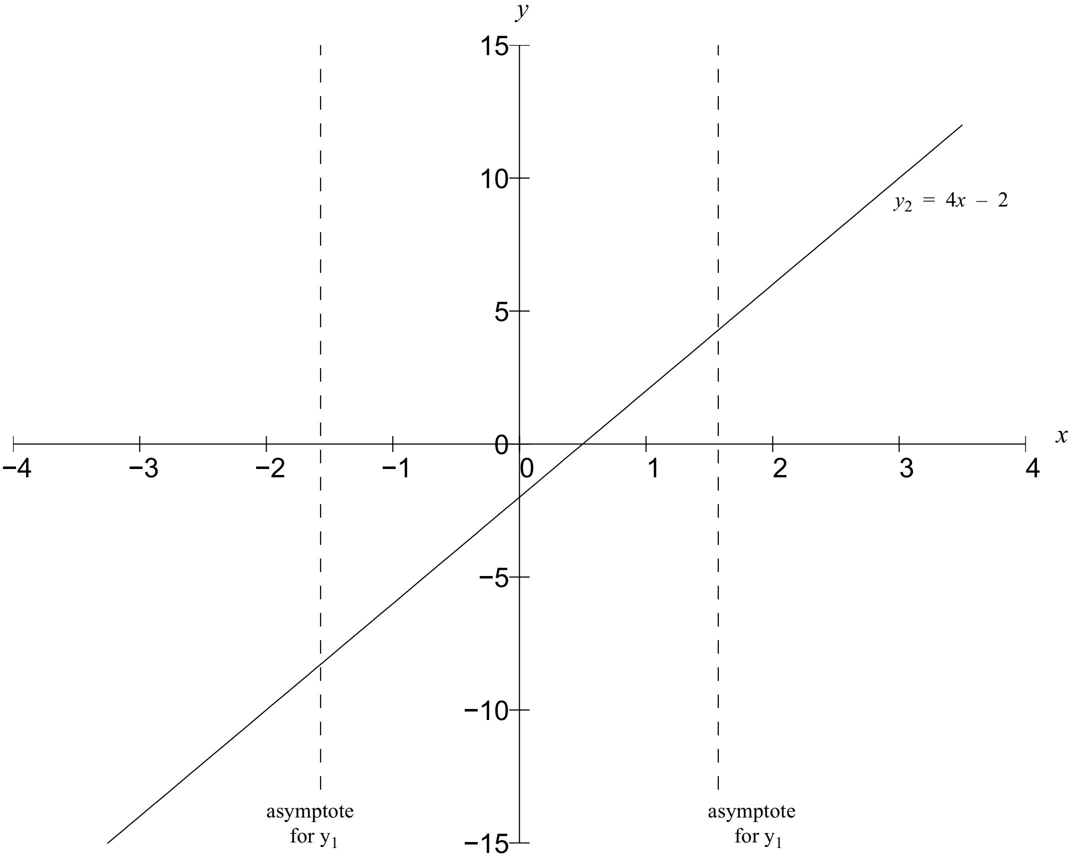
Note that

- $\sec x$ has asymptotes at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and has a period of π
- you really only need two values of y_2 because its graph is a straight line.

x	$-\pi$	$-\frac{7\pi}{8}$	$-\frac{3\pi}{4}$	$-\frac{5\pi}{8}$	$-\frac{\pi}{2}$	$-\frac{3\pi}{8}$	$-\frac{\pi}{4}$	$-\frac{\pi}{8}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$
$y_1 = \sec x$ $y_2 = 4x - 2$	 –14.57		 –11.42		und –8.28	2.61		1.08	1 –2		1.41		und	 5.85	

You may find it easier to draw the graphs if you convert the x values to decimals. (Round to 2 decimal places)

Draw the graph of $y_1 = \sec x$ on the axes provided below. ($y_2 = 4x - 2$ has already been done). Choose some more values of x to draw the graph accurately.



Examine the graphs. You should find three points of intersection. Thus the solutions of the equation $2 + \sec x = 4x$ for $-3 \le x \le 3$

are $x \simeq -1.7$, $x \simeq 0.9$ and $x \simeq 1.2$

(You can obtain these good approximations by using the graphing package and zooming in on the points of intersection).

Exercise Set 4.10

Find approximate solutions of the following equations

1. $x - \sin x = 0$ for $-6 \le x \le 6$
2. $x = \tan x$ for $-4 \le x \le 4$
3. $x^2 - 1 - 2\sin x = 0$
4. $x^2 + 4\cos x = -3x$
5. $|x| - \operatorname{cosec} x = 0$ for $-2 < x \le 2$
6. $e^{-x} = -1.5\cos x$ for $0 < x < 8$

This has been a long module with many important concepts. Make sure your notebook is up to date with the necessary formulae, (do not rote learn all the formulae) and then have a go at the assignment).

Solutions to Exercise Sets

Solutions Exercise Set 4.1 page 4.2

Using the rule 2π radians = 360°

(a) 360° = 2π radians

(b) π (radians) = 180°

(c) 45° = $\frac{\pi}{4}$ radians

(d) 9.2 (radians) = 527.12°

(e) 3π (radians) = 540°

(f) $(3\pi)°$ = 0.16 radians

Solutions Exercise Set 4.2 page 4.3

(a) $\sin \frac{\pi}{2} = 1$

(b) $\sin 2.15 = 0.837$

(c) $\sin 20.4° = 0.349$

(d) $\tan \frac{3\pi}{2}$ is undefined

(e) $\tan \frac{1}{8} = 0.126$

(f) $\cos -\frac{2\pi}{3} = -0.5$

(g) $\sec -40° = 1.305$

(h) $\operatorname{cosec} \frac{-3\pi}{2} = 1$

(i) $\cot \frac{8\pi}{3} = -0.577$

Solutions Exercise Set 4.3 page 4.5

1.

(i) $P(x,y) = P(4,2)$

Note that P is in the first quadrant $\therefore\ 0° \le \theta < 90°$ or $0 \le \theta < \frac{\pi}{2}$ radians

$r = \sqrt{x^2+y^2} = \sqrt{4^2+2^2} = \sqrt{20} = 2\sqrt{5}$

$\tan\ \theta = \frac{y}{x} = \frac{2}{4} = \frac{1}{2}$

$\therefore\ \theta = \tan^{-1}0.5 = 26.57°$ or 0.46 radians

Checking: θ is in the first quadrant ✓

$\tan\theta = \tan 26.57° \approx 0.5$ ✓

$\therefore P(r,\theta) = (2\sqrt{5}, 26.57°)$ or $(2\sqrt{5}, 0.46)$

(ii) $Q(x, y) = Q(-3, 3)$

Note that Q is in the second quadrant $\therefore\ 90° \le \theta < 180°$ or $\frac{\pi}{2} \le \theta < \pi$ radians

$r = \sqrt{(-3)^2+3^2} = \sqrt{18} = 3\sqrt{2}$

$\tan\ \theta = \frac{-3}{3} = -1$

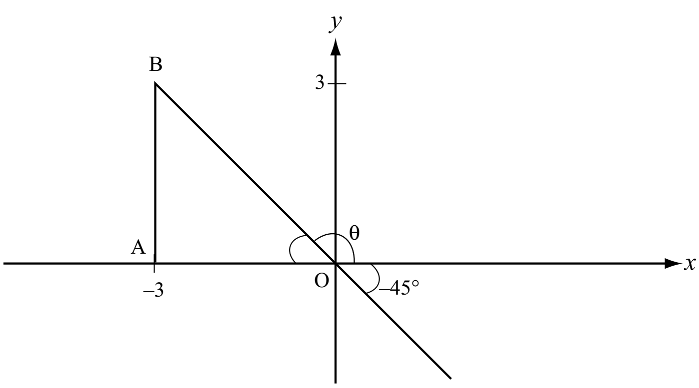
$\therefore\ \theta = \tan^{-1}-1 = -0.785$ radians. But this puts θ in the fourth quadrant and we want θ in the second quadrant.

$\therefore$ Correct $\theta = (\pi - 0.79)$ radians or $(180° - 45°)$

$= 2.35$ radians or $135°$

Checking: θ is in the second quadrant ✓

$\tan\theta = \tan 135° = -1$ ✓



In triangle ABO,
angle AOB = 45°
$\therefore$ the supplementary angle (which is what we want)
$\theta = 180° - 45° = 135°$

$\therefore\ Q(r, \theta) = (3\sqrt{2}, 135°)$

or $\left(3\sqrt{2}, \frac{3\pi}{4}\right)$

Solutions Exercise Set 4.3 cont.

1. continued

(iii) $R(x, y) = R(-5, -2)$

Note that R is in the third quadrant $\therefore$ $180° \leq \theta < 270°$ or $\pi \leq \theta < \frac{3\pi}{2}$ radians

$r = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$

$\tan\ \theta = \frac{-2}{-5} = 0.4$

$\therefore\ \theta = \tan^{-1}0.4$

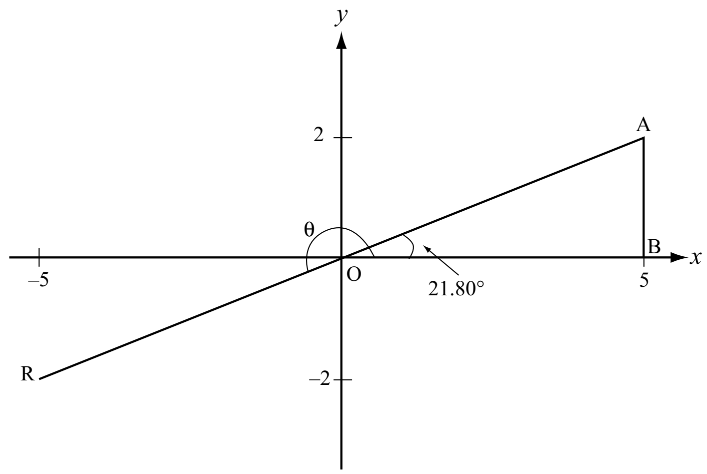
$= 21.80°$

From the calculator, $\theta = 21.80°$. But this puts θ in the first quadrant.

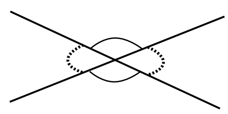
$\therefore$ Correct $\theta = (180° + 21.80°) = 201.80°$ or 3.52 radians

Checking: θ is in the third quadrant ✓

$\tan\ \theta = \tan\ 3.52 \approx 0.4$ ✓



In triangle ABO, angle AOB = 21.80°.



The angle we want is $\theta = 180° + 21.80° = 201.80°$.

(Vertically opposite angles are equal)

$\therefore\ R(r, \theta) = (\sqrt{29}, 201.80°)$ or $(\sqrt{29}, 3.52)$

Solutions Exercise Set 4.3 cont.

1. continued

(iv) $S(x, y) = S(1, -5)$

Note that S is in the fourth quadrant $\therefore$ $270° \leq \theta < 360°$ or $\frac{3\pi}{2} \leq \theta < 2\pi$ radians

$r = \sqrt{1^2 + (-5)^2} = \sqrt{26}$

$\tan\ \theta = \frac{-5}{1} = -5$

$\therefore\ \theta = \tan^{-1}-5$

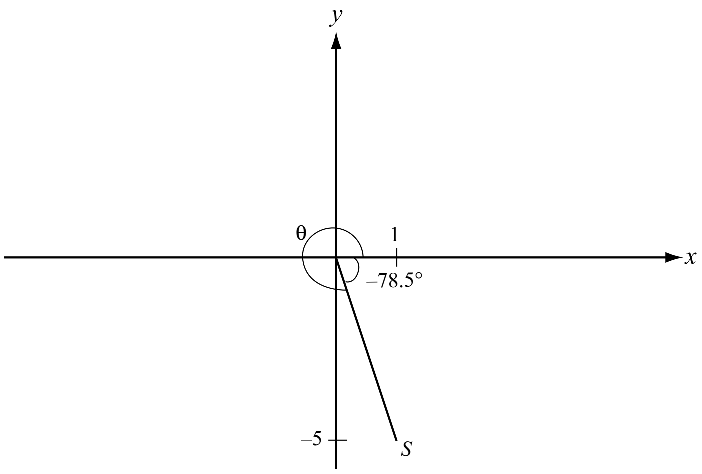
$\theta = -1.37$ radians

$= 2\pi - 1.37$ radians **See Note** 1

$= 4.91$ radians or $281.50°$

Checking: θ is in the fourth quadrant ✓

$\tan\ \theta = \tan\ 281.50° \approx -5$ ✓



$\therefore S(r, \theta) = (\sqrt{26}, 4.91)$ or $(\sqrt{26}, 281.50°)$

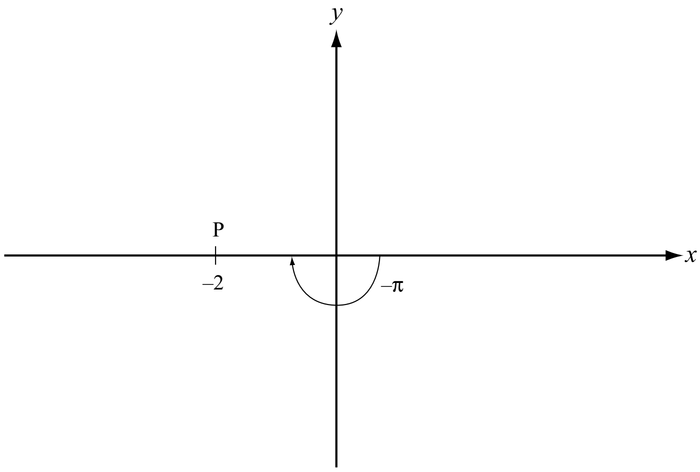
Notes

1. We measure angles positively, i.e. in the anti-clockwise direction.

Solutions Exercise Set 4.3 cont.

2. continued

(i)



Note: We expect x to be –ve and y to be 0.

$x = r\cos\theta$

$= 2 \times \cos(-\pi) = -2$

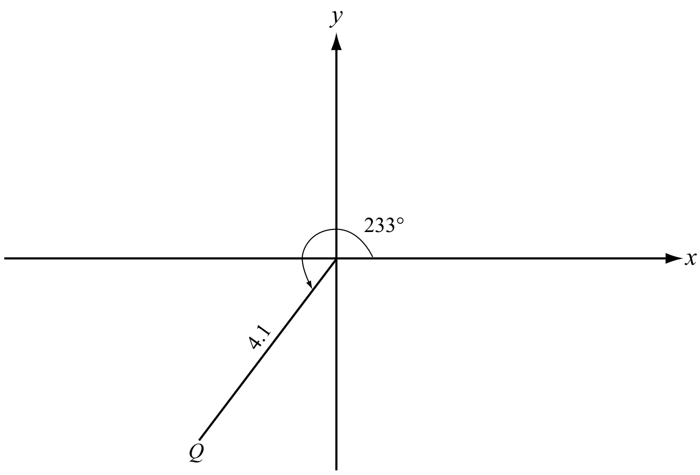
$y = r\sin\theta$

$= 2 \times \sin(-\pi) = 0$

Checking: $r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + 0^2} = 2$ ✓

$\therefore P(x, y) = (-2, 0)$

(ii)



Note: We expect both x and y to be –ve.

Solutions Exercise Set 4.3 cont.

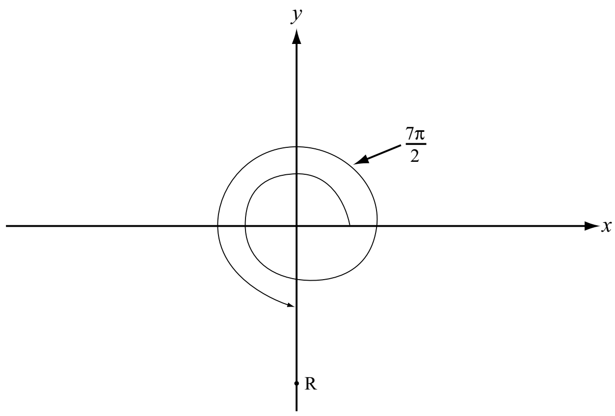
2. (ii) continued

$$
\begin{aligned}
x &= r\cos\theta \\
&= 4.1 \times \cos 233° \\
&= -2.47 \\
y &= r\sin\theta \\
&= 4.1 \sin 233° \\
&= -3.27
\end{aligned}
$$

Checking: $r = \sqrt{(-2.47)^2 + (-3.27)^2} = 4.10$ ✓

$\therefore Q(x, y) = (-2.47, -3.27)$

(iii)



Note: We expect x to be zero and y to be –ve.

$$
\begin{aligned}
x &= r\cos\theta \\
&= 5\cos\frac{7\pi}{2} \\
&= 0 \\
y &= 5\sin\frac{7\pi}{2} \\
&= -5
\end{aligned}
$$

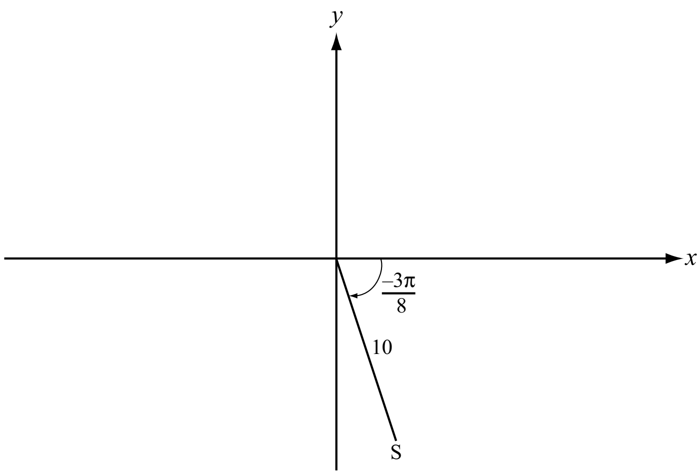
Checking: $r = \sqrt{(-5)^2 + 0^2} = 5$ ✓

$\therefore R(x, y) = (0, -5)$

Solutions Exercise Set 4.3 cont.

2. continued

(iv)



Note: We expect x to +ve and y to be –ve.

$$x = r\cos\theta$$
$$= 10\cos\left(\frac{-3\pi}{8}\right)$$
$$= 3.83$$
$$y = 10\sin\left(\frac{-3\pi}{8}\right)$$
$$= -9.24$$

Checking: $r = \sqrt{3.83^2 + (-9.24)^2} \approx 10$ ✓

$\therefore S(x, y) = (3.83, -9.24)$

Solutions Exercise Set 4.3 cont.

3.

(a) Length of arc of a circle, $l = r \times \theta$ where r is the radius of the circle and θ is the angle subtended by the arc at the centre of the circle in **RADIANS**.

$$36° = \frac{36 \times 2\pi}{360} \text{ radians}$$

$$= \frac{\pi}{5} \text{ radians}$$

Circumference of circle $= 2\pi r$ and we are given that the circumference $= 45\pi$.

$$\therefore 2\pi r = 45\pi$$

$$\therefore r = \frac{45}{2}$$

$$\text{Now } l = r\theta$$

$$\therefore l = \frac{45}{2} \times \frac{\pi}{5}$$

$$= \frac{9\pi}{2}$$

$$\approx 14.14 \text{ cm}$$

(b) Area of a sector of a circle $= \dfrac{r^2\theta}{2}$ where r is the radius of the circle and θ is the subtended angle in **RADIANS**.

$$360° = 2\pi \text{ radians}$$

$$\therefore \text{ For the area shaded, } \theta = \left(2\pi - \frac{\pi}{5}\right) \text{ radians } = \frac{9\pi}{5} \text{ radians}$$

$$\therefore \text{ Required area } = \frac{\left(\frac{45}{2}\right)^2 \times \frac{9\pi}{5}}{2}$$

$$\approx 1\,431.39 \text{ cm}^2$$

Solutions Exercise Set 4.3 cont.

4.

(a) Area of sector $= \dfrac{r^2\theta}{2} = 6\pi \text{ m}^2$

Area of circle $= \pi r^2 = 9\pi \text{ m}^2$

$\therefore r^2 = \dfrac{9\pi}{\pi} = 9$

Now area of sector $= \dfrac{r^2\theta}{2} = \dfrac{9\theta}{2}$

$\therefore 6\pi = \dfrac{9\theta}{2}$

$\therefore \theta = \dfrac{12\pi}{9} = \dfrac{4\pi}{3}$ or $\left(\dfrac{4\pi}{3} \times \dfrac{360}{2\pi}\right)^{\circ}$

i.e. $\theta = \dfrac{4\pi}{3}$ or 240°

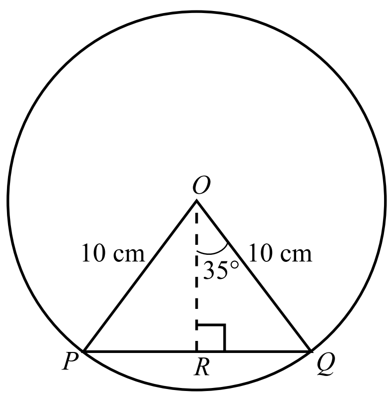
(b) Arc length, $l = r\theta$

$= 3 \times \dfrac{4\pi}{3}$

$= 4\pi$ m

≈ 12.57 m

5.



(i) A perpendicular from the centre O, to the chord PQ bisects angle POQ (because triangle POQ is isosceles)

$\therefore \sin 35^\circ = \dfrac{\overrightarrow{RQ}}{10}$ Note: There is no need to convert to radians for this calculation.

$\therefore \overrightarrow{RQ}\, RQ = 10 \times \sin 35^\circ$

$= 5.74$ cm

$\therefore$ Length of chord $= 2 \times 5.74$

$= 11.48$ cm

Solutions Exercise Set 4.3 cont.

5. continued

(ii) To find the arc length, 70° must be converted to radians.

$$70° = \frac{70 \times 2\pi}{360} \text{ radians}$$

$$= 1.22 \text{ radians}$$

$$\therefore \text{ Arc length } = r\theta$$

$$= 10 \times 1.22$$

$$= 12.2 \text{ cm}$$

(iii) $$\text{Area of sector} = \frac{r^2\theta}{2}$$

$$= \frac{10^2 \times 1.22}{2}$$

$$= 61 \text{ cm}^2$$

(iv) Shaded area = Area of sector – area of triangle *OPQ*

$$\text{Area of triangle } OPQ = \frac{1}{2} \times 11.48 \times \overrightarrow{OR}$$

$$= \frac{1}{2} \times 11.48 \times (10 \cos 35°)$$

$$= 47.02 \text{ cm}^2$$

$$\therefore \text{ Shaded area} = 61 - 47.02$$

$$= 13.98 \text{ cm}^2$$

Solutions Exercise Set 4.4 page 4.14

1.

There are several ways of doing these problems

(a) $\cos A \;\tan A \;\sec A = \cos A \times \dfrac{\sin A}{\cos A} \times \dfrac{1}{\cos A} = \tan A$

(b) $\dfrac{\sec A \;\tan A}{\operatorname{cosec} A} = \dfrac{\left(\dfrac{1}{\cos A}\right) \times \tan A}{\dfrac{1}{\sin A}} = \dfrac{1}{\cos A} \times \dfrac{\sin A}{1} \times \tan A = \tan^2 A$

(c) $\dfrac{1}{\cos^2\theta} - \tan^2\theta = \sec^2\theta - (\sec^2\theta - 1) = 1$ {Using the identity $1 + \tan^2\theta = \sec^2\theta$}

(d) $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$

$= \sin^2 x + 2\sin x \cos x + \cos^2 x + \sin^2 x - 2\sin x \cos x + \cos^2 x$

$= 2\sin^2 x + 2\cos^2 x = 2(\sin^2 x + \cos^2 x) = 2 \times 1 = 2$

2.

(a) $\sec^2\theta = 3 + \tan\theta$

$1 + \tan^2\theta = 3 + \tan\theta$

$\therefore\ \tan^2\theta - \tan\theta - 2 = 0$

$\therefore\ (\tan\theta - 2)(\tan\theta + 1) = 0$ *{Factorising - if you have trouble here replace tan θ by A}*

$\therefore\ \tan\theta = 2$ or $\tan\theta = -1$

$\therefore\ \theta = \tan^{-1} 2$ or $\theta = \tan^{-1} -1$

If $\theta = \tan^{-1} 2$

$\theta = 1.07$ radians or $(\pi + 1.107)$ radians

i.e. $\theta = 1.107$ radians or 4.249 radians

Solutions Exercise Set 4.4 cont.

2. continued

If $\theta = \tan^{-1} -1$

$\theta = -0.785$ radians or $(\pi + (-0.785))$ radians

$= -0.785$ radians or 2.356 radians

$= 5.498$ radians or 2.356 radians {Expressing -0.785 radians as a positive angle}

$\therefore \theta = 1.107, \; 2.356, \; 4.249$ and 5.498

Checking:

When $\theta = 1.107$

$$\text{LHS} = \sec^2\theta = \sec^2 1.107 = \frac{1}{\cos^2 1.107} = 4.997$$

$$\text{RHS} = 3 + \tan\theta = 3 + \tan 1.107 = 4.999 \approx \text{LHS} \quad \checkmark$$

When $\theta = 2.356$

$$\text{LHS} = \sec^2 2.356 = \frac{1}{\cos^2 2.356} = 2.001$$

$$\text{RHS} = 3 + \tan 2.356 = 2.000 \approx \text{LHS} \quad \checkmark$$

When $\theta = 4.249$

$$\text{LHS} = \sec^2 4.249 = \frac{1}{\cos^2 4.249} = 5.005$$

$$\text{RHS} = 3 + \tan 4.249 = 5.001 \approx \text{LHS} \quad \checkmark$$

When $\theta = 5.498$

$$\text{LHS} = \sec^2 5.498 = \frac{1}{\cos^2 5.498} = 1.999$$

$$\text{RHS} = 3 + \tan 5.498 = 2.000 \approx \text{LHS} \quad \checkmark$$

(b) $\cos^2\theta + 3\sin^2\theta = 2$

$$1 - \sin^2\theta + 3\sin^2\theta = 2$$

$$1 + 2\sin^2\theta - 2 = 0$$

$$2\sin^2\theta = 1$$

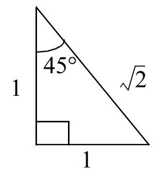
$$\therefore \sin^2\theta = \frac{1}{2}$$

$$\therefore \sin\theta = +\frac{1}{\sqrt{2}} \quad \text{or} \quad \sin\theta - \frac{1}{\sqrt{2}}$$

$$\therefore \theta = \sin^{-1}\frac{1}{\sqrt{2}} \quad \text{or} \quad \theta = \sin^{-1} -\frac{1}{\sqrt{2}}$$

Solutions Exercise Set 4.4 cont.

2. (b) continued



If $\theta = \sin^{-1}\dfrac{1}{\sqrt{2}}$

$$\theta = \frac{\pi}{4} \quad \text{or} \quad \pi - \frac{\pi}{4}$$

$$\therefore\ \theta = \frac{\pi}{4} \quad \text{or} \quad \frac{3\pi}{4}$$

If $\theta = \sin^{-1}-\left(\dfrac{1}{\sqrt{2}}\right)$

$$\theta = \frac{-\pi}{4} \quad \text{or} \quad \pi - \left(\frac{-\pi}{4}\right)$$

$$\therefore\ \theta = \frac{-\pi}{4} \quad \text{or} \quad \frac{5\pi}{4}$$

$$\therefore\ \theta = \left(2\pi - \frac{\pi}{4}\right) \quad \text{or} \quad \frac{5\pi}{4}$$

$$= \frac{7\pi}{4} \quad \text{or} \quad \frac{5\pi}{4}$$

$$\therefore\ \theta = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \text{and}\ \frac{7\pi}{4}$$

Checking:

When $\theta = \dfrac{\pi}{4}$

$$\begin{aligned} \text{LHS} &= \cos^2\theta + 3\sin^2\theta \\ &= \cos^2\left(\frac{\pi}{4}\right) + 3\sin^2\left(\frac{\pi}{4}\right) \\ &= \left(\frac{1}{\sqrt{2}}\right)^2 + 3\left(\frac{1}{\sqrt{2}}\right)^2 \\ &= \frac{1}{2} + \frac{3}{2} = 2 = \text{RHS} \quad \checkmark \end{aligned}$$

Checking of other angles is not shown but should be done.

Solutions Exercise Set 4.4 cont.

2. continued

(c) $2\sin^2\theta - 9\cos\theta + 3 = 0$

$2(1 - \cos^2\theta) - 9\cos\theta + 3 = 0$

$2 - 2\cos^2\theta - 9\cos\theta + 3 = 0$

$-2\cos^2\theta - 9\cos\theta + 5 = 0$

Using the quadratic formula

$$\cos\theta = \frac{-(-9) \pm \sqrt{(-9)^2 - 4 \times (-2) \times 5}}{2 \times (-2)}$$

$$= \frac{9 \pm \sqrt{81 + 40}}{-4}$$

$$= \frac{9 \pm 11}{-4}$$

$\therefore \cos\theta = -5 \qquad \text{or} \qquad \cos\theta = \frac{1}{2}$

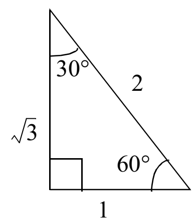
There is no value of θ for $\cos\theta = -5$, so we only need consider $\cos\theta = \frac{1}{2}$

If $\quad \cos\theta = \frac{1}{2}$

$\theta = \cos^{-1}\left(\frac{1}{2}\right)$

$\therefore \theta = \frac{\pi}{3} \qquad \text{or} \qquad \left(2\pi - \frac{\pi}{3}\right)$

i.e. $\quad \theta = \frac{\pi}{3} \qquad \text{or} \qquad \frac{5\pi}{3}$



Checking:

When $\theta = \frac{\pi}{3}$

$$\text{LHS} = 2\sin^2\theta - 9\cos\theta + 3$$

$$= 2\sin^2\left(\frac{\pi}{3}\right) - 9\cos\frac{\pi}{3} + 3$$

$$= 2 \times \left(\frac{\sqrt{3}}{2}\right)^2 - 9 \times \frac{1}{2} + 3$$

$$= 2 \times \frac{3}{4} - \frac{9 \times 1}{2} + 3$$

$$= 0 = \text{RHS} \quad ✓$$

Solutions Exercise Set 4.4 cont.

2. (c) continued

When $\theta = \dfrac{5\pi}{3}$

$$\text{LHS} = 2\sin^2\left(\frac{5\pi}{3}\right) - 9\cos\frac{5\pi}{3} + 3$$

$$= 1.5 - 4.5 + 3$$

$$= 0 = \text{RHS} \quad \checkmark$$

3. $N = 10 + \sin\dfrac{\pi}{24}t$ where t is the number of months after 1 January 1994

(i) In July 1995, $t = 12 + 6 = 18$

$$\therefore N = 10 + \sin\left(\frac{\pi}{24} \times 18\right)$$

$$= 10 + \sin\frac{3\pi}{4}$$

$$= 10 + 0.707$$

$$= 10.707$$

$\therefore$ In July 1995, there were about 10 700 rabbits in the area.

(ii) $\sin\dfrac{\pi}{24}t$ can **at most** be 1 and **at least** be –1 i.e. $-1 \leq \sin\dfrac{\pi}{24}t \leq 1$

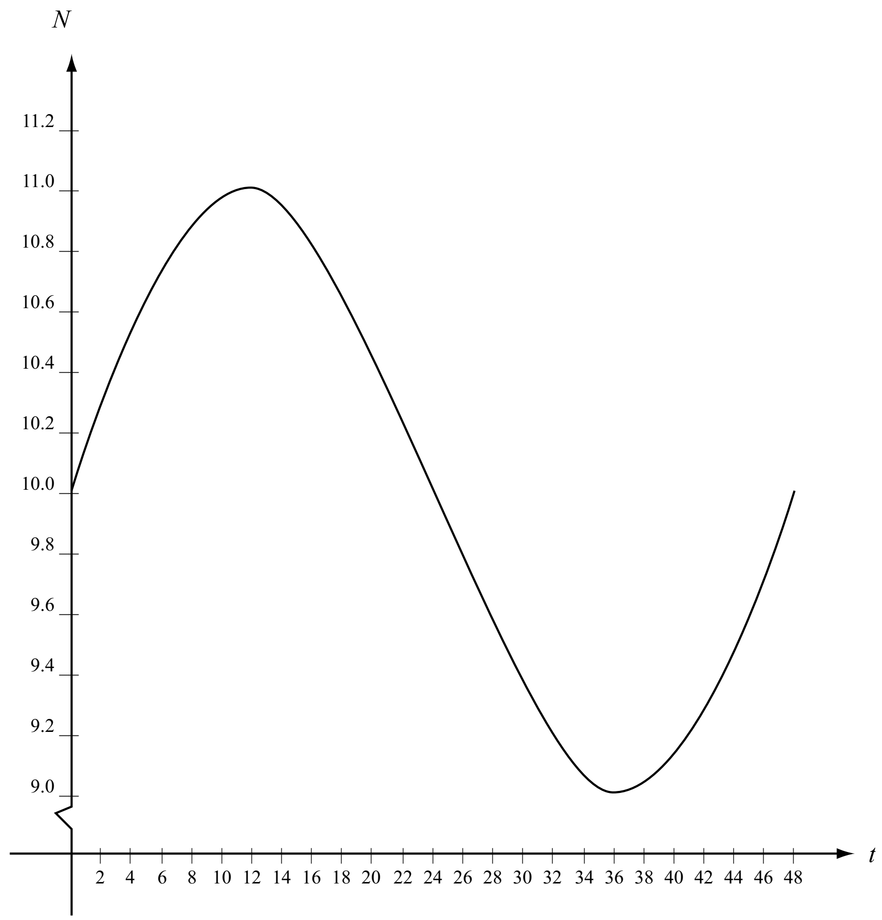
So N is at most 10 + 1, i.e. 11 000 rabbits is the maximum possible

and N is at least 10 – 1, i.e. 9 000 rabbits is the minimum possible.

Solutions Exercise Set 4.4 cont.

3. continued

(iii)



Solutions Exercise Set 4.4 cont.

3. continued

(iv) From 1 January 1994 to end December 1996 $\Rightarrow$ $0 \leq t \leq 48$

$N = 10 + \sin \frac{\pi}{24} t$

When population = 10 500, $N = 10.5$ {N is the thousands of rabbits in the population}

$$10.5 = 10 + \sin \frac{\pi}{24} t$$

$$\therefore 0.5 = \sin \frac{\pi}{24} t$$

$$\therefore \frac{\pi}{24} t = \sin^{-1}\left(\frac{1}{2}\right)$$

$$\frac{\pi}{24} t = \frac{\pi}{6} \quad \text{or} \quad \left(\pi - \frac{\pi}{6}\right)$$

$$= \frac{\pi}{6} \quad \text{or} \quad \frac{5\pi}{6}$$

$$\therefore t = \frac{\pi}{6} \times \frac{24}{\pi} \quad \text{or} \quad \frac{5\pi}{6} \times \frac{24}{\pi}$$

$$= 4 \quad \text{or} \quad 20$$

The population will be 10 500 on 1 May 1994 and 1 September 1995.

When population = 9 500, $N = 9.5$

$$9.5 = 10 + \sin \frac{\pi}{24} t$$

$$\therefore -0.5 = \sin \frac{\pi}{24} t$$

$$\therefore \frac{\pi}{24} t = \sin^{-1}\left(\frac{1}{2}\right)$$

$$\therefore \frac{\pi}{24} t = \frac{-\pi}{6} \quad \text{or} \quad \pi - \left(\frac{-\pi}{6}\right)$$

$$= \frac{-\pi}{6} \quad \text{or} \quad \frac{7\pi}{6}$$

$$= \left(2\pi - \frac{\pi}{6}\right) \quad \text{or} \quad \frac{7\pi}{6}$$

$$= \frac{11\pi}{6} \quad \text{or} \quad \frac{7\pi}{6}$$

$$\therefore t = \frac{11\pi}{6} \times \frac{24}{\pi} \quad \text{or} \quad \frac{7\pi}{6} \times \frac{24}{\pi}$$

$$= 44 \quad \text{or} \quad 28$$

Now $t = 44$ is not in the domain of t we are considering

$\therefore$ $t = 28$ is the only solution

The population will be 9 500 on 1 May 1996.

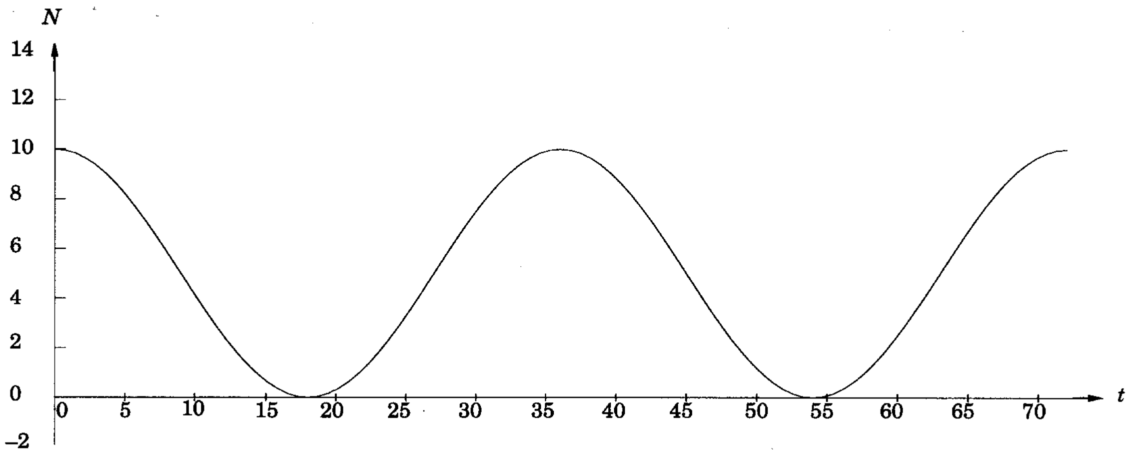
Solutions Exercise Set 4.4 cont.

3. continued

(v) Solutions are verified by reading from the graph in (iii)

When $t = 4$, $N = 10.5$ $\Rightarrow$ Pop $=$ 10 500 ✓
When $t = 20$, $N = 10.5$ $\Rightarrow$ Pop $=$ 10 500 ✓
When $t = 28$, $N = 9.5$ $\Rightarrow$ Pop $=$ 9 500 ✓

4. (i)



(vi) $N = 5 + 5\cos\dfrac{\pi}{18}t$

When the number of saplings = 8 000, $N = 8$

$$\therefore 8 = 5 + 5\cos\frac{\pi}{18}t$$

$$\therefore 3 = 5\cos\frac{\pi}{18}t$$

$$\therefore 0.6 = \cos\frac{\pi}{18}t$$

$$\therefore \frac{\pi}{18}t = \cos^{-1}0.6$$

$$\therefore \frac{\pi}{18}t = 0.93 \quad \text{or} \quad (2n\pi + 0.93)$$

If $\dfrac{\pi}{18}t = 0.93 \Rightarrow t = 5.3$

(We want the **first** occasion after $t = 0$ when the population is 8 000, $\therefore$ we do not have to consider other values of t which would result from solving $\dfrac{\pi}{18}t = (2n\pi + 0.93)$)

Solutions Exercise Set 4.4 cont.

4. (ii) continued

So the number of saplings will reach 8 000, 5.3 months after the given date.

Checking:

- From the graph, when $t = 5.3$, $N \approx 8$. $\therefore$ Number of saplings ≈ 5 000 ✓
- Substituting $t = 5.3$ in original equation yields

  $N = 5 + 5 \cos\left(\frac{\pi}{18} \times 5.3\right) = 8.009$

  $\therefore$ Number of saplings ≈ 8 000.

Solutions Exercise Set 4.5 page 4.18

1.

(a) $N = 10 + \sin\frac{\pi}{24}t$

(i) period $= \dfrac{2\pi}{\frac{\pi}{24}} = 2\pi \times \dfrac{24}{\pi} = 48$

(ii) amplitude $= 1$ (The constant 10 added to the sine function does not affect the amplitude)

(b) $N = 5 + 5\cos\frac{\pi}{18}t$

(i) period $= \dfrac{2\pi}{\frac{\pi}{18}} = 2\pi \times \dfrac{18}{\pi} = 36$

(ii) amplitude $= 5$

If you are unsure of these results check the graphs of these functions from Exercise Set 4.4.

2.

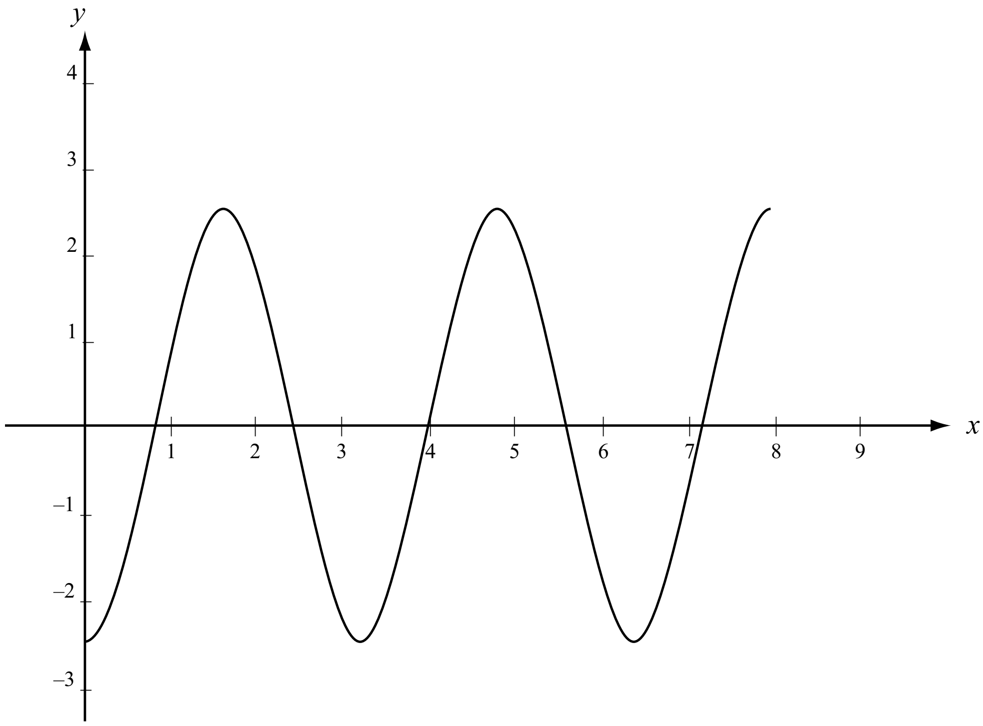
(a) frequency $= \dfrac{1}{\text{period}} = \dfrac{1}{48}$ i.e. in 1 unit of length along the t axis there is $\frac{1}{48}$ of a wave

(b) frequency $= \dfrac{1}{36}$

Check that these results make sense by again examining the graphs of the function.

Solutions Exercise Set 4.5 cont.

3. (i)



(iii) period $= \dfrac{2\pi}{2} = \pi$

amplitude $= 2.5$

frequency $= \dfrac{1}{\text{period}} = \dfrac{1}{\pi} \approx 0.32$

(iv) The negative coefficient of the cosine or sine function shows that the curve (or wave) starts at the lowest point of its cycle, i.e. the graph is the reflection across the x axis of the curve with positive A as the coefficient.

Solutions Exercise Set 4.5 cont.

4.

(a) $y = B + A\ \sin kx$ is a general sine function

In this case, $y = 4 + 3\ \sin kx$

Now the frequency $= \dfrac{1}{\text{period}}$

i.e. $\dfrac{1}{2} = \dfrac{1}{\text{period}}$

$\therefore$ period $= 2$

Now period $= \dfrac{2\pi}{k}$

$\therefore 2 = \dfrac{2\pi}{k}$

$\therefore k = \pi$

$\therefore$ function is $y = 4 + 3\ \sin \pi x$

(b) General form of this cosine function is $y = B + A\ \cos kt$

'Base' position is the t axis $\therefore B = 0$

$\therefore y = A\ \cos kt$

Now A must be negative because the wave starts at $(0, -1.5)$

$\therefore y = -1.5\ \cos kt$

period $= 4$

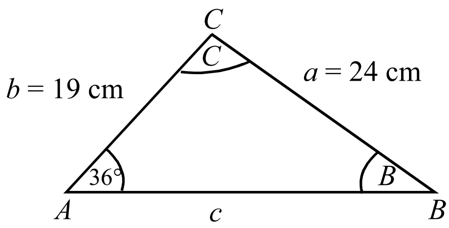
$\therefore 4 = \dfrac{2\pi}{k}$

$\therefore k = \dfrac{\pi}{2}$

$\therefore$ Function is $y = -1.5\ \cos \dfrac{\pi}{2} t$

Solutions Exercise Set 4.6 page 4.27

1. (a)



Given $A = 36°$, $a = 24$, $b = 19$ and need to find angles B and C and side length c.

Using sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{24}{\sin 36°} = \frac{19}{\sin B}$$

$$\therefore \sin B = \frac{19 \times \sin 36°}{24} = 0.4653 \qquad \text{(Calculator must be in degree mode)}$$

$$\therefore B = \sin^{-1} 0.4653 = 27.73° \quad \text{or} \quad 180° - 27.73°$$

$$= 27.73° \quad \text{or} \quad 152.27°$$

$B = 152.27°$ is impossible because then $A + B > 180°$

$\therefore B = 27.73°$

$\therefore C = 180° - (36° + 27.73°)$

$= 116.27°$

Again using sine rule:

$$\frac{24}{\sin 36°} = \frac{c}{\sin 116.27°}$$

$$\therefore c = \frac{24 \times \sin 116.27°}{\sin 36°} = 36.61 \text{ cm}$$

Solutions Exercise Set 4.6 cont.

1. continued

Checking:

$A + B + C = 36° + 27.73° + 116.27° = 180°$ ✓

Using cosine rule:

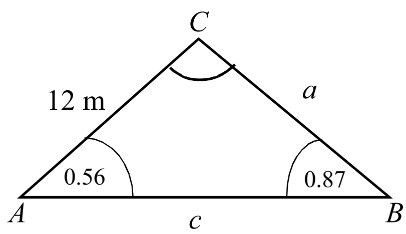
$c^2 = a^2 + b^2 - 2ab \cos C$

$c^2 = 24^2 + 19^2 - 2 \times 24 \times 19 \cos 116.27°$

$= 937 + 403.65 = 1\,340.22$

$\therefore\ c = \sqrt{1340.22} = 36.61$ cm ✓

(b)



π radians in a triangle

$\therefore\ C = \pi - (0.56 + 0.87)$

$= 1.71$ radians

Using sine rule:

$$\frac{a}{\sin 0.56} = \frac{12}{\sin 0.87}$$

$$\therefore\ a = \frac{12 \times \sin 0.56}{\sin 0.87}$$ (Check calculator is in radian mode)

$= 8.33$ m

Using sine rule:

$$\frac{8.33}{\sin 0.56} = \frac{c}{\sin 1.71}$$

$$\therefore\ c = \frac{8.33 \times \sin 1.71}{\sin 0.56}$$

$= 15.53$ m

Solutions Exercise Set 4.6 cont.

1. (b) continued

Checking:

$A + B + C = 0.56 + 0.87 + 1.71 = 3.14 \approx \pi$ ✓

Using cosine rule:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

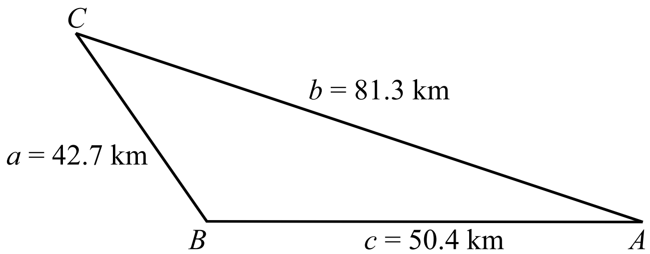
$$= 8.33^2 + 12^2 - 2 \times 8.33 \times 12\cos 1.71$$

$$= 213.3889 + 27.7398 = 241.1287$$

$$\therefore c = \sqrt{241.1287}$$

$$\approx 15.33$$ ✓

(c)



The hint given simplifies the calculation.

The longest side is b $\therefore$ we expect B to be the largest angle.

Using the cosine rule:

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$\therefore 81.3^2 = 42.7^2 + 50.4^2 - 2 \times 42.7 \times 50.4 \times \cos B$$

$$\therefore \cos B = \frac{6609.69 - 4363.45}{-4304.16} = -0.5219$$

$$\therefore B = \cos^{-1} -0.5219$$

$$= 2.120 \quad \text{or} \quad (2\pi - 2.120)$$

$$= 2.120 \quad \text{or} \quad 4.163 \text{ radians} \quad (4.163 \text{ is impossible as} > \pi)$$

$$\therefore B = 2.120$$

Solutions Exercise Set 4.6 cont.

1. (c) continued

Now both A and C must be acute angles, i.e. $< \frac{\pi}{2}$ as B is obtuse, i.e. $> \frac{\pi}{2}$,

so the rest of the calculations to find the angles are simplified.

Using sine rule:

$$\frac{42.7}{\sin A} = \frac{81.3}{\sin 2.120}$$

$$\therefore \sin A = \frac{42.7 \times \sin 2.120}{81.3}$$

$$= 0.4480$$

$$\therefore A = \sin^{-1} 0.448$$

$$= 0.465$$

$$\therefore B = \pi - (2.120 + 0.465)$$

$$= 0.557$$

$\therefore A = 0.465$ radians, $B = 0.557$ radians, $C = 2.120$ radians

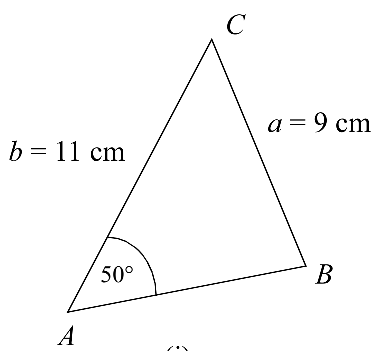
(or in degrees: $A = 26.64°$, $B = 31.91°$, $C = 121.47°$)

Checking:

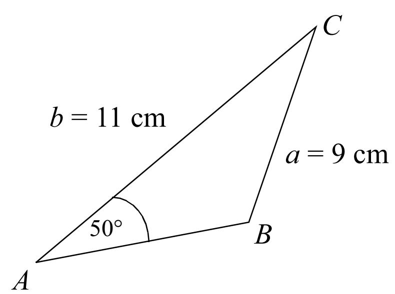
$A + B + C = \pi$ radians ✓

Use the cosine rule for two different angles. (Not shown here.)

2. We do not know the shape of this triangle. It may look like either of these.



(i)



(ii)

The fact that there are two possible solutions becomes apparent when we use the sine rule.

Solutions Exercise Set 4.6 cont.

2. continued

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{9}{\sin 50^\circ} = \frac{11}{\sin B}$$

$$\therefore \sin B = \frac{11 \times \sin 50^\circ}{9} = 0.9363 \qquad \text{(Check calculator is in degree mode)}$$

$$\therefore B = \sin^{-1} 0.9363$$

$$= 69.44^\circ \quad \text{or} \quad 180^\circ - 69.44^\circ$$

$$= 69.44^\circ \quad \text{or} \quad 110.56^\circ \qquad \text{(These are both possible values of } B\text{)}$$

If $B = 69.44^\circ$, Figure (i) shows triangle ABC

If $B = 110.50^\circ$, Figure (ii) shows triangle ABC

Case (i) $B = 69.44^\circ$

$$\therefore C = 180^\circ - (50^\circ + 69.44^\circ)$$

$$= 60.56^\circ$$

and $\dfrac{c}{\sin 60.56^\circ} = \dfrac{9}{\sin 50^\circ}$

$$\therefore c = \frac{9 \times \sin 60.56^\circ}{\sin 50^\circ}$$

$$= 10.23 \text{ cm}$$

Case (ii) $B = 110.56^\circ$

$$\therefore C = 180^\circ - (50^\circ + 110.56^\circ)$$

$$= 19.44^\circ$$

and $\dfrac{c}{\sin 19.44^\circ} = \dfrac{9}{\sin 50^\circ}$

$$\therefore c = \frac{9 \times \sin 19.44^\circ}{\sin 50^\circ}$$

$$= 3.91 \text{ cm}$$

Checking:

In each case $A + B + C = 180^\circ$ ✓

In each case use the cosine rule twice. (Not shown here.)

Solutions Exercise Set 4.6 cont.

3. Any two sides and their included angle can be used.

(a) Area of triangle $ABC = \frac{1}{2}(b\,c\,\sin A)$

$$= \frac{1}{2} \times 19 \times 36.61 \times \sin 36°$$

$$= 204 \text{ cm}^2$$

(b) Area of triangle $ABC = \frac{1}{2}(b\,a\,\sin C)$

$$= \frac{1}{2} \times 12 \times 8.33 \times \sin 1.71$$

$$= 49 \text{ m}^2$$

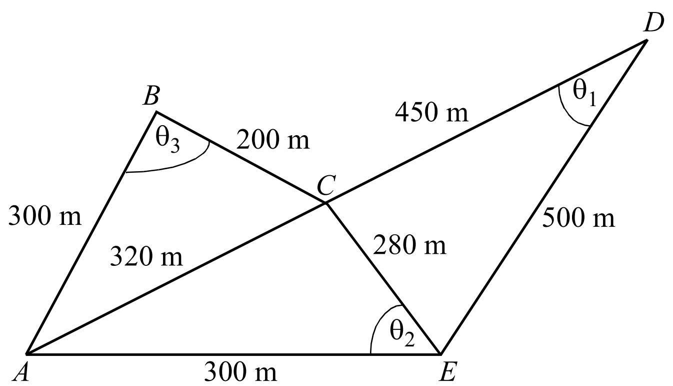
If you got a silly answer like 1.49 you do not have your calculator in radian mode.

(c) Area of triangle $ABC = \frac{1}{2}(a\,c\,\sin B)$

$$= \frac{1}{2} \times 42.7 \times 50.4 \times \sin 2.120$$

$$= 918 \text{ km}^2$$

4.



Consider triangle CDE

Using the cosine rule:

$$CE^2 = CD^2 + ED^2 - 2 \times CD \times ED \times \cos\theta_1$$

$$\therefore\ 280^2 = 450^2 + 500^2 - 2 \times 450 \times 500 \times \cos\theta_1$$

$$\therefore\ \cos\theta_1 = \frac{374100}{450000} = 0.8313$$

$$\therefore \theta_1 = \cos^{-1} 0.8313$$

$$= 33.764°$$

Solutions Exercise Set 4.6 cont.

4. continued

$$\text{Area of triangle } CDE = \frac{1}{2}(CD \times ED \sin\theta_1)$$
$$= \frac{1}{2} \times 450 \times 500 \times \sin 33.764°$$
$$= 62\,525 \text{ m}^2$$

Consider triangle ACE

Using the cosine rule:

$$AC^2 = CE^2 + AE^2 - 2 \times CE \times AE \cos\theta_2$$
$$320^2 = 280^2 + 300^2 - 2 \times 280 \times 300 \cos\theta_2$$
$$\therefore \cos\theta_2 = \frac{66000}{168000} = 0.3929$$
$$\therefore \theta_2 = \cos^{-1} 0.3929$$
$$= 66.868°$$

$$\text{Area of triangle } ACE = \frac{1}{2}(CE \times AE \times \sin\theta_2)$$
$$= \frac{1}{2} \times 280 \times 300 \times \sin 66.868°$$
$$= 38\,623 \text{ m}^2$$

Consider triangle ABC

Using cosine rule:

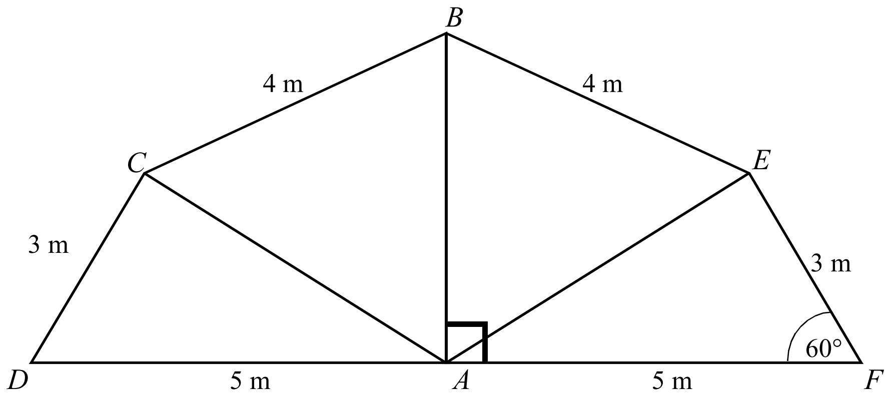
$$AC^2 = AB^2 + CB^2 - 2 \times AB \times CB \cos\theta_3$$
$$320^2 = 300^2 + 200^2 - 2 \times 300 \times 200 \cos\theta_3$$
$$\therefore \cos\theta_3 = \frac{27600}{120000} = 0.23$$
$$\therefore \theta_3 = \cos^{-1} 0.23$$
$$= 76.703°$$

$$\text{Area of triangle } ABC = \frac{1}{2}(AB \times CB \times \sin\theta_3)$$
$$= \frac{1}{2} \times 300 \times 200 \times \sin 76.703°$$
$$= 29\,196 \text{ m}^2$$

$$\therefore \text{Total area of lease} = 62\,525 + 38\,623 + 29\,196$$
$$= 130\,344 \text{ m}^2$$

Solutions Exercise Set 4.6 cont.

5.



Because the diagram is symmetrical about AB, angle $ADC = 60°$.

$\therefore$ In triangle ACD, using the cosine rule:

$$AC^2 = 3^2 + 5^2 - 2 \times 3 \times 5 \times \cos 60°$$

$$= 19$$

$$\therefore AC = \sqrt{19} = 4.36 \text{ m}$$

Using the sine rule:

$$\frac{AC}{\sin ADC} = \frac{CD}{\sin CAD}$$

$$\therefore \frac{\sqrt{19}}{\sin 60°} = \frac{3}{\sin CAD}$$

$$\therefore \sin CAD = \frac{3 \times \sin 60°}{\sqrt{19}} = 0.5960$$

$$\therefore CAD = \sin^{-1} 0.5960$$

$$= 36.6°$$ {It does not make sense to consider an angle > 90°}

Now BA is perpendicular to DF

$$\therefore CAB = 90° - 36.6°$$

$$= 53.4°$$

Solutions Exercise Set 4.6 cont.

5. continued

Consider triangle CAB, using the cosine rule:

$$\begin{aligned} CB^2 &= AC^2 + AB^2 - 2 \times AC \times AB \cos CAB \\ 4^2 &= (\sqrt{19})^2 + AB^2 - 2 \times \sqrt{19} \times AB \cos 53.4° \\ \therefore\ 16 &= 19 + AB^2 - 5.199 \times AB \\ \therefore\ AB^2 - 5.199AB + 3 &= 0 \end{aligned}$$

Using the quadratic formula

$$\begin{aligned} AB &= \frac{-(-5.199) \pm \sqrt{(-5.199)^2 - 4 \times 1 \times 3}}{2 \times 1} \\ &= \frac{5.199 \pm \sqrt{15.03}}{2} \\ &= 4.54 \text{ m} \quad \text{or} \quad 0.66 \text{ m} \end{aligned}$$

By considering the diagram and the size of angles and side lengths in triangle CAB it does not make sense for AB to be 0.66 m

$\therefore$ AB is 4.54 m

Solutions Exercise Set 4.7 page 4.30

I have not shown all the steps here.

(a) LHS $= \cos(173° + 28°) = \cos(201°) = -0.9336$

RHS $= \cos 173° \times \cos 28° - \sin 173° \times \sin 28°$

$= -0.9925 \times 0.8829 - 0.1219 \times 0.4695$

$= -0.8763 - 0.0572 = -0.9335 =$ LHS ✓

(b) LHS $= \sin(173° + 28°) = \sin(201°) = -0.3584$

RHS $= \sin 173° \times \cos 28° + \cos 173° \times \sin 28°$

$= 0.1219 \times 0.8829 + (-0.9925) \times 0.4695$

$= 0.1076 - 0.4660 = -0.3584 =$ LHS ✓

(c) LHS $= \sin(173° - 28°) = \sin(145°) = 0.5736$

RHS $= \sin 173° \times \cos 28° - \cos 173° \times \sin 28°$

$= 0.1219 \times 0.8829 - (-0.9925) \times 0.4695$

$= 0.1076 + 0.4660 = 0.5736 =$ LHS ✓

Solutions Exercise Set 4.8 page 4.34

1.

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

2.

(i)
$$\begin{aligned} \sin 2\theta &= \sin(\theta + \theta) \\ &= \sin\theta \cos\theta + \cos\theta \sin\theta \\ &= 2 \sin\theta \cos\theta \end{aligned}$$

(ii)
$$\begin{aligned} \cos 2\theta &= \cos(\theta + \theta) \\ &= \cos\theta \cos\theta - \sin\theta \sin\theta \\ &= \cos^2\theta - \sin^2\theta \end{aligned}$$

3.

(i)
$$\begin{aligned} \text{RHS} &= \frac{1}{2}(1 + \cos 2\theta) \\ &= \frac{1}{2}(\{\sin^2\theta + \cos^2\theta\} + \cos 2\theta) && \{\text{Using } \sin^2\theta + \cos^2\theta = 1\} \\ &= \frac{1}{2}(\sin^2\theta + \cos^2\theta + \{\cos^2\theta - \sin^2\theta\}) && \{\text{Using double angle formula for cosine}\} \\ &= \frac{1}{2} \times 2\cos^2\theta \\ &= \cos^2\theta = \text{LHS} \end{aligned}$$

(ii)
$$\begin{aligned} \text{RHS} &= \frac{1}{2}(1 - \cos 2\theta) \\ &= \frac{1}{2}(\{\sin^2\theta + \cos^2\theta\} - \{\cos^2\theta - \sin^2\theta\}) \\ &= \frac{1}{2}(\sin^2\theta + \cos^2\theta - \cos^2\theta + \sin^2\theta) \\ &= \frac{1}{2} \times 2\sin^2\theta \\ &= \sin^2\theta = \text{LHS} \end{aligned}$$

Solutions Exercise Set 4.8 cont.

4.

(i) $$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\therefore\ 2\cos^2\theta = 1 + \cos 2\theta$$

$$\therefore\ \cos 2\theta = -1 + 2\cos^2\theta$$

(ii) $$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\therefore\ 2\sin^2\theta = 1 - \cos 2\theta$$

$$\therefore\ \cos 2\theta = 1 - 2\sin^2\theta$$

Solutions Exercise Set 4.9 page 4.36

1. *There are many ways of doing these problems.*

(a) $$\text{LHS} = \frac{\sec A}{\operatorname{cosec} A} = \frac{\frac{1}{\cos A}}{\frac{1}{\sin A}}$$

$$= \frac{1}{\cos A} \times \frac{\sin A}{1} = \tan A = \text{RHS}$$

(b) $$\text{LHS} = \frac{\tan\theta}{\sqrt{1 + \tan^2\theta}} = \frac{\tan\theta}{\sqrt{\sec^2\theta}}$$

$$= \frac{\tan\theta}{\sec\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}$$

$$= \frac{\sin\theta}{\cos\theta} \times \frac{\cos\theta}{1}$$

$$= \sin\theta$$

$$\text{RHS} = \frac{1}{\operatorname{cosec}\theta} = \frac{1}{\frac{1}{\sin\theta}}$$

$$= 1 \times \frac{\sin\theta}{1} = \sin\theta = \text{LHS}$$

Solutions Exercise Set 4.9 cont.

2.

(a) $$\text{RHS} = \frac{\sin(A+B)}{\cos A\cos B}$$

$$= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B}$$

$$= \frac{\sin A\cos B}{\cos A\cos B} + \frac{\cos A\sin B}{\cos A\cos B}$$

$$= \tan A + \tan B = \text{LHS}$$

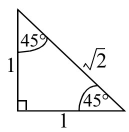
(b) $$\text{RHS} = \frac{1+\tan A}{1-\tan A}$$

$$\text{LHS} = \tan\left(\frac{\pi}{4}+A\right)$$

$$= \frac{\sin\left(\frac{\pi}{4}+A\right)}{\cos\left(\frac{\pi}{4}+A\right)}$$

$$= \frac{\sin\frac{\pi}{4}\cos A + \cos\frac{\pi}{4}\sin A}{\cos\frac{\pi}{4}\cos A - \sin\frac{\pi}{4}\sin A}$$

$$= \frac{\frac{1}{\sqrt{2}}\cos A + \frac{1}{\sqrt{2}}\sin A}{\frac{1}{\sqrt{2}}\cos A - \frac{1}{\sqrt{2}}\sin A}$$



$$= \frac{\frac{1}{\sqrt{2}}(\cos A + \sin A)}{\frac{1}{\sqrt{2}}(\cos A - \sin A)}$$

$$= \frac{\cos A + \sin A}{\cos A - \sin A}$$

$$= \frac{1+\frac{\sin A}{\cos A}}{1-\frac{\sin A}{\cos A}}$$ {Dividing each term in numerator and denominator by $\cos A$}

$$= \frac{1+\tan A}{1-\tan A} = \text{RHS}$$

Solutions Exercise Set 4.9 cont.

2. continued

(c) $$\text{LHS} = \frac{\cos(A+B)}{\sin(A+B)}$$

$$= \frac{\cos A\cos B - \sin A\sin B}{\sin A\cos B + \cos A\sin B}$$

$$= \frac{\cos A - \dfrac{\sin A\sin B}{\cos B}}{\sin A + \dfrac{\cos A\sin B}{\cos B}} \qquad \{\text{Divide each term by } \cos B\}$$

$$= \frac{\cos A - \sin A\tan B}{\sin A + \cos A\tan B}$$

$$= \frac{\dfrac{\cos A}{\sin A} - \tan B}{1 + \dfrac{\cos A\tan B}{\sin A}} \qquad \{\text{Divide each term by } \sin A\}$$

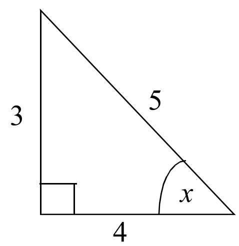
$$= \frac{\cot A - \tan B}{1 + \cot A\tan B} = \text{RHS}$$

(d) $$\text{LHS} = 2\sin^2\theta - 2$$

$$= 2(\sin^2\theta - 1)$$

$$= -2\cos^2\theta = \text{RHS} \qquad \{\text{Using } \sin^2\theta + \cos^2\theta = 1\}$$

3. If $\cos x = \dfrac{4}{5}$



(i) $$\cos 2x = -1 + 2\cos^2 x$$

$$= -1 + 2\times\left(\frac{4}{5}\right)^2$$

$$= -1 + \frac{32}{25} = \frac{7}{25}$$

(ii) $$\sin x = \frac{3}{5}$$

$$\therefore \sin 2x = 2\sin x\cos x$$

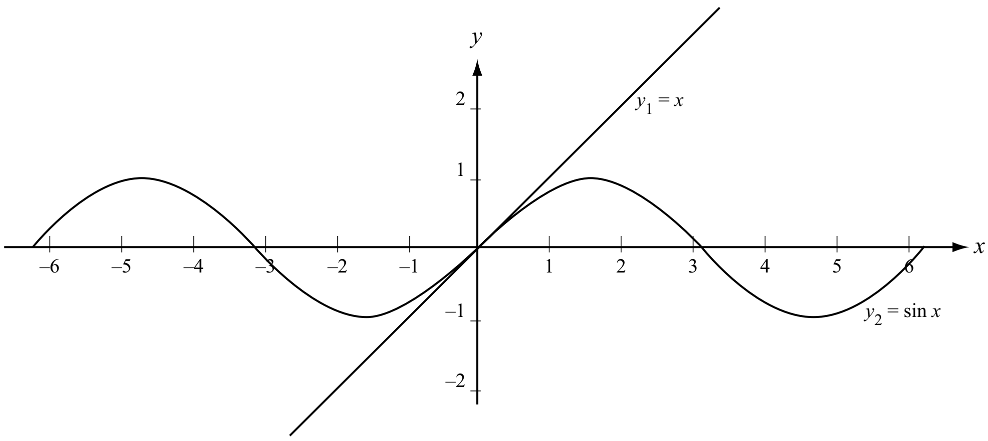
$$= 2\times\frac{3}{5}\times\frac{4}{5}$$

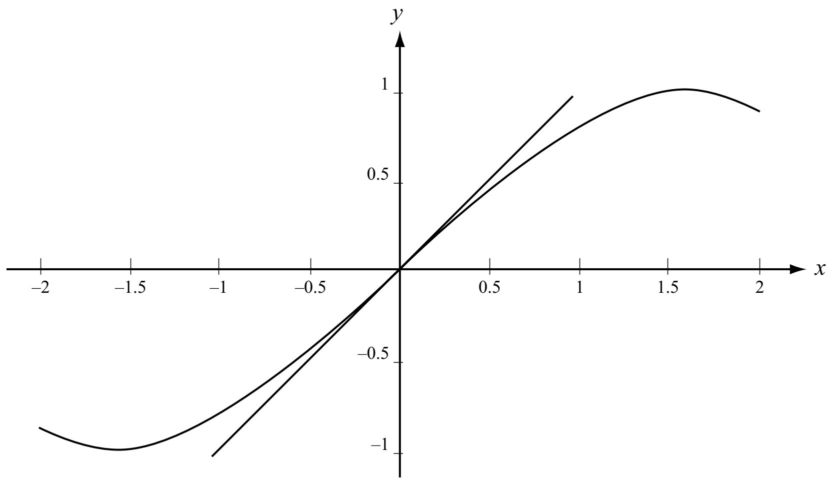
$$= \frac{24}{25}$$

Solutions Exercise Set 4.10 page 4.41

1. Draw the graphs of $y_1 = x$ and $y_2 = \sin x$ for $-6 \le x \le 6$.

   Remember to use radians.





Changing the scaling of the graph shows that there is only one point of intersection and this occurs when $x = 0$

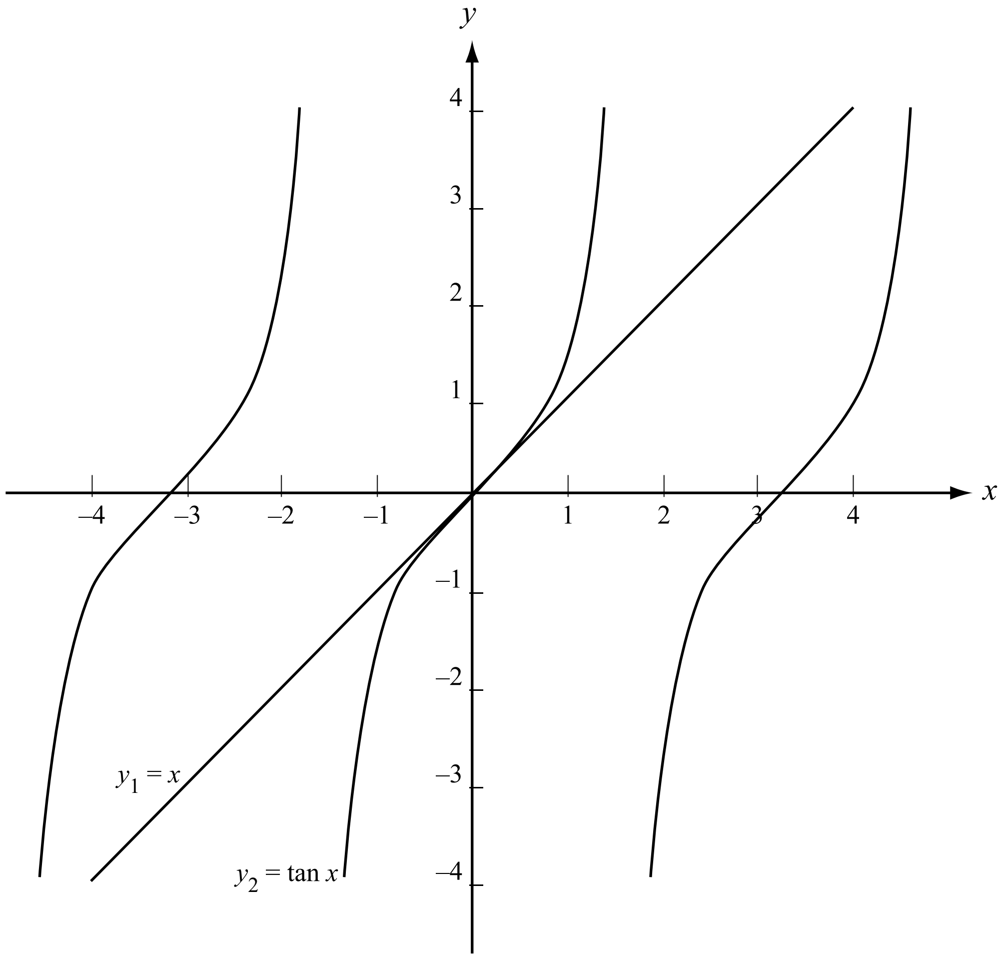
$\therefore$ The solution of $x - \sin x = 0$ for $-6 \le x \le 6$ is $x = 0$

Checking:

When $x = 0$, $x - \sin x = 0 - \sin 0 = 0$

Solutions Exercise Set 4.10 cont.

2. Draw the graphs of $y_1 = x$ and $y_2 = \tan x$ for $-4 \le x \le 4$



There is only one point of intersection of $y_1 = x$ and $y_2 = \tan x$ for $-4 \le x \le 4$

This occurs at $x = 0$

$\therefore$ Solution of $x = \tan x$ for $-4 \le x \le 4$ is $x = 0$

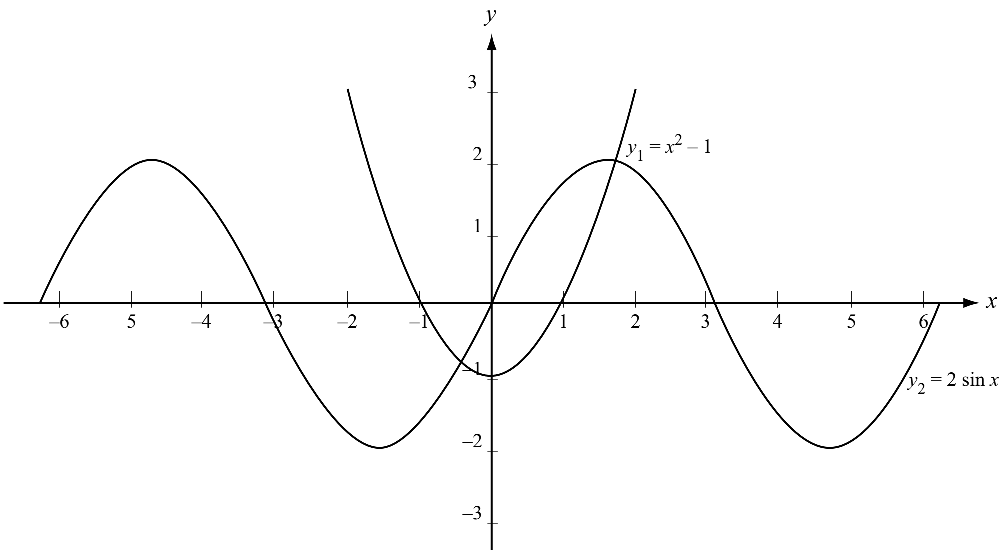
Checking:

When $x = 0$, LHS = 0

RHS = $\tan x = \tan 0$ = LHS ✓

Solutions Exercise Set 4.10 cont.

3. Draw the graphs of $y_1 = x^2 - 1$ and $y_2 = 2\sin x$



There are two points of intersection of $y_1 = x^2 - 1$ and $y_2 = 2\sin x$.

These occur when $x \approx -0.4$ and $x \approx 1.7$

So the solutions to $x^2 - 1 - 2\sin x = 0$ are $x \approx -0.4$ and $x \approx 1.7$

Checking:

When $x = -0.4$

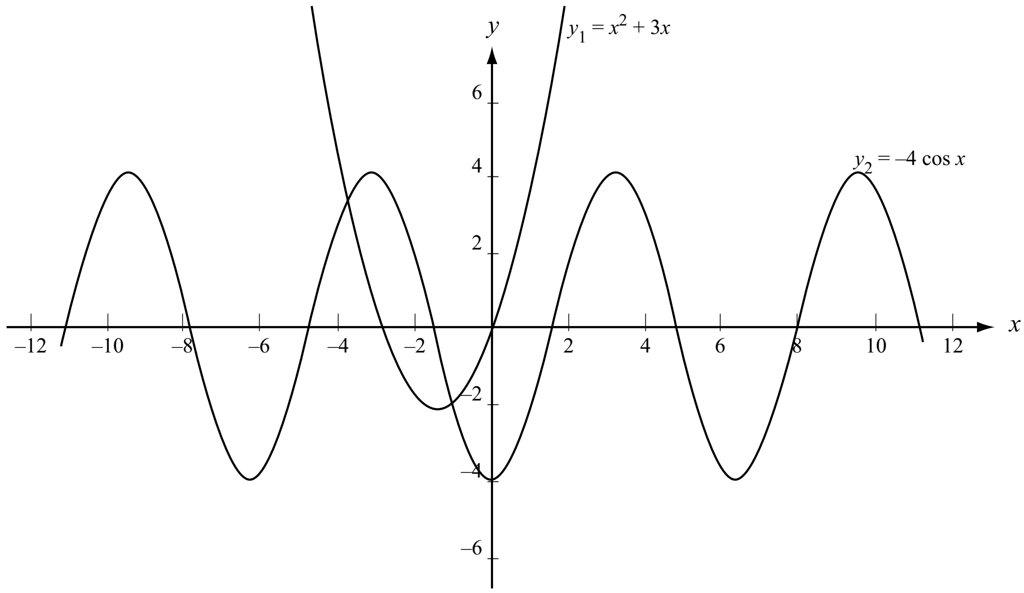
$x^2 - 1 - 2\sin x = (-0.4)^2 - 1 - 2\sin(-0.4) \approx 0$ ✓

When $x = 1.7$

$x^2 - 1 - 2\sin x = 1.7^2 - 1 - 2\sin 1.7 \approx 0$ ✓

Solutions Exercise Set 4.10 cont.

4. Draw the graphs of $y_1 = x^2 + 3x$ and $y_2 = -4 \cos x$



There are two points of intersection of $y_1 = x^2 + 3x$ and $y_2 = -4 \cos x$

These occur at $x \approx -3.8$ and $x \approx -1.0$ (you can zoom in and get more accurate x values)

So the solutions to $x^2 + 4 \cos x = -3x$ are $x \approx -3.8$ and $x \approx -1.0$

Checking:

When $x = -3.8$

LHS $= x^2 + 4 \cos x$

$= (-3.8)^2 + 4 \cos(-3.8) = 14.44 - 3.16 = 11.3$

RHS $= -3x$

$= -3 \times (-3.8) = 11.4 \approx \text{LHS}$ ✓

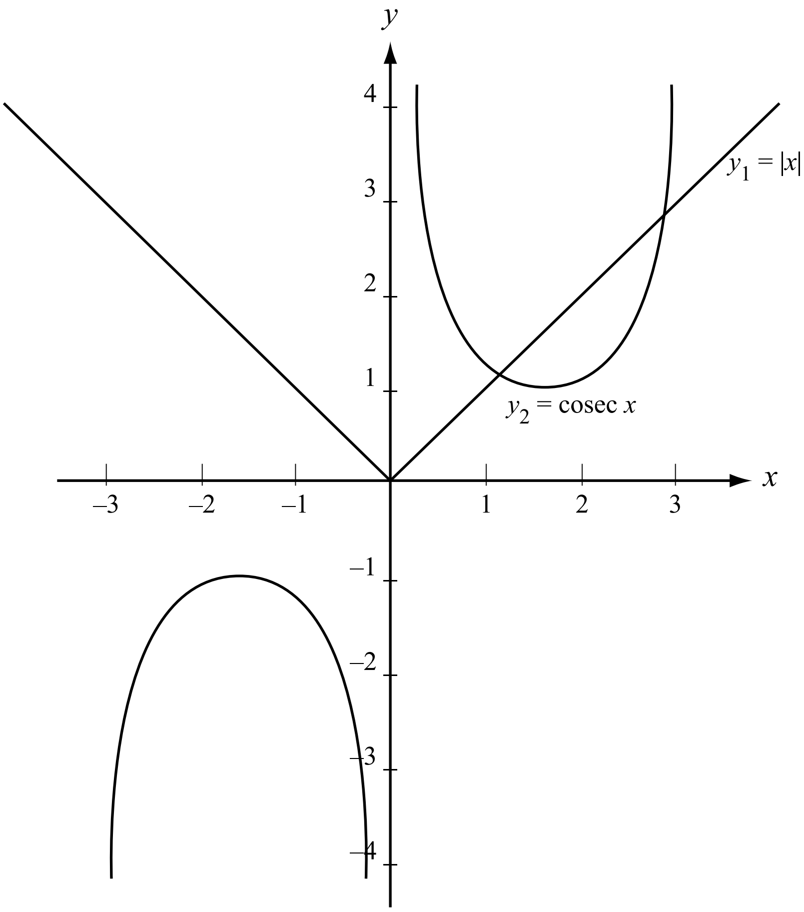
When $x = -1.0$

LHS $= (-1.0)^2 + 4 \cos(-1.0) = 1.0 + 2.16 = 3.2$

RHS $= -3 \times (-1.0) = -3.0 \approx \text{LHS}$ ✓

Solutions Exercise Set 4.10 cont.

5. Draw the graphs of $y_1 = |x|$ and $y_2 = \operatorname{cosec} x$ for $-2 \le x \le 2$



There is only one point of intersection of $y_1 = |x|$ and $y_2 = \operatorname{cosec} x$ in the domain $-2 \le x \le 2$

This occurs at $x \approx 1.1$

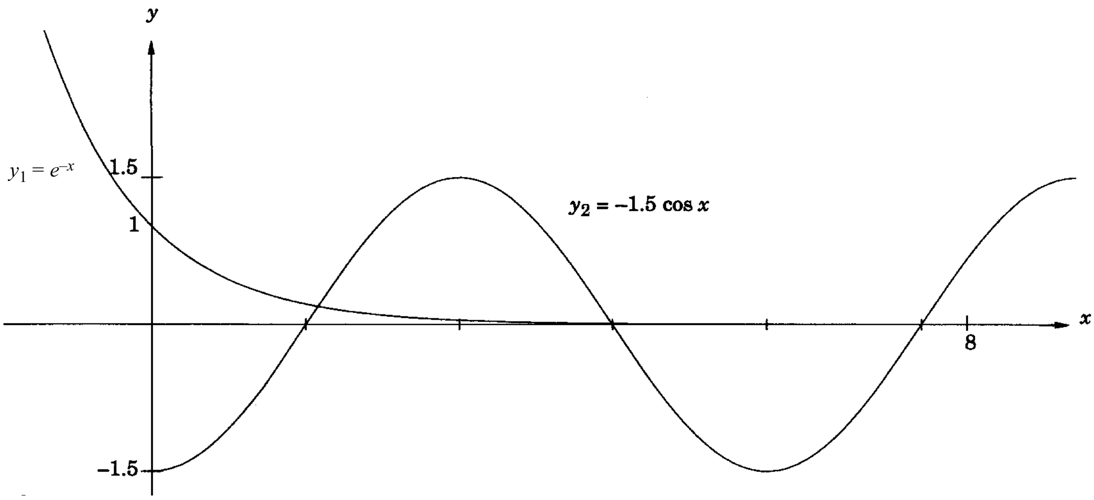
Checking:

When $x = 1.1$

$|x| - \operatorname{cosec} x = |1.1| - \operatorname{cosec} 1.1 = 1.1 - 1.12 \approx 0$ ✓

Solutions Exercise Set 4.10 cont.

6. Draw the graphs of $y_1 = e^{-x}$ and $y_2 = -1.5 \cos x$ for $0 < x < 8$



There are three points of intersection of $y_1 = e^{-x}$ and $y_2 = -1.5 \cos x$ for $0 < x < 8$

These occur when $x \approx 1.72$, $x \approx 4.71$ and $x \approx 7.84$

So the solutions to $e^{-x} = -1.5 \cos x$ are $x \approx 1.72$, $x \approx 4.71$ and $x \approx 7.84$

Checking:

When $x = 1.72$

$\text{LHS} = e^{-1.72} = 0.18$

$\text{RHS} = -1.5 \cos 1.72 = 0.22 \approx \text{LHS}$

When $x = 4.71$

$\text{LHS} = e^{-4.71} = 0.01$

$\text{RHS} = -1.5 \cos 4.71 = 0.00 \approx \text{LHS}$

When $x = 7.84$

$\text{LHS} = e^{-7.84} = 0.00$

$\text{RHS} = -1.5 \cos 7.84 = 0.02 \approx \text{LHS}$